Essay on a CDS Pricing Model with Joint Defaults

by

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Abstract

This essay discusses two resources of default, firm's default and counter-party's default in a CDS contract. We model the default in individual case and simultaneous case by infinitesimal Markov processes, and then deduce the CDS pricing equation by Itô formula under the general asset pricing framework. In the end, we introduce the correlation of defaults by Markov copula property, construct defaults with constant intensities and linear intensities respectively, and solve the pricing equation numerically.

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Last but not least, I wish to thank my parents and my twin sister. They supported me, taught me and love me. I want to dedicate this essay to them.

Dedication

To my parents

Chapter 1

Introduction

1.1 Credit Risk and CDS

Risk refers to the uncertainty of financial products. Usually, we consider the downside possibilities as the risk. There are various type of risks. The price going up or down is called market risk. The bonds not able to sell or buy is called liquidity risk. The default spreading mechanically is systemic risk. The borrower failing to make payment as promise is credit risk. In the essay, I focus on the credit risk. [2]

Credit risk varies the other types of risk. It is not easy to measure because of significant resources and asymmetric information (e.g. the counter-party's private information of the cash flow). Once credit risk occurs, it has been shown to be particularly large and damaging for the large investment projects. In 2008, the ever largest financial insurer AIG had to ask for a huge federal bailout to settle CDS contracts after the bankruptcy of Lehman Brothers.

In order to reduce credit risks, people design different credit derivatives these years. Credit default swap(CDS) is one of the them. The buyer of the CDS makes a series of payments to the CDS seller, in exchange, receives a payoff (protection) if a credit crisis occurs. In the ordinary way, CDS works well and can be thought as a form of insurance. However, in the extreme case, there is still risk in the CDS and it will cause great loss to either of the participants. These will be talked later.

1.2 CDS Worries and Development

Although CDS is treated as a tool to prevent credit risk, CDS itself still faces risks. The buyer takes the risk that the seller will default, which is also the case in the AIG's crisis. Without government's help, AIG's counter-parties would take on all the losses of CDS default. On the other hand, the seller takes the risk that the buyer will default on the contract, depriving the seller of the expected revenue stream. Furthermore, they has the potential risk of dealing with the existed hedging position by closing out the position or searching a new CDS buyer.

The market for CDS attracted considerable concern from regulators after a number of large scale incidents in 2008. In 2009, the introduction of central clearing houses, which act as the counter-party to both sides of a CDS transaction, and international standardization of CDS contracts change the way CDS operates. In the U.S., central clearing operations began in March 2009, operated by InterContinental Exchange(ICE). It's biggest competitor CME was also granted for entering CDS clearing sector in the end of 2009. In Europe, CDS index clearing was launched by ICE's European subsidiary ICE Clear Europe on July 31 2009, following by the Single Name clearing in the end of the year.

The appearance of clearing house transfer the risk between counter-parties to the central clearing house, thus reducing the risk of a counter-party defaulting on a transaction, which is the primary risk in the over-the-counter market. In addition, a clearinghouse provides one location for regulators to view traders' position and price, which increasing the transparency of the CDS market and breaking the information asymmetry.

Chapter 2

Description of CDS

2.1 CDS contract details

CDS is a contract that the buyer pays a consistent money (premium) to the seller, and get the compensation once the disaster arrive. To sketch out the counter-party risk on a CDS, we view it from either buyers' or sellers' standpoints.

• for the buyer, the seller may fail to pay the protection cash flow to the buyer in case of a default of the firm

• for the seller, he takes on the risk of losing premium in the contract period of validity.

In the essay, we consider only first case, which is the case the buyers will never default. The figure below shows the parties in the contract and their relationship.

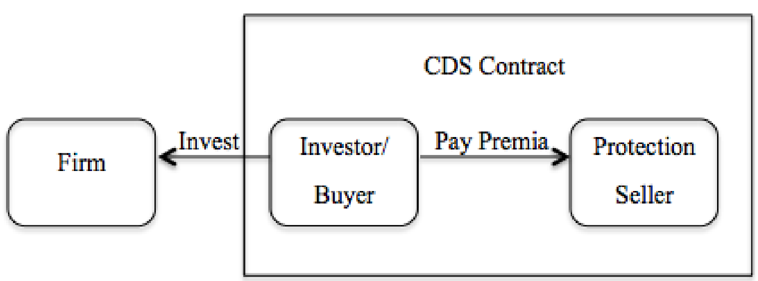

Figure 2.1: CDS Contract Structure

As shown in the illustration, there are three parties involved in the CDS contract. The firm, which is in the basic hierarchy, is the party who selling her risky financial products to the others. The investors of the firm, on one hand, invest the firm by buying her credit financial products (e.g. bonds), one the other hand, sign the CDS contract with the counter-party, which is a third party could be considered as a protection seller, to prevent the loss of investment once the firm defaults.

To use the mathematical tools, we need to clarify and parameterize the CDS contract and CDS risks at first. Since the CDS is a tradable contract, there is price for CDS. A fair CDS price Π_t, which we called risky CDS price in the essay, includes the influences not only from the CDS contract clauses, the firm default, but also from the default of counter-party. These factors will be clearer in the successive section. In a contrast, the CDS price without considering the counter-party's risk, the last factor in the risky CDS contract, is called risk-free CDS price P_t.

One principal clause in the CDS contract is about premia κ. The investor pays to the counter-party a stream of premia with spread κ from time 0 until the maturity T of the contract. If the default comes before maturity, the premia should be taken account into the loss of CDS.

Another important clause in the CDS contract talks about the protection against firm's default. Suppose the CDS buyers invest the bonds of the firm and the bonds default, then protection seller need to pay the par value of the bonds to the buyer, or to be more precise, to pay the difference between the par value and market value (at default time) of the bonds, meanwhile, the buyers deliver the bonds to the seller.

There are two types of risks associated with the CDS, the default of the firm, which is a concern in the CDS contract clause, and the default of CDS seller. The two aspect are the core factors leading CDS to different consequences. We use subscripts 1 and 2 to distinguish the quantities related to the firm from that to the seller of the CDS respectively and call them the firm and counter-party in the remaining part.

In addition, we denote default time of the firm and that of the counter-party as τ_1 and τ_2, they can take positive real value and positive infinity, which means they will never default. When either party defaults, in the real life, the court will intervene in this affair. They will sell all of the party's assets compulsorily by the market price of that time. The price will be much lower because of the negative news. (e.g. bankruptcy lead to a default) The assets sold are used to repay the investors by different priorities. To depict this fact, we introduce the concept of recovery rate of the firm and counter-party by R_1 and R_2. They can be thought as the ratio of the assets sold after bankruptcy to assets insured in the CDS contract before the bankruptcy(i.e a ratio of CDS face value). At last, we assume for simplicity that both process are adapted to the information available at time τ_1 and τ_2 and the face value of CDS is 1.

2.2 3 states of CDS contract

This section lists the three possible states of CDS contract. We discuss the buyers' situations in each case.

• **Scenario 1** : **No defaults**
This is the case that neither the firm nor counter-party will default. CDS contract is in this state most of time. The investor pays to the counterparty a stream of premia with spread κ from time 0 until default or the maturity T of the contract, whichever come first.

• **Scenario 2** : **Firm defaults first**
The firm defaults first and sells all the asset by R_1 of the CDS face value. In this case, the protection clause in the CDS contract takes effect. However, the protection cash flows gained by the buyer still depend on the counter-party's actions. We discuss the buyers' payoffs in the cases that the counter-party default and doesn't default.
Case 1: counter-party doesn't default and is still alive, she can fully compensate the loss of buyer, pay$(1-R_1)$ face value of CDS. In this context, payoff of the buyer is $(1-R_1)$.

Case 2: counter-party defaults at the same time as the firm($\tau_1 = \tau_2$). She supposed to pay the difference value of the CDS by $(1-R_1)$(as in *case1*), while her assets decrease to R_2 of the initial wealth in the bankruptcy, thus she could only pay a fraction of the difference to the buyer $R_2(1-R_1)$.

• **Scenario 3** : **Counter-party defaults first**

Usually, there are not obvious reasons that the counter-party will default earlier than the firm does. If it happens, it means the counter-party goes bankruptcy and cannot provide the protection any longer. However, since CDS are sold over-the-counter, it implies the two parties of the contract could negotiate the terms and even the ways to exit the trade in advance. In real life, counter-party stops the contract after paying the buyers a terminal cash flow (positive or negative). More precisely, the terminal cash flow is reasonably considered as a fair value $\chi(\tau_2)$ of CDS at counter-party default time τ_2. The complete story is, if the counter-party defaults in the life time of CDS while the firm is still alive, $\chi(\tau_2)$ is computed from the perspective of the buyer.

Case 1: if $\chi(\tau_2) < 0$, it means CDS is worthless to the buyer. Buyer need pay $-\chi(\tau_2)$ to counter-party in order to end the contract.

Case 2: if $\chi(\tau_2) > 0$, the CDS is still valuable to the buyer, but counter-party could only pay $R_2\chi(\tau_2)$ to the buyer because of its recovery rate R_2.

Remark:

There are some supplementary notes for the fair value $\chi(\tau_2)$ of CDS. $\chi(\tau_2)$ is actually the fair CDS price at time τ_2. Recall the two definitions of CDS price in the last section, risky CDS price Π_t and risk-free CDS price P_t, we can choose either of them. While it's better to choose $\chi(\tau_2) = \Pi_{\tau_2}$ for consistency since all things are discussed under risky CDS set-up. We will talk both alternatives later and still use $\chi(\tau_2)$ to skip the details.

To conclude this section, we list all the possibilities in the below table. Furthermore, we specify the periods of three scenarios, i.e. point out the order of default time τ_1 , τ_2 and maturity T. It is helpful to understand the next formulas.

Table 2.1: CDS Scenarios

Cash Flows in Different Scenarios (not discount)				
	No Defaults	Firm Defaults first		Counter-party Defaults first
Periods	$t < \tau_1 \wedge \tau_2 \wedge T$	$t = \tau_1$ $\tau_1 < T$ $\tau_1 < \tau_2$	$t = \tau_1$ $\tau_1 < T$ $\tau_1 = \tau_2$	$t = \tau_2$ $\tau_2 < T$ $\tau_2 < \tau_1$
Buyer	$-\kappa$	$1 - R_1$	$R_2(1 - R_1)$	$\chi(\tau_2)(\chi(\tau_2) < 0)$ $R_2\chi(\tau_2)(\chi(\tau_2) > 0)$

2.3 CDS pricing model framework

In the previous part, we have discussed most of the CDS details and modelled them in a mathematical way. In this section, we will follow the general asset pricing theory and apply it to the CDS case, thus give the framework of CDS pricing model. In addition, some CDS risk measurements will also be discussed at the same time.

In the general asset pricing theory, no arbitrage condition makes sure there is a risk-neutral pricing world $(\Omega, \mathbb{F}, \mathbb{P})$. If there is a risk-free asset, then we can find a discount factor β, for any self financing asset S, βS is a martingale. Without further precision, we assume all the conditions in the asset pricing theory hold. The filtration $\mathbb{F} = (\mathscr{F}_t)_{t \in [0,T]}$ is nice which makes τ_i's stopping times and discount factor β adapted. In shorthand, we denote $\mathbb{E}_t(\cdot) = \mathbb{E}(\cdot|\mathscr{F}_t)$ to stand for the conditional expectation at time t.

To price the risky CDS process Π_t and risk-free CDS process P_t, we need to find the value of the CDS at time t first, which is the total cash flow we can gain in the future. We let $\pi_T(t)$ is the risky CDS cumulative discounted cash flow on $(t, T]$ and similarly $p_T(t)$ is risk-free CDS cumulative discounted cash flow on $(t, T]$. The illustrations below describe all of the cash flows for $\pi_T(t)$ and $p_T(t)$. For the buyer, the premia produce negative cash flows and protection payoffs from counter-party produce positive cash flows. It is straightforward to get the expressions for both CDS cumulative discounted cash flows from these figures.

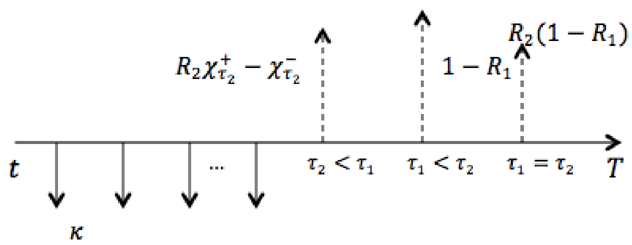

Figure 2.2: Risky CDS Cash Flow in $(t, T]$

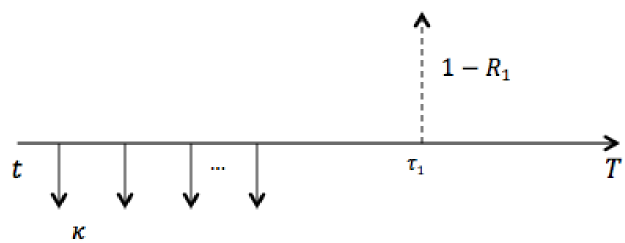

Figure 2.3: Risk-Free CDS Cash Flow in $(t, T]$

Accordingly, the value of the risky and risk-free CDS are:

$$\beta_t \pi_T(t) = -\kappa \int_{t\wedge\tau_1\wedge\tau_2\wedge T}^{\tau_1\wedge\tau_2\wedge T} \beta_s ds + \beta_{\tau_1}(1-R_1)\mathbb{1}_{t<\tau_1\leq T}[\mathbb{1}_{\tau_1<\tau_2} + R_2\mathbb{1}_{\tau_1=\tau_2}]$$
$$+ \beta_{\tau_2}\mathbb{1}_{t<\tau_2\leq T}\mathbb{1}_{\tau_2<\tau_1}[R_2\chi^+_{\tau_2} - \chi^-_{\tau_2}] \tag{2.1}$$
$$\beta_t p_T(t) = -\kappa \int_{t\wedge\tau_1\wedge T}^{\tau_1\wedge T} \beta_s ds + \beta_{\tau_1}(1-R_1)\mathbb{1}_{t<\tau_1\leq T} \tag{2.2}$$

and $\pi_T(t) = 0$ when $t \geq \tau_1 \wedge \tau_2 \wedge T$, $p_T(t) = 0$ when $t \geq \tau_1 \wedge T$ because there are no cash flows after any default. Since $\beta_t \Pi_t = \mathbb{E}_t$(discounted value of CDS) $= \mathbb{E}_t[\beta_t \pi_T(t)]$, LHS in (2.1), thus we can get $\Pi_t = \mathbb{E}_t[\pi_T(t)]$ and $P_t = \mathbb{E}_t[p_T(t)]$ in the same way.

Besides the CDS price, risk measurements are always concerned in the practical world. Exposure at default is a popular risk measurement in industry. It is the amount an investor has at risk or the amount she can lose. For a CDS, the exposure at default $\xi(\tau_2)$ is the loss caused by counter-party's default. It is easy to get from counter-party default scenarios in Table 2.1 by substracting the payoffs at default from CDS face value.

$$\xi(\tau_2) = \begin{cases} (1-R_1)(1-R_2) & \tau_2 = \tau_1 \leq T \\ P_{\tau_2} - (R_2\chi^+_{\tau_2} - \chi^-_{\tau_2}) & \tau_2 < \tau_1, \tau_2 \leq T \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

There is another explannation for exposure at default $\xi(\tau_2)$. Notice the value of risky CDS and risk-free CDS (in Figure 2.2 and Figure 2.3), the difference of each other is the exposure at default. It shows $\xi(\tau_2)$ is the added value from risk-free CDS to risky CDS. Thus we can treat $\xi(\tau_2)$ as an asset and also evaluate its' price. Following the idea, we define a process stopped at $\tau_1 \wedge \tau_2 \wedge T$ and call it Credit Valuation Adjustment(CVA).

$$\beta_t \text{CVA}_t = \mathbb{1}_{t<\tau_2}\mathbb{E}_t[\beta_{\tau_2}\xi(\tau_2)] \tag{2.4}$$

By intuition, $CVA_t = P_t - \Pi_t$ on $\{t < \tau_2\}$ because the difference in value results in the difference in price. The strict proof is given by S.Crepey' paper, **proposition 2.1**

The another index Expected Positive Exposure(EPE) is also considered in the article. It is the function of the time t and given by

$$\text{EPE}(t) = \mathbb{E}[\xi(\tau_2)|\tau_2 = t] \tag{2.5}$$

Chapter 3

Model of Defaults

We have already set up the framework of CDS model. However, we haven't dealt with defaults model yet. In the chapter, we will discuss the details of firm's default and counter-party's default to complete the description of the model.

3.1 Pair of Markov defaults

In the risk neutral world$(\Omega, \mathbb{F}, \mathbb{P})$, $(\Omega, \mathbb{F})$ decides all the possibilities of financial world. Assets are the random variables on this space, thus giving distinct payoffs under each case. What is the probability space for the CDS? Notice the possible cumulative discounted value of the CDS $\pi_T(t)$ in formula 2.1

$$\begin{aligned}\beta_t \pi_T(t) = &-\kappa \int_{t\wedge\tau_1\wedge\tau_2\wedge T}^{\tau_1\wedge\tau_2\wedge T} \beta_s ds + \beta_{\tau_1}(1-R_1)\mathbb{1}_{t<\tau_1\leq T}[\mathbb{1}_{\tau_1<\tau_2} + R_2\mathbb{1}_{\tau_1=\tau_2}] \\ &+ \beta_{\tau_2}\mathbb{1}_{t<\tau_2\leq T}\mathbb{1}_{\tau_2<\tau_1}[R_2\chi_{\tau_2}^{+} - \chi_{\tau_2}^{-}]\end{aligned}$$

only depends on the stopping times τ_1 and τ_2, so the events set $\mathbb{F}$ should be generated by τ_1 and τ_2. To be precise, we denote $H = (H^1, H^2) = (\mathbb{1}_{\tau_1\leq t}, \mathbb{1}_{\tau_2\leq t})$ the pair of the default indicator process of the firm and the counter-party. If the spaces of $\mathbb{1}_{\tau_1\leq t}$ and $\mathbb{1}_{\tau_2\leq t}$ are clear,then

$$\mathbb{F} = \mathbb{H} = (\sigma(H_s^1; 0\leq s\leq t) \vee \sigma(H_s^2; 0\leq s\leq t))_{t\in[0,T]}$$

The reason we use the pair default process $H = (H^1, H^2)$ instead of individual default processes H^is$(i = 1,2)$ is that despite of knowing each default process, we also need to pay attention to the relationship between them. The evidences can be shown in the formula 2.1 by the terms $\mathbb{1}_{t<\tau_1\leq T}\mathbb{1}_{\tau_1<\tau_2}$, $\mathbb{1}_{t<\tau_1\leq T}\mathbb{1}_{\tau_1=\tau_2}$ and $\mathbb{1}_{t<\tau_2\leq T}\mathbb{1}_{\tau_2<\tau_1}$. One of the way of formulating a multivariate distribution (pair process) from the marginal distributions is copula. We will see this point in the construction of the pair process.

We model $H = (H^1, H^2)$ as inhomogeneous Markov relative to its own filtration. Since the $H^i = \mathbb{1}_{\tau_i<t}$ are the processes with one jump in size 1, it's not hard to figure out the probability space $(\Omega, \mathbb{P})$, where state space is $\Omega = \{(0,0),(1,0),(0,1),(1,1)\}$.

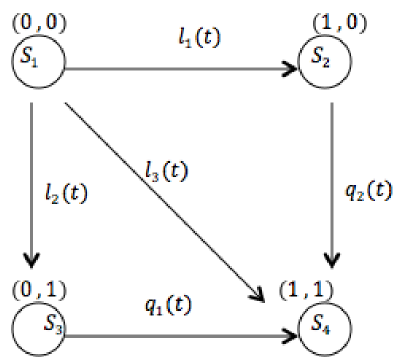

Figure 3.1: State Transition Diagram

The Figure 3.1 illustrate the state space and their transition probabilities in the infinitesimal time $[t, t+\Delta t]$. For example, the probability from state S_1 to state S_2 is $l_1(t)dt$, that is, the probability of default of the firm alone in time $[t, t+\Delta t]$ is $l_1(t)dt$. Another thing to mention here, there is no routes from S_4 $(1,1)$ to the other states. This is based on the fact that once a company defaults(going bankruptcy), she can not return to the state it is alive. Using the information in the Figure 3.1, we could define the infinitesimal generator 4×4 matrix $A(t)$ at time t for this markov process.

$$A(t) = \begin{bmatrix} -l(t) & l_1(t) & l_2(t) & l_3(t) \\ 0 & -q_2(t) & 0 & q_2(t) \\ 0 & 0 & -q_1(t) & q_1(t) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The columns (or rows) of $A(t)$ correspond with four possible states $(0,0),(1,0)$, $(0,1),(1,1)$ in order. The entry of $A_{ij}(i \neq j)$ gives the transition probability $A_{ij}dt$ from state S_i to S_j in a small time interval dt. For example, $A_{24}(t) = q_2(t)$ shows there is a probability $q_2(t)dt$ of counter-party defaulting after knowing firm's default (i.e. $S_2 = (1,0) \rightarrow S_4 = (1,1)$) in the time interval $(t, t+dt)$. We assume l's and q's in $A(t)$ denote deterministic functions of time integrable over $[0,T]$ and $l(t) = l_1(t) + l_2(t) + l_3(t)$.

Under the last assumption, we find the the sum of each line is equal to zero. This is from the requirement of the infinitesimal generator of a countinuous Markov process. The assumption is reasonable because the interpretation of $A(t)$ is that, if we start from a probability distribution $\vec{p}(0) = (p_1(0), p_2(0), p_3(0), p_4(0))$ at time $t = 0$, then the probability distribution $\vec{p}(t)$ at time $t > 0$ solves the equation

$$\frac{d}{dt}\vec{p}(t) = \vec{p}A(t), \text{ with } \vec{p}(0) \text{ given}$$

and hence

$$\vec{p}(t) = \vec{p}(0)e^{A(t)}$$

Note that distribution forces $\sum_{i=1}^{4} p_i(t) = 1$, so:

$$\sum_{i=1}^{4} \frac{dp_i}{dt} = \sum_{i=1}^{4} (\sum_{j=1}^{4} a_{ij}) p_i \quad \text{for all } (p_1, \cdots, p_4)$$

which is equivalent to the assumption:

$$\sum_{j=1}^{4} a_{ij} = 0 \quad \text{for all } i$$

3.2 Marginal Markov processes

Besides the Markov generator assumption, we will give the Markov copula assumptions here.

(i) $q_1(t) = l_1(t) + l_3(t)$

(ii) $q_2(t) = l_2(t) + l_3(t)$

This property help us to decouple the marginal from the pair process. To see this, we discuss in two cases:

Case 1: Conditional on H_t^1 being 0, for the pair process, it could be either in state $(0,0)$ or $(0,1)$. When (H_t^1, H_t^2) is $(0,0)$, the probability of H_t^1 changing from 0 to 1 is made up of two parts $(0,0) \to (1,0)$ and $(0,0) \to (1,1)$, which is $(l_1(t) + l_3(t))\,dt$. When (H_t^1, H_t^2) is $(0,1)$, the probability of firm defaulting is the probability of the event $(0,1) \to (1,1)$, which is $q_1(t)dt$. With the copula assumption, we know whatever the initial state of H_t^2 is, the probability of H_t^1 defaulting is $q_1(t)dt$.

Case 2: Conditional on H_t^2 being 0, in the same argument, the default probability of H_t^2 alone, whatever the start state is $(0,0)$ or $(1,0)$, is $q_2(t)dt$ in the next time interval $(t, t+dt)$.

By the Markov copula assumptions and analysis above, we could isolate H_t^1 and H_t^2 from the pair process. They are also the Markov processes on the space $\{0,1\}$ with the generator matrices $A_1(t)$ and $A_2(t)$ for default indicator processes H^1 and H^2 respectively.

$$A_1(t) = \begin{bmatrix} -q_1(t) & q_1(t) \\ 0 & 0 \end{bmatrix} \qquad\qquad A_2(t) = \begin{bmatrix} -q_2(t) & q_2(t) \\ 0 & 0 \end{bmatrix}$$

This decomposition makes the default events (the relation between the firm and counter-party) easy to describe. We introduce the new processes and restate the previous pair process in another way. Define

$$H_t^{\{1,2\}} = [H^1, H^2]$$
$$H_t^{\{1\}} = H^1 - H_t^{\{1,2\}}, H_t^{\{2\}} = H^2 - H_t^{\{1,2\}}$$

The $[\cdot,\cdot]$ is the quadratic covariation of two processes in the first expression. The new indicator processes $H^{\{1\}}$, $H^{\{2\}}$ and $H^{\{1,2\}}$ stand for the events of a default of the firm alone, of the counter-party alone and a simultaneous default of the firm and the counter-party which corresponding to the three scenarios in Section 2.2. In other words:

$$H_t^{\{1\}} = \mathbb{1}_{\tau_1 \le t, \tau_1 \ne \tau_2}$$
$$H_t^{\{2\}} = \mathbb{1}_{\tau_2 \le t, \tau_1 \ne \tau_2}$$
$$H_t^{\{1,2\}} = \mathbb{1}_{\tau_1 = \tau_2 \le t}$$

The new constructions for the defaults give the probability distribution of three events directly. Furthermore, without proof here, the natural filtration of $(H^{\iota})_{\iota \in I}, I = \{\{1\},\{2\},\{1,2\}\}$ is equal to $\mathbb{H}$, the original filtration.

Chapter 4

CDS Pricing Equation

The whole picture of CDS is described in the previous three chapters. In this chapter, we will follow the asset pricing framework which works for any asset, and give the pricing equation for CDS. Although the framework is the same, the evolution of the asset is no longer the diffusion like European Option. We will give some properties of the default processes and Itô formula for the jump process first and then find the CDS equation.

4.1 $H^\iota s$' properties and extended Itô formula

We have introduced the $(H^\iota)_{\iota \in I}, I = \{\{1\},\{2\},\{1,2\}\}$, three jump processes, in the end of last chapter. The below property will give their intensities and construct martingales from compensation of jumps.

Default processes property The $\mathbb{H}-$intensity of H^ι is of the form $q_\iota(t, H_t)$,

$$q_{\{1\}}(t,H_t) = (1-H_t^1)\left((1-H_t^2)l_1(t)+H_t^2 q_1(t)\right) \tag{4.1}$$

$$q_{\{2\}}(t,H_t) = (1-H_t^2)\left((1-H_t^1)l_2(t)+H_t^1 q_2(t)\right) \tag{4.2}$$

$$q_{\{1,2\}}(t,H_t) = (1-H_t^1)(1-H_t^2)l_3(t) \tag{4.3}$$

and $M_t^\iota = H_t^\iota - \int_0^t q_\iota(s,H_s)ds$ are $\mathbb{H}-$ martingales for every $\iota \in I$.

The proof of this property could be found in **proposition 3.1** in paper [4]. We will not refer to the probability theory here.

In the derivative pricing issues, Itô formula is a powerful tool. Our familiar form of Itô formula is usually for diffusion process. Here we extend Itô formula to a more general category of processes, jump process.

Two-dimensional Itô-Doeblin Formula for Processes with Jumps
Let $X_1(t)$ and $X_2(t)$ be jump processes, and let $f(t,x_1,x_2)$ be a function whose first and second partial derivatives appearing in the following are defined and are continuous. Then

$$\begin{aligned}
& f(t,X_1(t),X_2(t)) \\
& = f(0,X_1(0),X_2(0)) + \int_0^t f_t(s,X_1(s),X_2(s))ds \\
& + \int_0^t f_{x_1}(s,X_1(s),X_2(s))dX_1^c(s) + \int_0^t f_{x_2}(s,X_1(s),X_2(s))dX_2^c(s) \\
& + \frac{1}{2}\int_0^t f_{x_1,x_1}(s,X_1(s),X_2(s))dX_1^c(s)dX_1^c(s) \\
& + \int_0^t f_{x_1,x_2}(s,X_1(s),X_2(s))dX_1^c(s)dX_2^c(s) \\
& + \frac{1}{2}\int_0^t f_{x_2,x_2}(s,X_1(s),X_2(s))dX_2^c(s)dX_2^c(s) \\
& + \sum_{0<s\leq t}[f(s,X_1(s),X_2(s)) - f(s,X_1(s-),X_2(s-))]
\end{aligned} \tag{4.4}$$

where $X_i(t)$ are adapted, right-continuous jump processes. $X_i(t) = X_i(0) + I_i(t) + R_i(t) + J_i(t) = X_i(0) + X_i^c(t) + J_i(t)$, which are Itô integral $I_i(t) = \int_0^t \Gamma_i(s)dW(s)$, Riemann integral $R_i(t) = \int_0^t \Theta_i(s)ds$, pure jump process $J_i(t)$. We define $X_i^c(t) = I_i(t) + R_i(t)$ is seen as continuous process.

Remark: The notion $X_1(t-)$ is strictly defined by $X_1(t-) = \lim_{s\uparrow t} X_1(s)$ for left-continuous point t. The related notions and proof are on Shreve's book, *chapter 11, Introduction to Jump Processes*.

4.2 Pricing equation

In this section, we will deduce several equations for the CDS price and risk measurements. At the beginning, we assume discount factor $\beta_t = \exp(-\int_0^t r(s)ds)$ and recovery rates R_1 and R_2 are constant.

Risky CDS Pricing Equation

The pricing funciton u of the risky CDS is given by

$$\beta_t u(t) = \int_t^T \beta_s e^{-\int_t^s l(u)du} \pi(s) ds$$
$$\pi(s) = (1-R_1)[l_1(s) + R_2 l_3(s)] + l_2(s)[R_2 \tilde{\chi}(s)^+ - \tilde{\chi}(s)^-] - \kappa$$

The function u satisfies the following *ODE*:

$$\begin{cases} u(T) = 0 \\ \frac{du}{dt}(t) - [r(t) + l(t)]u(t) + \pi(t) = 0, t \in [0,T) \end{cases}$$

proof: We divide the proof into four steps.

i) Claim $\Pi_t = (1-H_t^1)(1-H_t^2)u(t)$

We have known $\Pi_t = \mathbb{E}_t[\pi_T(t)]$ from equation 2.1. Since $\pi_T(t)$ is 0 on $\{t > \tau_1 \wedge \tau_2\}$, equivalently $\Pi_t = \mathbb{E}_t[\mathbb{1}_{t<\tau_1\wedge\tau_2} \pi_T(t)]$. We notice $\{\{t > \tau_1 \wedge \tau_2\}, \{t < \tau_1 \wedge \tau_2\}\}$ is a partition of the Ω, in this case, conditional expectation could be expressed as:

$$\begin{aligned} \Pi_t =& \mathbb{1}_{t<\tau_1\wedge\tau_2} \frac{\mathbb{E}[\mathbb{1}_{t<\tau_1\wedge\tau_2} \pi_T(t)]}{\mathbb{P}(t < \tau_1 \wedge \tau_2)} \\ =& \mathbb{1}_{t<\tau_1\wedge\tau_2} \frac{\mathbb{E}[\pi_T(t)]}{\mathbb{P}(t < \tau_1 \wedge \tau_2)} \\ =& (1-H_t^1)(1-H_t^2) \frac{\mathbb{E}[\pi_T(t)]}{\mathbb{P}(t < \tau_1 \wedge \tau_2)} \\ =& (1-H_t^1)(1-H_t^2)u(t) \end{aligned}$$

Here, the initial pricing function $\Pi(t,0,0) = u(t) := \frac{\mathbb{E}[\pi_T(t)]}{\mathbb{P}(t<\tau_1\wedge\tau_2)}$ is a deterministic function.

ii) Apply Itô's formula to $\beta_t \Pi_t$

Let $f(t, X_1(t), X_2(t)) = \beta_t \Pi_t = \beta_t(1-H_t^1)(1-H_t^2)u(t) = \beta_t u(t)(1-H_t^1)(1-H_t^2)$, where $\beta_t = \exp(-\int_0^t r(s)ds)$. We apply itô's formula and get:

$$\begin{aligned}
&\beta_t \Pi_t \\
&= u(0) + \int_0^t (1-H_s^1)(1-H_s^2)[u'(s)\beta_s - u(s)\beta_s r(s)]ds \\
&+ \sum_{0<s\leq t} [\beta_s u(s)(1-H_s^1)(1-H_s^2) - \beta_s u(s)(1-H_{s-}^1)(1-H_{s-}^2)] \qquad (4.5)
\end{aligned}$$

Since H_t^is are pure jump processes without continuous parts, so we do not have Itô integrals in Itô's formula 4.4.

iii) Find out the jumps

We will figure out the third term(jumps) in the equation 4.5 by discussing three cases.

***Case 1:* $H^{\{1\}}$ jumps (H^1 jumps, H^2 doesn't)**

Before $H^{\{1\}}$ jumps, there is no variation of the price of contract. So $\beta_s u(s)(1-H_{s-}^1)(1-H_{s-}^2) = \beta_s u(s)$ by the continuity of $u(t)$. At the time $H^{\{1\}}$ jumps, the payoff is given by the counter-party, which is $1-R_1$. Thus,

$$\begin{aligned}
&\beta_s u(s)(1-H_s^1)(1-H_s^2) - \beta_s u(s)(1-H_{s-}^1)(1-H_{s-}^2) \\
&= \beta_s[(1-R_1) - u(s)]dH_s^{\{1\}} \\
&= \beta_s(1-H_s^2)[(1-R_1) - u(s)]dH_s^{\{1\}}
\end{aligned}$$

The last equality holds because this event only occur when H^1 defaults and H^2 doesn't. Recall the property 4.1 and rewrite it in a differential form.

$$\begin{aligned}
dH_s^{\{1\}} &= dM_s^{\{1\}} + q_{\{1\}}(s, H_s)ds \\
&= dM_s^{\{1\}} + [(1-H_s^1)\left((1-H_s^2)l_1(s) + H_s^2 q_{\{1\}}(s)\right)]ds
\end{aligned}$$

Then

$$\beta_s(1-H_s^2)[(1-R_1)-u(s)]dH_s^{\{1\}}$$
$$=\beta_s(1-H_s^2)[(1-R_1)-u(s)]dM_s^{\{1\}}$$
$$+\beta_s(1-H_s^1)(1-H_s^2)[(1-R_1)-u(s)]l_1(s)ds$$

Sum up them in the interval $[0,t]$ and notice $dM_s^{\{1\}}$ is a martingale.

$$\sum_{0<s\leq t}[\beta_s u(s)(1-H_s^1)(1-H_s^2)-\beta_s u(s)(1-H_{s-}^1)(1-H_{s-}^2)]$$
$$=\int_0^t \beta_s(1-H_s^2)[(1-R_1)-u(s)]dH_s^{\{1\}}$$
$$=\int_0^t \beta_s(1-H_s^2)[(1-R_1)-u(s)]dM_s^{\{1\}}$$
$$+\int_0^t \beta_s(1-H_s^1)(1-H_s^2)[(1-R_1)-u(s)]l_1(s)ds$$
$$+\int_0^t \beta_s(1-H_s^1)(1-H_s^2)[(1-R_1)-u(s)]l_1(s)ds+\mathbb{H}-\text{martingale}$$

Case 2: **$H^{\{2\}}$ jumps (H^2 jumps, H^1 doesn't)**
We adopt the same analysis as *case 1*, $\beta_s u(s)(1-H_{s-}^1)(1-H_{s-}^2)=\beta_s u(s)$ before the default, and $\beta_s u(s)(1-H_s^1)(1-H_s^2)=R_2\chi_{\tau_2}^+-\chi_{\tau_2}^-$ at the default.

$$\beta_s u(s)(1-H_s^1)(1-H_s^2)-\beta_s u(s)(1-H_{s-}^1)(1-H_{s-}^2)$$
$$=\beta_s(1-H_s^1)[(R_2\chi_{\tau_2}^+-\chi_{\tau_2}^-)-u(s)]dH_s^{\{2\}}$$

by property 4.2

$$\sum_{0<s\leq t}[\beta_s u(s)(1-H_s^1)(1-H_s^2)-\beta_s u(s)(1-H_{s-}^1)(1-H_{s-}^2)]$$
$$=\int_0^t \beta_s(1-H_s^1)(1-H_s^2)[(R_2\chi_{\tau_2}^+-\chi_{\tau_2}^-)-u(s)]l_2(s)ds+\mathbb{H}-\text{martingale}$$

Case 3: **$H^{\{1,2\}}$ jumps (H^1 and H^2 jump at the same time)**

In this case, the pre-default price is still $\beta_s u(s)$. The value of the CDS at default is the compensation $R_2(1-R_1)$ according to table 2.1. Then the jump is:

$$\begin{aligned}&\beta_s u(s)(1-H_s^1)(1-H_s^2)-\beta_s u(s)(1-H_{s-}^1)(1-H_{s-}^2)\\&=\beta_s[R_2(1-R_1)-u(s)]dH_s^{\{1,2\}}\end{aligned}$$

From property 4.3, $dH_s^{\{1,2\}}=dM_s^{\{1,2\}}+(1-H_s^1)(1-H_s^2)l_3(s)ds$, we get

$$\begin{aligned}&\sum_{0<s\le t}[\beta_s u(s)(1-H_s^1)(1-H_s^2)-\beta_s u(s)(1-H_{s-}^1)(1-H_{s-}^2)]\\&=\int_0^t\beta_s(1-H_s^1)(1-H_s^2)[R_2(1-R_1)-u(s)]l_3(s)ds+\mathbb{H}-\text{martingale}\end{aligned}$$

In conclusion, we put three cases together and get

$$\begin{aligned}&\sum_{0<s\le t}[\beta_s u(s)(1-H_s^1)(1-H_s^2)-\beta_s u(s)(1-H_{s-}^1)(1-H_{s-}^2)]\\&=\int_0^t\beta_s(1-H_s^1)(1-H_s^2)[(1-R_1)-u(s)]l_1(s)ds\\&+\int_0^t\beta_s(1-H_s^1)(1-H_s^2)[(R_2\chi_{\tau_2}^+-\chi_{\tau_2}^-)-u(s)]l_2(s)ds\\&+\int_0^t\beta_s(1-H_s^1)(1-H_s^2)[R_2(1-R_1)-u(s)]l_3(s)ds+\mathbb{H}-\text{martingale}\end{aligned}$$

iv) Claim $\beta_t\Pi_t-\kappa\int_0^t\beta_s(1-H_s^1)(1-H_s^2)ds$ is a martingale

CDS is self-financing, to be precise, the evolution of CDS made up by the spread κ in $[0,t]$ and fair CDS price at time t, $\beta_t\Pi_t-\kappa\int_0^t\beta_s(1-H_s^1)(1-H_s^2)ds$ is self-financing. The classical theory shows it is a martingale under $\mathbb{P}$

$$\begin{aligned}&\beta_t\Pi_t-\kappa\int_0^t\beta_s(1-H_s^1)(1-H_s^2)ds\\&=u(0)+\int_0^t\beta_s(1-H_s^1)(1-H_s^2)\{u'(s)-u(s)r(s)+[(1-R_1)-u(s)]l_1(s)\\&+[(R_2\chi_{\tau_2}^+-\chi_{\tau_2}^-)-u(s)]l_2(s)+[R_2(1-R_1)-u(s)]l_3(s)-\kappa\}ds+\mathbb{H}-\text{martingale}\end{aligned}$$

Since martingale requires no drift term and we have $l(s) = l_1(s) + l_2(s) + l_3(s)$ assumption for Markov pair process, then

$$u' - ur + (1-R_1)l_1 + (R_2\chi^+_{\tau_2} - \chi^-_{\tau_2})l_2 + R_2(1-R_1)l_3 - ul - \kappa = 0 \tag{4.6}$$

This is the ODE at the beginning in the time interval $[0,T)$. We also add the final condition $u(T) = 0$ because the CDS contract is worthless at the maturity.

Remark: As discussed before, for consistency reason, the fair price of CDS at time τ_2 is $\chi_{\tau_2} = \Pi_{\tau_2-}$, by continuity of u, $\Pi_{\tau_2-} = \lim_{t\to\tau_2-} u(t) = u(\tau_2)$, then the solution is not explicit any longer since of the non-linear terms, while we still can solve the ODE. Another proof could be found in paper [4], **proposition 3.2**.

Risk-free CDS Pricing Equation

The pricing funciton v of the risk-free CDS is given by

$$\begin{aligned} \beta_t v(t) &= \int_t^T \beta_s e^{-\int_t^s q_1(x)dx} p(s)ds \\ p(s) &= (1-R_1)q_1(s) - \kappa \end{aligned} \tag{4.7}$$

The function v satisfies the following *ODE*:

$$\begin{cases} v(T) = 0 \\ \frac{dv}{dt}(t) - [r(t) + q_1(t)]v(t) + p(t) = 0, t \in [0,T) \end{cases}$$

proof: Since the idea of the proof is the same as the risky CDS case. We will only list the key parts here.

i) $P_t = P(t, H_t^1) = (1-H_t^1)v(t)$ since

$$\begin{aligned} P_t =& \mathbb{E}_t[p_T(t)] = \mathbb{E}_t[\mathbb{1}_{t<\tau_1} p_T(t)] \\ =& \mathbb{1}_{t<\tau_1} \frac{\mathbb{E}[\mathbb{1}_{t<\tau_1} p_T(t)]}{\mathbb{P}(t<\tau_1)} \\ =& (1-H_t^1)\frac{\mathbb{E}[p_T(t)]}{\mathbb{P}(t<\tau_1)} \end{aligned}$$

ii) Apply Itô's formula to $\beta_t P_t = \beta_t v(t)(1-H_t^1)$

$$\beta_t P_t = v(0) + \int_0^t (1-H_s^1)[v'(s)\beta_s - v(s)\beta_s r(s)]ds$$
$$+ \sum_{0<s\leq t} [\beta_s v(s)(1-H_s^1) - \beta_s v(s)(1-H_{s-}^1)]$$

iii) Figure out the jump term

$$\sum_{0<s\leq t} [\beta_s v(s)(1-H_s^1) - \beta_s v(s)(1-H_{s-}^1)]$$
$$= \int_0^t \beta_s[(1-R_1) - v(s)]dH_s^1$$
$$= \int_0^t \beta_s[(1-R_1) - v(s)]dM_s^1 + \int_0^t \beta_s(1-H_s^1)[(1-R_1) - v(s)]q_1(s)ds$$

Notice H_s^1, M_s^1 are different from $H_s^{\{1\}}$, $M_s^{\{1\}}$, the last equation holds from the **proposition 3.1 (ii)** in the paper [4] which is similar to the property 4.1.

iv) $\beta_t P_t - \kappa \int_0^t \beta_s(1-H_s^1)ds$ is a martingale
The necessity condition for it require the drift to be 0, which is

$$v' - vr + (1-R_1)q_1 - vq_1 - \kappa = 0$$

In addition the final condition is $v(T) = 0$

4.3 Price dynamics

From the proof the CDS pricing equations by Itô formula, we could directly get an additional conclusion, CDS price dynamic. This is the martingale terms in the Itô formula. Let $\hat{\Pi}$ denote the discounted cumulative price of the risky CDS in time interval $[0,t]$.(the definition of $\pi_t(0)$ is given by Equation 2.1)

$$\hat{\Pi}_t = \beta_t \Pi_t + \pi_t(0)$$

One thing to mention is that there is no default when $t \in [0, \tau_1 \wedge \tau_2 \wedge T]$, $\beta_0 \pi_t(0) = \pi_t(0) = -\kappa \int_0^t \beta_s ds$. Thus the dynamic is meaningful on $[0, \tau_1 \wedge \tau_2 \wedge T]$:

$$d\hat{\Pi}_t = \beta_t \left(N_t^1 dM_t^{\{1\}} + N_t^2 dM_t^{\{2\}} + N_t^3 dM_t^{\{1,2\}} \right)$$

$$N_t^1 = 1 - R_1 - u(t), N_t^2 = R_2 \tilde{\chi}^+(t) - \tilde{\chi}^-(t) - u(t), N_t^3 = R_2(1 - R_1) - u(t)$$

Similarly, we could discuss the dynamic for risk-free CDS

$$\hat{P}_t = \beta_t P_t + p_t(0)$$

$$d\hat{P}_t = \beta_t L_t^1 dM_t^1$$

$$L_t^1 = 1 - R_1 - v(t)$$

The dynamics of the CDS not only give us an idea how the CDS evolves, but also benefit us for hedging the CDS.

Chapter 5

Realization of the model

We have already finished the proof of the pricing equations for the most general situations. In this chapter, we will apply our previous results to a simple case and get some intuitive ideas by seeing some numerical results.

5.1 Linear intensities model

We will specify the intensity functions q_1 and q_2 by linear functions first,

$$q_1(t) = a_1 + b_1 t$$
$$q_2(t) = a_2 + b_2 t$$

and then define the linear simultaneous default intensity as well.

$$l_3(t) = a_3 + b_3 t$$
$$a_3 = \alpha \min\{a_1, a_2\},\ b_3 = \alpha \min\{b_1, b_2\}$$

The $\min(\cdot)$ reveals that simultaneous default probability can not excess either marginal default probability. α is the dependence parameter in $[0,1]$ and can be seen as an another form of correlation of marginal defaults later. Following our assumptions, we could get the some propositions for the premia κ and correlation ρ immediately. Notice they are not exogenous parameters, once the intensities are decided, they are determined internally.

Proposition (premia κ) : The premia κ_i $(i = 1, 2)$ of a risk-free CDS with default intensity q_i is given by

$$\kappa_i = (1 - R_i)\frac{\int_0^T \beta_t(a_i + b_i t)\exp(-a_i t - \frac{b_i}{2}t^2)dt}{\int_0^T \beta_t \exp(-a_i t - \frac{b_i}{2}t^2)dt} \tag{5.1}$$

proof: Since there is no cost to enter the CDS contract, so the price for risk-free CDS at the initial time is 0. We adopt the result from risk-free CDS pricing equation and got

$$\beta_0 v(0) = \int_0^T \beta_s e^{-\int_0^s q_i(x)dx} p(s)ds = 0$$
$$p(s) = (1 - R_i)q_i(s) - \kappa$$

We solve this algebra equation under linear assumptions and got the κ_i directly.

Proposition (correlation ρ) : The correlation of H_t^1 and H_t^2 is

$$\rho := \rho(T) = \frac{e^{a_3 T + b_3 T^2/2} - 1}{\sqrt{\left(e^{a_1 T + b_1 T^2/2} - 1\right)\left(e^{a_2 T + b_2 T^2/2} - 1\right)}} \tag{5.2}$$

or, equivalently,

$$\alpha = \frac{\ln\left(1 + \rho\sqrt{\left(e^{a_1 T + b_1 T^2/2} - 1\right)\left(e^{a_2 T + b_2 T^2/2} - 1\right)}\right)}{aT + bT^2/2} \tag{5.3}$$

where $a = \min\{a_1, a_2\}$ and $b = \min\{b_1, b_2\}$

Remark: This proposition is very useful for our implementation of model since the simultaneous default intensity only depends on the marginal intensity and their correlation. Because this proposition belongs to Markov copula topics, we will just introduce the formula without proof. It's proof and reference could be find in paper [4] **proposition 3.1 (v)** and paper **[1]**. In conclusion, by giving the a_1, b_1, a_2, b_2 and ρ, we could compute κ_i and ρ to determine all the parameters.

5.2 Numerical results

We do the numerical tests from the most simple case(constant intensities case) where $b_1 = b_2 = b_3 = 0$ and $r(t) = r$. Since it is a particular case of the linear intensity model, we naturally get:

$$
\begin{aligned}
& q_1(t) = a_1,\ q_2(t) = a_2,\ l_3(t) = a_3 \\
& \rho = \frac{e^{a_3 T} - 1}{\sqrt{(e^{a_1 T} - 1)(e^{a_2 T} - 1)}} \\
& a_3 = \frac{1}{T} \ln\left(1 + \rho\sqrt{(e^{a_1 T} - 1)(e^{a_2 T} - 1)}\right)
\end{aligned}
$$

In the tests, we first give the general data and firm's default data. We fix them, and then choose different counter-party's data and correlations to see the difference of risky CDS from risk-free CDS in various situations.

Table 5.1: Fixed Data — Constant Intensities

r	R_1	R_2	T	a_1	κ_1
5%	40%	40%	10 years	.0140	84bp

Table 5.2: Variable Data — Constant Intensities

a_2	κ_2	a_3	α	ρ	|CVA(0)|
.0083	50bp	.0011	.1366	10%	.0030
.0125	75bp	.0014	.1124	10%	.0037
.0167	100bp	.0016	.1171	10%	.0044
.0250	150bp	.0020	.1461	10%	.0054
.0083	50bp	.0045	.5374	40%	.0119
.0125	75bp	.0055	.4403	40%	.0146
.0167	100bp	.0064	.4572	40%	.0170
.0250	150bp	.0079	.5671	40%	.0210
.0083	50bp	.0077	.9253	70%	.0204
.0125	75bp	.0094	.7553	70%	.0251
.0167	100bp	.0109	.7819	70%	.0291
.0250	150bp	.0135	.9648	70%	.0359

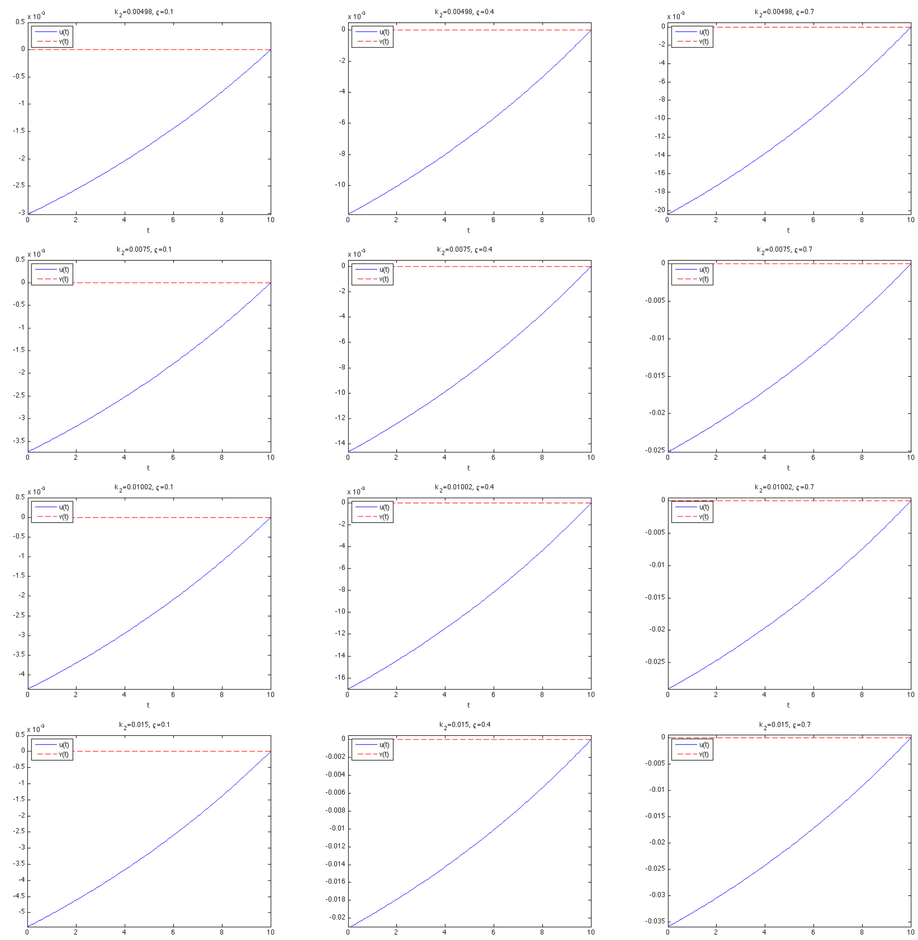

Figure 5.1: Pricing functions of risky CDS $u(t)$ and risk-free CDS $v(t)$ with constant intensities

The data in the Table 5.1 are predefined and κ_1 is computed by formula 5.1. In the Table 5.2 , we only control the counter-party's intensity a_3 and correlation ρ(the 1_{st} and 5_{th} column), and get κ_2, a_3, α by formula 5.1, 5.3, 5.2. For the $|CVA(0)|$, it is computed by the difference of risky CDS and risk-free CDS at time 0, i.e. $|u(0) - v(0)|$. $u(0)$, $v(0)$ are one of the results from solving two pricing ODE (in Section 4.2) numerically. In addition, Figure 5.1 is another numerical result. It depict risky CDS price $u(t)$ (blue curve) and risk-free CDS price $v(t)$ (red curve) in the time interval $[0, T]$ under different circumstances by choosing κ_2 and ρ.

Remark: i) Since the risk-free pricing equation only considers the firm's default, so there is no counter-party's data in it. All the parameters are taken in Table 5.1 , in particular, $\kappa = \kappa_1$. Risk-free CDS price $v(t)$ is given by the explicit solution 4.7.

ii) There is no explicit solution for risky CDS ODE because of the non-linear terms $u^+(t)$ and $u^-(t)$. While we could solve it numerically. We let $w(t) = u(T-t)$ and transform final value problem to initial value problem.

$$\begin{cases} w(0) = 0 \\ \frac{dw}{dt}(t) + [r(T-t) + l(T-t)]w(t) - \pi(T-t) = 0, t \in [0,T) \end{cases}$$

We pick $\kappa = \kappa_1$ in the ODE. Although it is not the true premia, we would like to see what the influence could be after integrating counter-party default into the risk-free CDS contract. We apply matlab function *ode45*(book [3]) to find $w(t)$ and $u(t)$.

Table 5.3: Fixed Data — Linear Intensities

r	R_1	R_2	T	a_1	b_1	κ_1
5%	40%	40%	10 years	.0095	.0010	84bp

Table 5.4: Variable Data — Linear Intensities

a_2	b_2	κ_2	a_3	b_3	α	ρ	$\lvert$CVA(0)$\rvert$
.0056	.0006	50bp	.00077	.00008	.1368	10%	.0031
.0085	.0009	75bp	.00096	.00010	.1124	10%	.0039
.0122	.0010	100bp	.00110	.00012	.1171	10%	.0045
.0189	.0014	150bp	.00140	.00015	.1466	10%	.0057
.0056	.0006	50bp	.0030	.0003	.5380	40%	.0119
.0085	.0009	75bp	.0037	.0004	.4400	40%	.0147
.0122	.0010	100bp	.0043	.0005	.4566	40%	.0171
.0189	.0014	150bp	.0054	.0006	.5684	40%	.0212
.0056	.0006	50bp	.0052	.0006	.9257	70%	.0205
.0085	.0009	75bp	.0064	.0007	.7543	70%	.0253
.0122	.0010	100bp	.0074	.0008	.7803	70%	.0292
.0189	.0014	150bp	.0092	.0010	.9660	70%	.0361

Similarly, we extend the constant intensities case to the linear intensities case through the same steps. The Table 5.3 and Table 5.4 are the fixed firm's default data and 12 alternative counter-party's default data. The Figure 5.2 is the risky and risk -free CDS prices with linear intensities in the 12 scenarios by choosing different κ_2 and ρ.

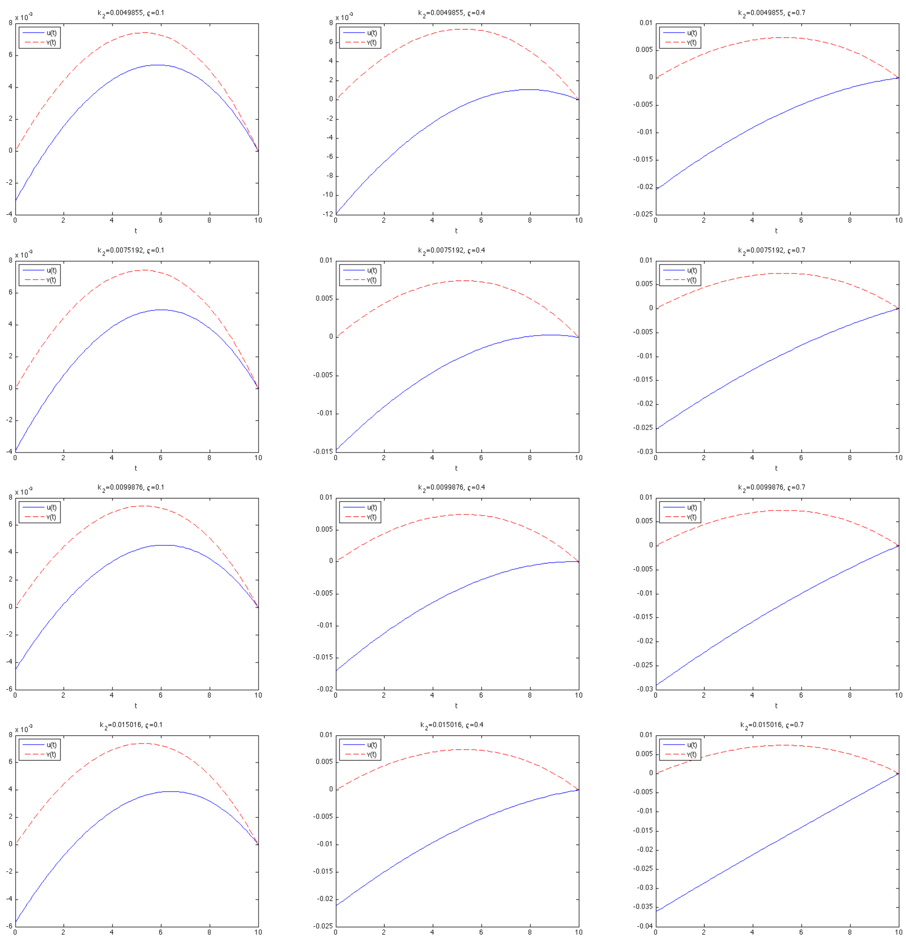

Figure 5.2: Pricing functions of risky CDS $u(t)$ and risk-free CDS $v(t)$ with linear intensities

In both constant intensities situation and linear constant situation, we could find some several consistent conclusions which agree with our intuition.

i) From Figure 5.1 and Figure 5.2, the risky CDS price is always lower than the risk-free CDS price when we pay the same premia κ_1. The reason is that the risk-free CDS price does not include the counter-party risk. The fair CDS price should be lower because the seller of the CDS (counter-party), the party of insurer, could default.

ii) Both the figures and tables show that with the correlation increase, the difference of CDS prices (CVA(t)) increases. It reflects that the counter-party is more likely to default when the firm defaults with a higher correlation between each other. The higher default possibility results in the bigger difference.

Remark: For choosing the value of parameters, we set them the same as the inputs in paper [4], **Table 1** $\sim$ **4**. We run the numerical test independently and find most results are quite close while there are some mistakes in paper [4].

Chapter 6

Conclusions

The essay provides a complete description of the CDS pricing model, from introducing the credit risk and CDS contract, modelling the cash flows and the defaults to deducing the CDS pricing equations and numerical solutions. All the things are under the framework of general asset pricing theory and the evolution of the assets extend from diffusion to jump process. To point out, the core of the model is the default processes. Different default models lead to different CDS pricing models. In the essay, we use the infinitesimal Markov processes to illustrate firm's default and counter-party's default.

There are still something that could be considered in the future:

i) As we discussed in the Chapter 2, since there are three parties in the event, it is possible all of them could default. We have already give the whole story of defaults from two parties. It could extend to the case of three parties.

ii) In the implementation of the model, although the copula property is a great help for us to model the simultaneous default from marginals, however, we still need to be careful to specify the intensities. It would be the key part to apply the model to reality successfully.

Bibliography

[1] V. A. V. L. Bielecki, T.R. and J. Jakubowski. Study of dependence for some stochastic processes. *Journal of Stochastic Analysis and Applications*, 26: 903–924, July 2008. → pages 25

[2] K. J. S. Darrell Duffie. *Credit Risk Pricing, Measurement, and Management*. 2002. → pages 1

[3] J. Polking. *Ordinary Differential Equations Using MATLAB*. Prentice Hall, 3rd edition, 2003. → pages 28

[4] B. S.Crepey, M.Jeanblanc. Counterparty risk on a cds in a markov chain copula model with joint defaults. 2009. → pages 9, 16, 21, 22, 25, 30

[5] S. E. Shreve. *Stochastic Calculus for Finance*, volume 2. Springer, 2004. → pages 16

Appendix A

Matlab Codes

Function "CDS.m"

```
function f = CDS(s,u)
%input fix parameters
r = 0.05;
R1 = 0.4;
R2 = 0.4;
T = 10;
a1 = 0.0095;
b1 = 0.001;
K1U = @(x)exp(-r*x).*(a1+b1*x).*exp(-a1.*x-b1/2*x.^2);
K1D = @(x)exp(-r*x).*exp(-a1*x-b1/2*x.^2);
%get k1
k1 = (1-R1)*quad(K1U,0,T)/quad(K1D,0,T)
%input variable parameters
a2 = 0.0189;
b2 = 0.0014;
p = 0.7;
%get k2,a3,alpha
K2U = @(x)exp(-r*x).*(a2+b2*x).*exp(-a2.*x-b2/2*x.^2);
K2D = @(x)exp(-r*x).*exp(-a2*x-b2/2*x.^2);
k2 = (1-R2)*quad(K2U,0,T)/quad(K2D,0,T)
```

```
a = min(a1,a2);
b = min(b1,b2);
alpha = log(1+p*sqrt((exp(a1*T+b1*T^2/2)-1)* ...
    (exp(a2*T+b2*T^2/2)-1)))/(a*T+b*T^2/2)
a3 = alpha * a
b3 = alpha * b
a3_check =1/T*log(1+p*sqrt((exp(a1*T)-1)*(exp(a2*T)-1)))
%intensity
l1 = a1-a3 + (b1-b3)*(T-s);
l2 = a2-a3 + (b2-b3)*(T-s);
l3 = a3 + b3*(T-s);
l = l1 + l2 + l3;
%function
f = -((r + l) * u-((1-R1)*(l1+R2*l3)+l2*...
    (R2*max(0,u)+min(0,u))-k1));
```

Code "ODE.m"

```
%clean
clc;
clear;
%input fix parameters
r = 0.05;
R1 = 0.4;
R2 = 0.4;
T = 10;
a1 = 0.0095;
b1 = 0.001;
K1U = @(x)exp(-r*x).*(a1+b1*x).*exp(-a1.*x-b1/2*x.^2);
K1D = @(x)exp(-r*x).*exp(-a1*x-b1/2*x.^2);
%get k1
k1 = (1-R1)*quad(K1U,0,T)/quad(K1D,0,T)
%input variable parameters
a2 = 0.0189;
```

```
b2 = 0.0014;
p = 0.7;
%get k2,a3,alpha
K2U = @(x)exp(-r*x).*(a2+b2*x).*exp(-a2.*x-b2/2*x.^2);
K2D = @(x)exp(-r*x).*exp(-a2*x-b2/2*x.^2);
k2 = (1-R2)*quad(K2U,0,T)/quad(K2D,0,T)
a = min(a1,a2);
b = min(b1,b2);
alpha = log(1+p*sqrt((exp(a1*T+b1*T^2/2)-1)*...
    (exp(a2*T+b2*T^2/2)-1)))/(a*T+b*T^2/2)
a3 = alpha * a
b3 = alpha * b
a3_check =1/T*log(1+p*sqrt((exp(a1*T)-1)*(exp(a2*T)-1)))
%CDS risk-free
time2 = [0:0.1:10];
price2 = zeros(1,length(time2));
for i = 1 : length(time2)
    iter_t = 0.1*(i-1);
    V1 = @(x)exp(-r*x).*exp(-a1*(x-iter_t)-b1/2*...
        (x.^2-iter_t.^2)).*((1-R1)*(a1+b1*x)-k1);
    price2(i)=exp(r*iter_t)*quad(V1,iter_t,T);
end
V1 = @(x)exp(-r*x).*exp(-a1*x-b1/2*x.^2);
% %solve ODE
[time,price]=ode45('CDS',[0:0.1:10],0);
plot(10-time,price,'-',time2,price2,'r--');
title(['k_2=',num2str(k2),', \rho =',num2str(p)]);
xlabel('t');
%axis([0 T price(101) 0.0005])
h = legend('u(t)','v(t)',2);
set(h,'Interpreter','none')
%CVA
CVA = price(101)-price2(101)
```