Valuation of Lease Contract and Credit Risk

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Abstract

Leasing has become a very important financial instrument today. A large amount of literature discusses the determination of lease rate. In this essay, we focus on the valuations of lease contracts and credit risk. First, we introduce both discrete and continuous time models to determine the equilibrium rate of risk-free lease contracts. Then, by relaxing the restriction of riskless leases, a model for determining the equilibrium lease rate subject to credit risk is introduced. The model is flexible enough to be applied to a variety of real-world leasing structures, including security deposit, up-front prepayment, and lease credit insurance contracts.

1. Introduction

Leasing plays a very important roll in the global economy nowadays. At the end of 2004, the global leasing volume was reported to be USD 579 billion with a roughly 10% annual growth rate. The European leasing market, as represented by LEASEUROPE, has more than tripled in size from less than € 90 billion in 1994 to € 297.5 billion in 2006. New equipment and vehicle leasing contracts amounted to 250.7 billion euros, with real estate leasing volumes being worth 46.8 billion euros. The equipment leasing penetration, or the percentage of equipment acquired through leasing rather than cash or credit reached a level of 29.9%. In U.S, according to the U.S. Equipment Leasing Association, leasing organizations financed $248 billion of a total of $800 billion in business equipment investments in 2005. This amount equaled 1.99% of GDP. The market penetration was around 31%.

A lease is a contractual agreement between two counter parties, the lessor and the lessee, for the use of a certain asset for a specific period of time. It offers a mechanism that allows the separation between ownership and use. Thus, the lessee receives the benefits of use and the lessor receives the lease payment flow plus the residual value of the asset.

Numerous papers in finance discuss the determination of lease rates. Starting with Miller and Upton (1976), lease rates and the lease versus buy polices are modeled under an equilibrium structure. McConnell and Schallheim (1983) uses an no-arbitrage framework to price asset leases and the associated insurance premium for leased assets. Grenadier (1995) presents a thorough analysis of the valuation of options embedded in lease contracts based on fundamental economic principles. Moreover, Grenadier (1996) provides a unified framework for determining the equilibrium credit spread on leases subject to default risk. Ambrose and Yildirim (2007) extends the work by Grenadier who uses a structural model to value the credit risk to a reduced form model similar to that used in valuation of credit default swap. In addition to the various theoretical researches, there are also a few empirical developments. For instance, Schallheim, Johnson, Lease, and McConnell (1986), using a sample of 363 contracts, find that lease yields are set to compensate the lessor for the default potential inherent in leasing contracts. Schmit and Stuyck (2002) analyses the severity of loss when a lease defaults to show that leasing benefits from high recovery rates in the event of default.

This essay goes through some theoretical models mentioned above to introduce the methods of valuating lease rate. Next section will introduce both discrete and continuous time model for valuation of default-free lease contracts. Section 3 will consider credit risk into valuations and apply to some real-world leasing arrangements such as security deposits, required up-front prepayments, and lease credit insurance contracts. The final section will conclude.

2. Valuation of default-free lease contract

This section introduces three kinds of valuation of lease contracts under a default-free setting: single-period model by Miller and Upton (1976); Multi-period model from McConnell and Schallheim (1983); and a continuous time framework by Grenadier (1996). There are some common assumptions of these models must be emphasized. These assumptions will also be applied in Section 3 when we talk about valuation of lease contracts with credit risk.

(1) The risk-free rate of interest is assumed to be constant over time. An extension of present model to consider stochastic interest rate could be accomplished in a manner similar to Brennan and Schwartz (1980) and Kim, Ramaswamy, and Sundaresan (1993).

(2) The contract commitment itself is assumed noncancelable over the agreed term of the lease in general.

(3) The analysis focuses on the nontax and nontransaction cost aspect of leasing, although for realism a vast literature include taxes (e.g., Myers, Dill, and Bautista, 1976; Huang and Yildirim, 2006). Unfortunately, incorporating tax considerations into those models would dramatically increase their complexity.

2.1. Single-period valuation

Miller and Upton (1976) modeled the equilibrium rental payment on a one-period lease in a capital asset pricing model (CAPM) framework. Recall that under CAPM framework, the expected rate of return on any asset j is given by the relation

$$E(R_j) = R_f + \beta_j[E(R_m) - R_f] \tag{1}$$

where R_f is the riskless rate of interest; $E(R_m)$ is the expected rate of return on total wealth; and $\beta_j = Cov(R_j, R_m)/Var(R_m)$ is a measure of the relative, non-diversifiable risk of asset *j*.

Then in a leasing company, machine *i* is an asset like any other and its rate of return is

$$R_{it} = \frac{L_{it}}{A_{it}} - d_{it} \tag{2}$$

where L_{it} is the rental payment which is paid at the beginning of the period and assumed to be a sure payment once the lease is signed; A_{it} is the beginning-of-period market value of asset *i*; d_{it} is the economic depreciation of asset *i* during the period *t*, which remains uncertain and can be expressed as

$$d_{it} = E(d_{it}) + \beta_{it}(R_m - E(R_m)) + \varepsilon_{it} \tag{3}$$

where $E(d_{it})$ is the expected depreciation; $\beta_{it} = Cov(d_{it}, R_m)/Var(R_m)$ is the standard capital asset pricing measure of the relative non-diversifiable risk of asset *i* in period *t*, $\beta_{it}(R_m - E(R_m))$ is any depreciation above (or below) $E(d_{it})$ due to unexpected variations in the rate of return on total wealth; and ε_{it} represents all other random deviations from expected depreciation, which has a mean of zero. Taking the expectations in equation (2) and using the equilibrium condition of (1), we have

$$E(R_{it}) = \frac{L_{it}}{A_{it}} - E(d_{it}) = R_f - \beta_{it}(E(R_m) - R_f) \tag{4}$$

the sign of β_{it} is negative because the change in asset value is measured as capital depreciation rather than capital appreciation. Hence by organizing the equation above, the equilibrium lease payment is

$$L_{it}^{*} = [E(R_{it}) + E(d_{it})]A_{it} = [R_f - \beta_{it}[E(R_m) - R_f] + E(d_{it})]A_{it} \tag{5}$$

Thus, the equilibrium single-period rental consists of the risk-free return on the capital invested ($R_f A_{it}$); plus the non-diversifiable risk borne ($-\beta_{it}[E(R_m) - R_f]A_{it}$); and plus the expected depreciation of the capital.

2.2. Multi-period valuation

McConnell and Schallheim (1983) extends the Miller and Upton (1976) analysis to a multi-period framework. The lease that we will consider here calls for n rental payments of L each. The n payments are due at equal intervals prior to the maturity date of the contract, T. Before illustration, there are two important assumptions to mention

(1) The distribution of the rate of economic depreciation of the leased asset is stationary over time.

(2) Lognormality of returns: The return on the underlying (leased) asset and the return on aggregate wealth are jointly lognormally distributed.

Now consider a one-period case first. Combining the analysis of Miller and Upton (1976) and the valuation techniques of Rubinstein (1976), the equilibrium rental at time $T - 1$ for a single-period lease that is due at time T is

$$L = A_{T-1} - \frac{[1 - E(d)]e^{\text{cov}(l,y)} A_{T-1}}{(1 + r_f)} \tag{6}$$

where A_{T-1} is the market value of the leased asset at time $T - 1$; $E(d)$ is the expected rate of economic depreciation of the asset; $Cov(l, y)$ [1] is the covariance between the logarithm of one minus the rate of economic depreciation and the 'market factor' y; and r_f is the risk-free rate of interest.

For convenience, let $\lambda = [1 - E(d)]e^{\mathrm{cov}(l,y)}$. Then the rental on a single-payment lease that rents the asset over the period from $T-2$ to $T-1$ is

$$L = A_{T-2} - \frac{\lambda A_{T-2}}{(1+r_f)} \tag{7}$$

here

$$\lambda A_{T-2} = A_{T-1} = L + \frac{\lambda A_{T-1}}{(1+r_f)} \tag{8}$$

by (8) we have $\lambda A_{T-1} = \lambda^2 A_{T-2}$, and plugging the equation back into (8) we have

$$\lambda A_{T-2} = L + \frac{\lambda^2 A_{T-2}}{(1+r_f)} \tag{9}$$

then plug equation (9) into (7) and reorganize it

$$A_{T-2} = L + \frac{L}{(1+r_f)} + \frac{\lambda^2 A_{T-2}}{(1+r_f)^2} \tag{10}$$

the last part in the equation above is the current (time *T - 2*) market value of the residual part of the leased asset at the maturity date of the lease. We can write it as S_{T-2}^2.

By iterating the procedures above, we can find the rental payment at time *0*. The equilibrium condition for a multi-period lease contract can be written as

$$A_0 = \sum_{t=0}^{n-1} \frac{L}{(1+r_f)^t} + S_0^n \tag{11}$$

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[1] The rate of economic depreciation is defined in market value terms as $(A_{T-1} - A_T)/A_{T-1}$, so $Cov(l, y)$ can be written as $Cov[\ln(A_T / A_{T-1}), y]$.

$$S_0^n = \frac{\lambda^n A_0}{(1+r_f)^n} \tag{12}$$

In this analysis, the riskless rate, expected rate of economic depreciation, and covariance term are assumed to be constant over time. Thus, as in the single-period case, risk enters into the determination of the equilibrium rental rate of a lease only because the end-of-lease residual value of the asset is uncertain. Furthermore, only the non-diversifiable risk associated with the asset's residual value is relevant to the determination of the rental payments on the lease. Furthermore, because the lessor bears the residual-value risk only at the termination of the lease, only the discounted value of residual-value risk is relevant to the determination of the rental payment *L*.

2.3. Continuous time valuation

In this subsection, we will present a continuous time valuation to determine the rental payment of a *T*-year default-free lease according to the model of Grenadier (1996). This valuation will also be used in the next section when we talk about valuation of defaultable leases. In this model, the economic benefits from the service flow are realized by the user of the asset, while the owner of the asset retains the right to sell this service flow to potential lessees. At each point in time, the value of the service flow from the asset, or the instantaneous lease rate *S(t)* follows a diffusion process:

$$dS = \alpha_s S dt + \sigma_s S dz_s \tag{13}$$

where α_s is the instantaneous conditional expected percentage change in S per unit time; σ_s is the volatility; and dz_s is the increment of a standard Brownian motion. α_s here can be either positive or negative since the service flow can either appreciate or depreciate over time.

Denote the present value of the use of the asset for *T* years as *Y(S, T)*. It can be expressed as follows:

$$Y(S,T) = E\left[\int_0^T e^{-rt} S(t)dt\right] \tag{14}$$

Using the properties of log-normal variables, we can find an explicit expression for *Y(S, T)*. First we know

$$S(t) = S\exp\left[\left(\alpha_s - \frac{\sigma_s^2}{2}\right)t + \sigma_s z_s\right] \tag{15}$$

then *Y(S, T)* can be written as

$$Y(S,T) = \int_0^T E\left[e^{-rt} S\exp\left[\left(\alpha_s - \frac{\sigma_s^2}{2}\right)t + \sigma_s z_s\right]\right]dt \tag{16}$$

By Girsanov Theorem, we can change the probability measure to a new measure Q^θ. Let θ be equal to $-\sigma_s$, then $\xi_t^\theta = \exp\left(\sigma_s z_s - \frac{\sigma_s^2}{2}t\right)$ is the Radon Nikodym derivative.

Therefore

$$\begin{aligned} Y(S,T) &= \int_0^T E\left[e^{-rt} S e^{\alpha_s t} \cdot \xi_t^\theta\right]dt \\ &= \int_0^T E_{Q^\theta}(S e^{(-r+\alpha_s)t})dt \end{aligned} \tag{17}$$

By solving equation (17), finally we have

$$Y(S,T) = \frac{S}{r-\alpha_s}\left[1 - e^{-(r-\alpha_s)T}\right] \tag{18}$$

The equilibrium lease rate, denoted by *R(T)*, will then be the payment flow whose annuity value equals $Y(S, T)$. Thus, $R(T)$ can be expressed as

$$R(T)=\left(\frac{r}{1-e^{-rT}}\right)Y(S,T) \tag{19}$$

Plugging equation (18) into $Y(S, T)$ above, we have the explicit expression

$$R(T)=\left(\frac{1-e^{-(r-\alpha_s)T}}{1-e^{-rT}}\right)\frac{rS}{r-\alpha_s} \tag{20}$$

Notice that the value of the underlying asset, $V(S)$, is the perpetuity value of $S(t)$

$$V(S)=\lim_{T\to\infty}Y(S,T) \tag{21}$$

3. Valuation of leasing contracts under default risk and its extensions

Although a vast amount of literature focuses on valuations of default-free lease contracts, lease is not born to be riskless. Empirical evidence has pointed to a strong similarity between lease contracts and junk bonds. Therefore, the determination of lease rate must consider not only the forgone use of the asset but also the potential consequences of default. In this section, we will introduce an equilibrium valuation of lease rate when the lease payments are subject to credit risk based on the work of Grenadier (1996), then extend the results to several real-world leasing arrangements.

3.1. Determination of lease rates under credit risk

The equilibrium lease valuation will take two sources of uncertainty into account:

(1) The service flow of the leased asset is stochastic; (2) The timing and consequences of default are also stochastic. One assumption here is risk neutrality: all assets are priced to yield an expected rate of return equal to the risk-free rate.

A *T*-year lease is said to be subject to the risk of default because there exists possibility that the promised rental payment *P(T)* may not be realized by the lessee. Lease default can be broken into two parts: the occurrence of default and the consequences of default. First, we will model the occurrence of default. It is similar to the Merton's setting. The default occurs when a financial state variable of the lessee, *X(t)*, falls to a lower threshold *K*. *X(t)* here can be interpreted as cash flow or asset value. It is assumed to follow a diffusion process that is correlated with the value of the leased asset (see Section 2.3) as follows:

$$dX = \alpha_x X dt + \sigma_x X dz_x \tag{22}$$

where α_x is the instantaneous conditional expected percentage change in X per unit time; σ_x is the volatility; and dz_x is the increment of a standard Brownian motion. Let $\rho(S, X, t)$ denote the instantaneous correlation between the Brownian motions dz_s and dz_x. Thus, default occurs on a *T*-year lease if the first passage time of *X(t)* to the boundary K is less than *T*. Mathematically, default occurs at a stopping time t^* where $t^* = \inf[t < T : X(t) \leq K]$, where $t^* = \infty$ if no such t exists.

After that the default occurrence has been defined, we will model the consequences of default. When a default occurs, the value of the asset may not be fully recovered due to different reason. For example, the asset is damaged or some other losses due to delay, legal costs, and brokerage cost. If the asset could be fully recovered and immediately re-leased at time t^*, the remaining value of the lease would be $Y[S(t^*), T-t^*]$ (see Section 2.3). However, due to losses, the lessor is only able to receive a fraction, $1-\omega$, of this remaining value, where $\omega \in [0,1]$. For simplicity, ω is assumed to be constant.

Next, we will move on to the determination of the equilibrium rent on a risky T-year lease, $P(T)$. Grenadier (1996) derives the equilibrium rent based on the underlying concept: any two methods of selling the service flow of the asset for T year must have the same value. The two alternative methods are the followings:

Alternative 1: Lease the asset for T years to a lessee free of credit risk. The value of this alternative is $Y(S,T)$.

Alternative 2: Lease the asset under a T-year lease to a risky lessee, at the rental rate $P(T)$. If the event of default occurs at time $t < T$, lease the asset out for the remainder of the term $(T - t)$ to a riskless lessee. Then the remainder value at that point will be $(1-\omega)\cdot Y[S(t), T-t]$.

Let $F(S,X,t;P,T)$ denote the value of *Alternative 2*, where t is the current time and S and X are the current values of $S(t)$ and $X(t)$, respectively. Let P be a given rent rate which will be determined in equilibrium. By Itô's lemma, the instantaneous change in F over a region without occurrence of default, i.e. $X(\upsilon) > K, \forall \upsilon \le t$, is

$$dF = \left[\tfrac{1}{2}\sigma_s^2(S,t)S^2F_{SS} + \rho(S,X,t)\sigma_s(S,t)\sigma_x(X,t)SXF_{SX} \right.$$
$$\left. + \tfrac{1}{2}\sigma_x^2(X,t)X^2F_{XX} + \alpha_s(S,t)SF_S + \alpha_x(X,t)XF_X + F_t\right]dt \tag{23}$$
$$+ \sigma_s(S,t)SF_S dz_s + \sigma_x(X,t)XF_X dz_x$$

The return on F is composed of two elements: the capital gain and lease payment per unit time. Therefore, the expected return on F per unit time, $\mu_F dt$, is define as

$$\mu_F dt = E\left[\frac{dF + Pdt}{F}\right] \tag{24}$$

Setting the expected return equal to the risk-free rate of return, we will have the following partial differential equation:

$$0 = \tfrac{1}{2}\sigma_s^2(S,t)S^2F_{SS} + \rho(S,X,t)\sigma_s(S,t)\sigma_x(X,t)SXF_{SX} + \tfrac{1}{2}\sigma_x^2(X,t)X^2F_{XX}$$
$$+ \alpha_s(S,t)SF_S + \alpha_x(X,t)XF_X + F_t + P - rF \tag{25}$$

When $X(t) = K$, default occurs. This default boundary condition can be written as

$$F(S,K,t;P,T) = (1-\omega)\cdot Y[S(t),T-t] \tag{26}$$

If no default occurs over the life of the lease, the terminal condition ensures that the value of the remaining rental payments equals zero at maturity:

$$F(S,X,T;P,T) = 0 \tag{27}$$

The partial differential equation (25) can be solved subject to boundary conditions (26) and (27). In general, closed-form solutions will not be available. However, we will assume that $S(t)$ and $X(t)$ follow correlated geometric Brownian motions. Thus $\alpha_s(S,t)$, $\alpha_x(X,t)$, and $\rho(S,X,t)$ are constants denoted by α_s, α_x, and ρ respectively. This assumption will permit a closed-form solution to be obtained as the following.

$$F(S,X,t;P,T) = \Phi_1(P,\tau) - \Phi_2(X,\tau)\cdot\frac{P}{r} + \Phi_3(X,\tau)\cdot(1-\omega)\frac{S}{r-\alpha_s} \tag{28}$$

where

$$\Phi_1(P,\tau) = \frac{P}{r}\left[1 - e^{-r\tau}\right],$$

$$\Phi_2(X,\tau) = \left(\frac{X}{K}\right)^{c_1} G(X,\tau,a_2) - e^{-r\tau} G(X,\tau,a_1),$$

$$\Phi_3(X,\tau) = \left(\frac{X}{K}\right)^{c_2} G(X,\tau,a_4) - e^{-(r-\alpha_s)\tau} G(X,\tau,a_3),$$

$$G(X,\tau,y) = N\left[\frac{\ln(K/X) - y\cdot\tau}{\sigma_x\sqrt{\tau}}\right] + \left(\frac{K}{X}\right)^{2y/\sigma_x^2} \cdot N\left[\frac{\ln(K/X) + y\cdot\tau}{\sigma_x\sqrt{\tau}}\right],$$

$$c_1 = \frac{a_2 - a_1}{\sigma_x^2},$$

$$c_2 = \frac{a_4 - a_3}{\sigma_x^2},$$

$$a_1 = \alpha_x - \tfrac{1}{2}\sigma_x^2,$$

$$a_2 = \sqrt{a_1^2 + 2r\sigma_x^2},$$

$$a_3 = \alpha_x + \rho\sigma_x\sigma_s - \tfrac{1}{2}\sigma_x^2,$$

$$a_4 = \sqrt{a_3^2 + 2(r - \alpha_s)\sigma_x^2},$$

$$\tau = N - t,$$

and where $N(\cdot)$ denotes the cumulative standard normal distribution function.

The first term, $\Phi_1(P,\tau)$, is equal to the value of the rental flow if the lease is risk-free. The second term, $\Phi_2(X,\tau)\cdot\frac{P}{r}$, is the value of the potential loss of all contracted rental if the lessee defaults. The third term, $\Phi_3(X,\tau)\cdot(1-\omega)\frac{S}{r-\alpha_s}$, is the value of the rent that is recovered if the lessee default. Therefore, the value of *Alternative 2* is same as a portfolio consisting of a riskless lease, a short position in a contract which pays out the credit loss under a lease, and a long position in a contract which pays the amount recovered from a lease default.

Finally, the equilibrium risky rent on a T-year lease can be derived according to the condition that the values of *Alternative 1* and *2* must be equal. Therefore, at time *0*, the following condition must be satisfied:

$$F(S,X,0;P(T),T)=Y(S,T) \tag{29}$$

By applying above condition to equation (28), the equilibrium rent on a lease subject to credit risk is

$$P(T)=\left(\frac{r\cdot S}{r-\alpha_s}\right)\left[\frac{1-e^{-(r-\alpha_s)T}-\Phi_3(X,T)\cdot(1-\omega)}{1-e^{-rT}-\Phi_2(X,T)}\right] \tag{30}$$

3.2. Extension to real-world leasing arrangements

Many real-world lease contracts contain clauses to protect the lessor against lessee default. The clauses that we will present here include security deposit, up-front prepayment, and credit insurance. All the methods here are useful to lessen credit risk. In return, the contractual lease payment must be lower. Next, we will introduce the determinations of lease rate with these clauses.

(I). Equilibrium lease rate with security deposit

A lessee and lessor agree to a *T*-year lease with a security deposit of *\$M* and a rental rate of $P^D(T, M)$. The deposit generates interest at the rate *r*. If a default occurs during the life of the lease, the lessor can use the deposit to cover some potential credit losses. If the lessee does not default, the deposit will be returned back to the lessee. Denote the value of cash flows from leases with security deposit as $W(S,X,t;P^D,T,M)$. As in the basic model (see Section 3.1), the value must satisfy the following partial differential equation in equilibrium:

$$\begin{aligned}0 &= \tfrac{1}{2}\sigma_s^2(S,t)S^2W_{SS} + \rho(S,X,t)\sigma_s(S,t)\sigma_x(X,t)SXW_{SX} + \tfrac{1}{2}\sigma_x^2(X,t)X^2W_{XX} \\ &\quad + \alpha_s(S,t)SW_S + \alpha_x(X,t)XW_X + W_t + P^D - rW\end{aligned} \tag{31}$$

and the boundary conditions will be

$$\begin{aligned}&\text{No Default:} \quad W(S,X,T;P^D,T,M) = 0 \\ &\text{Default at } t\text{:} \quad W(S,K,t;P^D,T,M) = (1-\omega)\cdot Y\big[S(t),T-t\big] + Me^{rt} \\ &\qquad\qquad - \max\big\{Me^{rt} - \omega\cdot Y\big[S(t),T-t\big],0\big\}\end{aligned} \tag{32}$$

The second boundary condition shows that the security deposit can be used to cover the loss, but the excess amount must be return to the lessee when the full remaining value, $Y\big[S(t),T-t\big]$, is retained. Then the solution of the differential equation (31) can be found subject to the boundary conditions (32). Grenadier (1996) gives a solution evaluated at time $t = 0$ by assuming that S and X follow correlated geometric Brownian motion.

$$W(S,X,0;P^D,T,M) = F(S,X,0;P^D,T) + M\cdot Q(X,T) - B(S,X;T,M,0,\omega) \tag{33}$$

where

$$Q(X,T) = N\left[\frac{\ln(K/X) - a_1\cdot T}{\sigma_x\sqrt{T}}\right] + \left(\frac{K}{X}\right)^{2a_1/\sigma_x^2}\cdot N\left[\frac{\ln(K/X) + a_1\cdot T}{\sigma_x\sqrt{T}}\right],$$

$$B(S,X;T,y_1,y_2,y_3) = \int_0^T b(S,X,T,y_1,y_2,y_3,\upsilon)d\upsilon,$$

$$\begin{aligned}b(S,X,T,y_1,y_2,y_3,\upsilon) &= e^{-r\upsilon}g(X,\upsilon)[b_1(S,X,T,y_1,y_2,y_3,\upsilon) \\ &\quad - b_2(S,X,T,y_1,y_2,y_3,\upsilon)],\end{aligned}$$

$$b_1(S,X,T,y_1,y_2,y_3,\upsilon) = f_1(T,y_1,y_2,\upsilon)N\left[\left(\ln\left(\frac{f_1(T,y_1,y_2,\upsilon)}{f_2(T,y_3,\upsilon)K^\lambda}\right)\right.\right.$$

$$\left.\left.-\mu_z(S,X,\upsilon)\right)\frac{1}{\sigma_z(\upsilon)}\right],$$

$$b_2(S,X,T,y_1,y_2,y_3,\upsilon) = K^\lambda \exp\left[\mu_z(S,X,\upsilon)+\tfrac{1}{2}\sigma_z(\upsilon)^2\right]$$

$$\cdot f_2(T,y_3,\upsilon)N\left[\left(\ln\left(\frac{f_1(T,y_1,y_2,\upsilon)}{f_2(T,y_3,\upsilon)K^\lambda}\right)\right.\right.$$

$$\left.\left.-\mu_z(S,X,\upsilon)-\sigma_z(\upsilon)^2\right)\frac{1}{\sigma_z(\upsilon)}\right],$$

$$f_1(T,y_1,y_2,\upsilon) = e^{r\upsilon}\left[y_1 + y_2\frac{P(T)}{r}\left(e^{-r\upsilon}-e^{-rT}\right)\right],$$

$$f_2(T,y_3,\upsilon) = y_3\left[1-e^{-(r-\alpha_s)(T-\upsilon)}\right],$$

$$g(X,\upsilon) = \frac{\ln(X/K)}{\sqrt{2\pi\sigma_x^2\upsilon^3}}\exp\left[\frac{(\ln(X/K)+a_1\upsilon)^2}{-2\sigma_x^2\upsilon}\right],$$

$$a_1 = \alpha_x - \tfrac{1}{2}\sigma_x^2,$$

$$\mu_z(S,X,\upsilon) = \ln\left(\frac{S}{r-\alpha_s}\right) - \lambda\ln(X) + \left[\alpha_s - \sigma_s^2/2 - \lambda(\alpha_x-\sigma_x^2/2)\right]\upsilon,$$

$$\sigma_z(\upsilon) = \sqrt{(\sigma_s^2-\lambda^2\sigma_x^2)\upsilon},$$

$$\lambda = \rho\sigma_s/\sigma_x,$$

Finally, selling the use of the asset for T years with or without security deposit must have the same value in equilibrium. Thus, $P^D(T, M)$ can be determined by the following equality:

$$F(S,X,0;P(T),T) = W(S,X,0;P^D(T,M),T,M) \qquad (34)$$

where $F(S,X,0;P(T),T)$ is already known in Section 3.1. Then by applying the above equality into equation (33), we can find $P^D(T, M)$ in the following form

$$P^D(T,M) = P(T) - \frac{r}{1-e^{-rT}-\Phi_2(X,T)}\cdot\left(M\cdot Q(X,T) - B(S,X;T,M,0,\omega)\right) \quad (35)$$

where $P(T)$ and $\Phi_2(X,T)$ are given in Section 3.1.

(II). Equilibrium lease rate with up-front prepayment clause

Up-front prepayment is very common in leases, especially in real-estate leasing. According to the contract, if a lessee pays n periods' rent in advance, the final n period of lease are rent-free. Prepayments provide protection to the lessor, and in return the lease payment should be lower than usual. Therefore, Grenadier (1996) assume the rent to be equal to the risk-free lease payment, *R(T)*, and find out how many years of rent should be prepaid. We will follow his way and also show that the same method can be used to find the equilibrium rental payment if the number of periods is given.

Assume the rent is *R(T)*, and δ years of rent is required in advance (unit of time is measured in years in a *T*-year lease). Then the up-front prepayment will be $\delta \cdot R(T)$. Once again the same argument that any two methods of selling the service flow of an asset for *T* years should have the same value will be used here. The first alternative is same as the risk-free case in Section 2.3, and the value is *Y(S, T)*. The second alternative is to charge the rent *R(T)*, and collect the prepayment $\delta \cdot R(T)$. If the lessee pays all the rents up to time $T-\delta$, then lessee can use the asset for the remainder of the term with no payments. If the lessee defaults at any time $t < T-\delta$, then the lessor will re-lease the asset to a riskless lessee for the remainder of the term.

The value of the second alternative is only slightly different from *Alternative 2* in Section 3.1. Here we consider a $(T+\tilde{T})$-year risky lease with a rental payment flow of P for the first T years, and no payments required for the rest $\tilde{T}$ years. The value of this lease is denoted by $\hat{F}(S,X,t;P,T,\tilde{T})$. It will solve the same differential equation as $F(S,X,t;P,T)$, with the exception that the right-hand side of boundary condition (26) will be $(1-\omega)\cdot Y[S(t),T+\tilde{T}-t]$. Under the assumption that S and X follow correlated geometric Brownian motions, $\hat{F}(S,X,t;P,T,\tilde{T})$ will have a closed-from solution identical to that of $F(S,X,t;P,T)$ in equation (28) with one exception that $\Phi_3(X,\tau)$ will be equal to $\left(\frac{X}{K}\right)^{c_2} G(X,\tau,a_4)-e^{-(r-\alpha_s)(\tau+\tilde{T})}G(X,\tau,a_3)$. Then, at time *0*, the value of the second alternative is $\delta\cdot R(T)+\hat{F}(S,X,0;R(T),T-\delta,\delta)$.

Thus, the number of periods of prepayment, δ, satisfies the equilibrium relation condition below:

$$Y(S,T)=\delta\cdot R(T)+\hat{F}(S,X,0;R(T),T-\delta,\delta) \qquad (36)$$

If a constant δ, say $\tilde{\delta}$, is given, we can also use equation (36) to find the equilibrium rental payment $\tilde{P}(T)$ other than *R(T)*. The solution is given below:

$$\tilde{P}(T)=\left(\frac{r\cdot S}{r-\alpha_s}\right)\left[\frac{1-e^{-(r-\alpha_s)T}-(1-\omega)\Phi_3(X,T-\tilde{\delta})}{1+r\tilde{\delta}-e^{-r(T-\tilde{\delta})}-\Phi_2(X,T-\tilde{\delta})}\right] \qquad (37)$$

(III). Valuation of lease credit insurance

Lease credit insurance protects the lessor against the loss of payments due to default. There are a variety of forms of such a protection, and one example that we will focus here is called credit enhancement provisions of lease-backed securities.

Lease-backed securities (LBS) are securitized leases. The lessor bundles a poor of leases and sells to the market to generate cash. In return, buyers of LBS can get steady cash flows which come from lease payments. Since buyers prefer to buy safe securities, lessors need to purchase credit insurance to improve the security's credit rating. Thus, in this subsection, we will focus on the value of the insurance contract instead of lease rate.

Consider a *T*-year risky lease with an equilibrium lease payment *P(T)*, as defined in equation (30). The insurance contract guarantees the investor a payoff of at least a fraction γ of the promised value of the remaining lease payments, $P(T)(1-e^{-r(T-t^*)})/r$, if a default occurs at time t^*. Since the amount recovered on the underlying lease is $(1-\omega)Y[S(t^*),T-t^*]$, the investor is assured of receiving an amount of $\max\left\{\gamma\cdot P(T)(1-e^{-r(T-t^*)})/r,(1-\omega)Y[S(t^*),T-t^*]\right\}$. One important assumption here is that the insurer's obligation is without risk. Then let $\Psi(S,X,t;T,\gamma)$ be the value of this insurance contract. It must solve the following partial differential equation:

$$\begin{aligned}0 &= \tfrac{1}{2}\sigma_s^2(S,t)S^2\Psi_{SS} + \rho(S,X,t)\sigma_s(S,t)\sigma_x(X,t)SX\Psi_{SX} + \tfrac{1}{2}\sigma_x^2(X,t)X^2\Psi_{XX} \\ &\quad + \alpha_s(S,t)S\Psi_S + \alpha_x(X,t)X\Psi_X + \Psi_t - r\Psi\end{aligned} \tag{38}$$

subject to the boundary conditions:

$$\begin{aligned}&\Psi(S,X,T;T,\gamma) = 0 \\ &\Psi(S,K,t;T,\gamma) = \max\left\{\frac{\gamma\cdot P(T)(1-e^{-r(T-t)})}{r} - (1-\omega)Y[S(t),T-t],0\right\}\end{aligned} \tag{39}$$

The cost of this insurance contract paid at time *0* is equal to $\Psi(S,X,0;T,\gamma)$. Under the assumption that *S* and *X* follow correlated geometric Brownian motions,

$$\Psi(S,X,0;T,\gamma) = B(S,X;T,0,\gamma,1-\omega) \tag{40}$$

where the function $B(S,X;T,y_1,y_2,y_3)$ is defined in equation (33).

4. Conclusion

In this essay, we have presented the valuation of lease contracts and credit risk. We start from the introduction of valuations of risk-free leases. Based on the work of Miller and Upton (1976), we show that the determination of the equilibrium rental payment on a one-period lease in a capital asset pricing model framework. Miller and Upton (1976) model can also be used iteratively to extend to a new discrete time model to valuate the rental payment for a *T*-year lease contract. Furthermore, by assuming that the value of using the lease asset follows a stochastic process, a continuous time valuation of lease payment has been introduced according Grenadier (1996).

The most important part in this essay is the presentation of Grenadier's (1996) approach to determine the equilibrium lease rate subject to credit risk. The model is flexible enough to be extended a variety of realistic leasing structures. Such structures include security deposits, up-front prepayments, and lease credit insurance.

Some extensions of the model would make it more adaptive to the real world. First, the risk free rate of interest could be stochastic. Leasing contracts are usually long commitments. Interest rate will change in long term. A large amount of literature has considered stochastic interest rate into valuations of corporate bonds. Same method could be used in valuations of leasing contracts. Second, the recovery rate could be stochastic too. Many papers, for an instance Guo, Jarrow and Zeng (2007), model the recovery rate process in credit risk literature. The methods could also be used in leasing valuations. However the difficulties may arise in the leasing valuations, because the availability of large and reliable samples of individual leasing contracts is quite difficult to obtain. The model might not be empirically tested on actual data.

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