Default Contagion in Intensity-Based Models

by

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Abstract

In this essay we introduce three intensity-based credit risk models: the copula approach, "the total hazard construction" approach and the Markov jump approach. While the three approaches model default dependence structure in different ways, they all capture the credit contagion property of the market. The copula approach introduces default dependence structure through a copula function and derives the dynamics of default intensities. Both the "total hazard construction" approach and the Markov jump approach are capable of handling "looping default" intensities, while we find that the "total hazard construction" approach actually implies the latter.

Contents

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Chapter 1

Introduction

In the credit risk literature, there are basically two main approaches of modeling the default processes of obligated firms: structural approach and reduced form approach.

The structural approach models the dynamics of the firm's asset value process and debt value process directly and defines default as the first time that the asset value reaches the debt value. The structural approach suffers from the disadvantage that both the asset value and the debt value are not continuously observable to the market. The framework of this approach is based on Merton (1974) and Black and Cox (1976). Intensity based approach, the most extended reduced form approach[1], models the intensity process of a single jump process, and defines default time as the jump time of the single jump process. This allows one to model default intensity and then derive the law of default time from it. In this essay, we concentrate on the intensity based approach.

In the past decade, the credit derivatives market have been growing rapidly both in volume and in the type of instruments it offers. Several multi-name credit derivatives, whose payoffs depend on credit events of a portfolio of credits have been

[1]There are other type of reduced form models that are not intensity based, e.g. Brody, Hughston and Macrina (2005)

traded largely. Typical multi-name structures are Basket Default Swaps (BDSs) and Collateralized Debt Obligations (CDOs). The default dependencies structure among many obligors in the reference portfolio of a multi-name derivative plays crucial role in the pricing problem, thus it draws great attention to both practitioners and researchers.

Duffee (1999) among others introduces default dependency by making the default intensities of different obligors dependent on a common set of state variables and on some firm specific factors. The common state variables represent the general economic conditions and may include macro and micro economic factors. However, this approach has been criticized for the low correlation it can reach. In fact, default clustering have been observed empirically. One firm's default often causes some other firms' credit risk to increase. This calls for models that can generate realistic correlation structures, i.e. capture the credit contagion.

In this essay, we present three intensity based approaches which capture the credit contagion: Schönbucher and Schubert (2001), Yu (2007) and Herbertsson and Rootzén (2007). Schönbucher and Schubert (2001) use pseudo default intensities and trigger levels to define default times. The default correlations are introduced through the copula function of the trigger levels. The joint law of default times and dynamics of default intensities are derived through the copula and the pseudo default intensities. Yu (2007) uses an algorithm called "total hazard construction" by Norros (1987) and Shaked and Shanthikumar (1987) to generate a set of default times that admit the pre-specified default intensities. These models are suitable for various types of default intensities and appeals to Monte-Carlo simulation. Herbertsson and Rootzén (2007) use a Markov jump process to define default times and deals with a special but intuitive type of default intensities. This model is actually also derived by Yu (2007), but it gives explicit joint default probabilities even for a large number of firms.

The rest of the essay is organized as follows. Chapter 2 introduces some preliminary knowledge that are for better understanding of this essay. Chapter 3 introduces Schönbucher and Schubert's copula approach. Chapter 4 discusses looping default problem and an interesting example by Kusuoka. Chapter 5 introduces Yu's "total hazard construction" approach. Chapter 6 introduces Herbertsson and Rootzén's Markov jump process approach and discusses its links to Yu's "total hazard construction" approach. Chapter 7 reviews this essay and includes some remarks on each of the three models.

Chapter 2

Preliminaries

2.1 Probability space and filtrations

Here we specify the probability space and information sets that would be needed in the following discussion. We would like to carefully introduce the information sets, since we are likely to get different default intensities under different information sets.

We consider an economy with I obligors and a probability space $(\Omega, \widetilde{\mathcal{F}}, \mathbb{P})$. All the filtrations introduced below are subsets of $\widetilde{\mathcal{F}}$ and contain the null-sets of $\widetilde{\mathcal{F}}$. The probability measure $\mathbb{P}$ may or may not be the martingale measure[1]. On this probability space lives a background process X and stopping times $\tau_i, i = 1, ..., I$, which are the default time of the ith obligor. We introduce filtrations generated by these processes as follows:

- The background filtration $\mathcal{G}_t$ is generated by the background process X_t. This filtration represents the general market information. Define $\mathcal{G}_\infty := \sigma\left(\bigcup_{t\geq 0} \mathcal{G}_t\right)$.

[1] $\mathbb{P}$ may vary between physical measure and risk-neutral measure, due to different applications. Unless we tackle the pricing problem, we need not specify it.

- $\mathcal{F}_t^i$ is generated by $N_i(t) := \mathbf{1}_{\{\tau_i \leq t\}}$, containing default information of obligor i up to time t.

- $\mathcal{F}_t := \sigma\left(\bigcup_{i=1}^{I} \mathcal{F}_t^i\right) = \sigma\left(N_i(s), 1 \leq i \leq I, 0 \leq s \leq t\right)$ contains default information of all the I obligors up to time t.

- $\mathcal{H}_t^i := \sigma\left(\mathcal{F}_t^i \bigcup \mathcal{G}_t\right)$ contains the default information of obligor i as well as the general economic information up to time t.

- $\mathcal{H}_t = \sigma\left(\bigcup_{i=1}^{I} \mathcal{H}_t^i\right) = \sigma\left(\mathcal{F}_t \bigcup \mathcal{G}_t\right)$ contains all the information that is available to market participants at time t, including the default information of all the I obligors in the market as well as the general economic information.

- $\tilde{\mathcal{H}}_t = \sigma\left(\mathcal{F}_t \bigcup \mathcal{G}_\infty\right)$ contains default or survival information of all the obligors until t and complete information about the background process.

2.2 Hazard rate and intensity

Fundamentally, reduced form model is built on the specification of the hazard rate or the intensity of the default time. The formal definitions in the mathematical sense are as follows.

Definition 2.1. *Let τ be a stopping time with cumulative distribution function $F(t)$ given by $F(t) = \mathbb{P}[\tau \leq t]$ and probability density function $f(t)$ given by $f(t) = \frac{dF(t)}{dt}$. Assume $F(t) < 1$ for all t. Then the hazard rate h of τ is defined as:*

$$h(t) := \frac{f(t)}{1 - F(t)}. \tag{2.1}$$

At later point in time $t > 0$ with $\tau > t$, the conditional hazard rate at time $T > t$ is defined as:

$$h(t,T) := \frac{f(t,T)}{1 - F(t,T)}, \tag{2.2}$$

where $F(t,T) := \mathbb{P}[\tau \leq T \mid \tau > t]$ is the conditional distribution of τ given the information of $\tau > t$ and $f(t,T)$ is the corresponding density.

It turns out that

$$h(t,T) = \lim_{\Delta t \searrow 0} \frac{1}{\Delta t}\mathbb{P}(T < \tau \leq T + \Delta t \mid \tau > t) \tag{2.3}$$

$$= -\frac{\partial}{\partial T}\mathbb{P}(\tau > T \mid \tau > t), \tag{2.4}$$

and

$$h(t) = \lim_{\Delta t \searrow 0} \frac{1}{\Delta t}\mathbb{P}(t < \tau \leq t + \Delta t \mid \tau > t)$$

$$= -\frac{\partial}{\partial T}\mathbb{P}(\tau > T \mid \tau > t) \mid_{T=t}. \tag{2.5}$$

Equation (2.3) gives the the following interpretation of hazard rate: if τ is the default time of an obligor, then hazard rate $h(t,T)$ of τ is the default probability per unit of time at time T, which gives the finest possible resolution of the likelihood of default in an infinitesimally small time interval $[T, T + \Delta t]$.

Consider a filtered probability space $(\Omega, \widetilde{\mathcal{F}}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and a stopping time τ adapted to it. Let $N(t) = \mathbf{1}_{\{\tau \leq t\}}$ be the single jump process associated with τ. From the *Doob-Meyer Decomposition Theorem*, the process $N(t)$ can be represented as:

$$N(t) = M(t) + A(t) \qquad \forall\, t > 0,$$

where $M(t)$ is a continuous martingale and $A(t)$ is a continuous increasing process.

Definition 2.2. *The single jump process N(t) has an $\mathcal{F}_t$-intensity λ, if λ is non-negative and $\mathcal{F}_t$-adapted and the compensator A(t) can be represented by*

$$A(t) = \int_0^{t\wedge\tau} \lambda_s ds \qquad \forall\, t \in [0,T].$$

Li (2000) describes default time by a survival function $S(t) = \mathbb{P}(\tau > t)$, which indicates the probability that an obligor can "survive" up to time t. Intensity, hazard rate and survival function are closely related. Under some regularity conditions[2],

[2]See e.g. Aven(1985)

we can obtain $\lambda(t) = h(t)$. Moreover, since $S(t) = 1 - F(t)$, from the definition of hazard rate function one can easily get

$$h(t) = -\frac{S'(t)}{S(t)},$$

and equivalently,

$$S(t) = e^{-\int_0^t h(s)ds}.$$

In a model with constant intensity $\lambda(t) = \lambda$, the single jump process $N(t)$ is the first arrival of a Poisson process with intensity λ. If we consider $\lambda(t)$ to be a stochastic process, then $N(t)$ is said to follow what is called a *Cox process* or a *doubly stochastic Poisson process*. Lando (1998) shows that, under a *Cox process* the default time τ can be equivalently characterized in the following way:

$$\tau := \inf\{t : \int_0^t \lambda(s)ds \geq \theta\}, \tag{2.6}$$

where θ is an exponential random variable of parameter 1 and independent of λ.

2.3 Copula function

A copula function is used as a general way of formulating a multivariate distribution in such a way that various types of dependence structure can be represented. In credit risk area, copulas are used to model the joint distribution and dependence structure of default times.

Definition 2.3. *A copula is a multivariate joint distribution function* $\boldsymbol{C}: [0,1]^N \to [0,1]$ *, such that every marginal distribution is uniform on [0,1].*

Since a copula function $\mathbf{C}$ has uniform marginal distributions, it has the following property:

$$\mathbf{C}(1, ..., 1, u_i, 1, ..., 1) = u_i, \tag{2.7}$$

for all $i = 1, ..., N$ and $u_i \in [0, 1]$.

Recall that, if a function F(x) is a distribution function of some random variable X, then F(X) is uniform on [0,1]. Sklar (1959) shows that every joint distribution function can be represented as some copula function, which takes as inputs the marginal distributions.

Theorem. (Sklar)
Let $X_1, X_2, ..., X_n$ be random variables with marginal distribution function $F_1, F_2, ..., F_n$ and joint distribution $\mathbf{F}$. Then there exits an N-dimensional copula function $\mathbf{C}$, such that $\mathbf{F}(x_1, x_2, ..., x_n) = \mathbf{C}(F_1(x_1), F_2(x_2), ..., F_n(x_n))$ for all $(x_1, x_2, ..., x_n) \in \mathbb{R}^N$. Moreover, if each F_i is continuous, then the copula $\mathbf{C}$ is unique.

Sklar's Theorem allows one to separate the modeling of the marginal distributions of individual default times from the modeling of their dependence structure. Unfortunately, how to choose the copula function is still a very challenging issue to the literature, and this difficulty has been the major drawback of the copula approach.

Chapter 3

Schönbucher and Schubert's copula model

3.1 Model setup

Schönbucher and Schubert (2001) use copula function to model the dependence structure in default times of different companies. While different from other models in copula approach, the copula function is not directly applied to default times or default intensities, but rather to the so called "trigger variables". Before we introduce the model, let us recall Lando's definition of default time:

$$\tau_i := \inf\{t : \int_0^t \lambda_i(s)ds \geq \theta_i\}, \tag{3.1}$$

where θ_i is an exponential random variable of parameter 1 and independent of λ_i[1]. If we multiply by -1 and take exponential to both sides of the inequality above, we end up with

$$\exp(-\int_0^t \lambda_i(s)ds) \leq \exp(-\theta). \tag{3.2}$$

Now define countdown process γ_i and trigger variable U_i as:

$$\gamma_i(t) := \exp(-\int_0^t \lambda_i(s)ds), \tag{3.3}$$

[1] The subscript i refers to obligor i.

and

$$U_i := \exp(-\theta). \tag{3.4}$$

It can be easily prove that U_i is uniformly distributed on [0,1].

After substitutions, we end up with an equivalent definition of default time as (3.1):

$$\tau_i := \inf\{t : \gamma_i(t) \leq U_i\}. \tag{3.5}$$

Now, default of obligor i occurs when the countdown process $\gamma_i(t)$ reaches the trigger level U_i.

Schönbucher and Schubert(2001) define the individual default time τ_i as in (3.5) and make further assumptions:

(i) The processes λ_i, $i = 1, 2, ..., I$, the *pseudo default intensities* called by the authors, are non-negative stochastic processes adapted to the background filtration $\mathcal{G}_t$.

As explained by the authors:" the pseudo default intensity $\lambda_i(t)$ controls the speed of the countdown process $\gamma_i(t)$, and thus the likelihood of an early default." The specification of λ_i in this model is not restricted, thus gives readers freedom to choose their favorite one. The reason why the authors call λ_i the pseudo default intensity is that it is the default intensity of τ_i only when the underlying filtration is $\mathcal{H}_t^i$. The detailed discussion of this is shown in next section.

(ii) $\mathcal{U}_i := \sigma(U_i)$, the filtration generated by U_i, is independent of the total background information set $\mathcal{G}_\infty$. The realization of U_i is only revealed to the economy when the default of obligor i occurs.

(iii) Under $\mathcal{H}_0^i$, $i = 1, ..., I$, the trigger level U_i is uniformly distributed on [0,1]. Let $\mathbf{U} = (U_1, ..., U_I)$ and the distribution of $\mathbf{U}$ is given by the I-dimensional copula $C(\mathbf{u})$, where $\mathbf{u} \in [0,1]^I$ and C is twice continuously differentiable.

3.2 Conclusion

Under these assumptions, we reach the following conclusions. First, as mentioned above, when only one obligor and the general state of economy are observed, the default intensity of that obligor is just its pseudo default intensity.

Theorem 3.1. *Given the filtration $\mathcal{H}_t^i$, the default intensity $h_i(t)$ of the single jump process $N_i(t) := \mathbf{1}_{\{\tau_i \le t\}}$ is given by:*

$$h_i(t) = \mathbf{1}_{\{\tau_i > t\}} \lambda_i(t). \tag{3.6}$$

For a proof see appendix B.

If assuming larger information set $\mathcal{H}_t$ is available, that is all the obligors as well as the general state of economy are observed, the default intensity of each obligor is no longer the pseudo default intensity but depends on the default times of the defaulted obligors. Furthermore, the joint default probability, the default intensities and their dynamics are derived in the following theorem.

Theorem 3.2. *Assume that $\lambda_i(t)$ follows some diffusion process and the filtration $\mathcal{H}_t$ is observed. At time t, if no default has occurred, then the joint distribution of default times $\boldsymbol{\tau} = (\tau_1, ..., \tau_I)$ is given by:*

$$\mathbb{P}(\boldsymbol{\tau} > \mathbf{T} \mid \mathcal{H}_t) = \frac{\mathbf{E}\left[\, C(\gamma_1(T_1), ..., \gamma_I(T_I)) \mid \mathcal{H}_t\right]}{C(\gamma_1(t), ..., \gamma_I(t))}, \tag{3.7}$$

where $\mathbf{T} = (T_1, ..., T_I)$, and $T_i \ge t$ for all $i = 1, ..., I$;

The $\mathcal{H}_t$-intensity of $N_i(t)$ is given by:

$$h_i(t) = \lambda_i(t)\gamma_i(t)\frac{\frac{\partial}{\partial x_i}C(\boldsymbol{\gamma}(t))}{C(\boldsymbol{\gamma}(t))} = \lambda_i(t)\gamma_i(t)\frac{\partial}{\partial x_i}\ln C(\boldsymbol{\gamma}(t)), \tag{3.8}$$

where $\boldsymbol{\gamma}(t) := (\gamma_1(t), ..., \gamma_I(t))$.

The dynamics of h_i is given by:

$$dh_i = \frac{C_{x_i}}{C}\gamma_i\lambda_i\left[\left(\frac{d\lambda_i}{\lambda_i} - \lambda_i dt\right) - \sum_{j=1}^{I}\left(\frac{C_{x_ix_j}}{C_{x_i}} - \frac{C_{x_j}}{C}\right)\gamma_j\lambda_j dt\right], \tag{3.9}$$

where the subscript notation of C means partial derivatives and the argument t of λ and the argument $\boldsymbol{\gamma}(t)$ of C are suppressed .

If k obligors ($k \leq I$) have defaulted before time t, w.l.o.g. assuming these are the first k ones, the conditional distribution function of $\boldsymbol{\tau}$ is given by:

$$\mathbb{P}(\boldsymbol{\tau} > \mathbf{T} \mid \mathcal{H}_t) = \frac{\mathbf{E}\left[\, C_{x_1...x_k}(\gamma_1(t_1), ..., \gamma_k(t_k), \gamma_{k+1}(T_{k+1}), ..., \gamma_I(T_I)) \mid \mathcal{H}_t\right]}{C_{x_1...x_k}(\gamma_1(t_1), ..., \gamma_k(t_k), \gamma_{k+1}(t), ..., \gamma_I(t))}, \tag{3.10}$$

where $\tau_i = t_i$ and $T_i < \tau_i$ for $i \leq k$ and $T_i > t$ for $i > k$.

In this case the $\mathcal{H}_t$-intensity of $N_i(t)$ is given by:

$$h_i(t) = \frac{C_{x_1...x_kx_i}(\gamma_1(t_1), ..., \gamma_k(t_k), \gamma_{k+1}(t), ..., \gamma_I(t))}{C_{x_1...x_k}(\gamma_1(t_1), ..., \gamma_k(t_k), \gamma_{k+1}(t), ..., \gamma_I(t))}\lambda_i(t)\gamma_i(t), \tag{3.11}$$

and the dynamics of h_i is given by:

$$dh_i = \frac{C_{x_1...x_kx_i}}{C_{x_1...x_k}}\gamma_i\lambda_i\left[\left(\frac{d\lambda_i}{\lambda_i} - \lambda_i dt\right) - \sum_{j=k+1}^{I}\left(\frac{C_{x_1...x_kx_ix_j}}{C_{x_1...x_kx_i}} - \frac{C_{x_1...x_kx_j}}{Cx_1...x_k}\right)\gamma_j\lambda_j dt\right]. \tag{3.12}$$

For a proof see appendix C.

As seen from theorem 3.2, when more information is available, not only the occurrence or non-occurrence of default of the underlying obligor i but also that of the other obligors, the default intensity $h_i(t)$ is no longer the pseudo default intensity $\lambda_i(t)$. This is because the default times of obligors are correlated (through the copula function), then the information of the default or non-default statuses of the other obligors would be "bad news" or "good news" for the concerned obligor, i.e. the default intensity of the concerned obligor would deviate from the pseudo intensity, the default intensity in the isolated information case.

We notice from above that, under the filtration $\mathcal{H}_t$ the default statuses of other obligors may affect the default intensity of the target obligor. We are interested in what kind of effect it would be. For this purpose, we compare the default intensity of the target obligor i right after the default of another obligor j with its ex-ante default intensity. The results are shown in the following theorem.

Theorem 3.3. *Since we assume the copula function C to be differentiable, at each point of time there is at most one default that could happen. Assume the first k obligors ($k \leq I$) have defaulted before time t and obligor j (different from i and $j \geq k$) defaults at time t, i.e. $\tau_j = t$, then from theorem 3.2 the default intensity of obligor i is given by:*

$$h_i(t) = \frac{C_{x_1 \ldots x_k x_i x_j}(\gamma_1(t_1), \ldots, \gamma_k(t_k), \gamma_{k+1}(t), \ldots, \gamma_I(t))}{C_{x_1 \ldots x_k x_j}(\gamma_1(t_1), \ldots, \gamma_k(t_k), \gamma_{k+1}(t), \ldots, \gamma_I(t))} \lambda_i(t)\gamma_i(t), \tag{3.13}$$

where $\tau_n = t_n$ for $n \leq k$.

Compared (3.13) with the default intensity before the default of obligor j (3.11), we can see a jump in the default intensity and the jump size is given by:

$$\Delta h_i = \lambda_i \gamma_i \frac{C_{x_1 \ldots x_k x_i}}{C_{x_1 \ldots x_k}} \left[\frac{C_{x_1 \ldots x_k x_i x_j} C_{x_1 \ldots x_k}}{C_{x_1 \ldots x_k x_j} C_{x_1 \ldots x_k x_i}} - 1 \right]. \tag{3.14}$$

What about the sign of the jump? Intuitively, we would expect to see a positive jump if the default times τ_i and τ_j are positively correlated and vice versa. To be more rigorous, the sign of the jump involves in the measure of dependence between U_i and U_j. In fact, if we use *Kendall's tau* as the dependence measure, a positive dependence between U_i and U_j would lead to a positive jump and a negative dependence would lead to a negative jump. And a positive (negative) dependence between U_i and U_j means positive (negative) dependence between τ_i and τ_j. This means the default of obligor j would increase (decrease) the default probability of obligor i, thus the contagion effect is captured. In the next chapter we will introduce a different approach of capturing the contagion effect.

Chapter 4

Looping default and Kusuoka's example

4.1 Looping default

In order to capture the default clustering phenomenon, Jarrow and Yu (2001) among others consider a very intuitive and straightforward intensity model. This approach introduces single jump processes to constant default intensities[1]. It assumes the default intensities remain constant if no default happens, while the default of a correlated firm[2] causes a jump in default intensity. Under this assumption, the default intensities have the following form:

$$\lambda_i = (a_i + \sum_{j \neq i} b_{i,j} \mathbf{1}_{\{\tau_j \leq t\}}) \mathbf{1}_{\{\tau_i \geq t\}}, \quad i = 1, ..., I \tag{4.1}$$

where a_i and $b_{i,j}$ are constants and $a_i > 0$. Here, in order to make it more general, we allow $b_{i,j}$ to be negative or zero: a negative $b_{i,j}$ means obligor i benefits from the default of obligor j and $b_{i,j} = 0$ means i is not affected by the default of j.

While intuitively convenient, (4.1) defines default times recursively through each

[1]Unless explicitly stated otherwise, we refer to default intensity as $\mathcal{H}_t$-intensity.
[2]Here we mean non-zero correlation between default times.

other, which makes it difficult to derive both joint and marginal default probabilities. In order to avoid the "looping defaults" problem, Jarrow and Yu (2001) assume the so called " primary-secondary firm" structure, where they classify obligors into primary and secondary firms and assume the default of primary firms may increase the default intensities of secondary firms, but not the other way round. They also give real-life financial arguments of two kinds to support the assumption. First, the secondary firm may be seen as a financial institution that has a long or a short position in the debt of a primary firm, so that the likelihood of its default depends on the status of the primary firm. Second, a primary firm may be seen as a large corporation which have many suppliers and a secondary firm may be seen as one of its small dependent supplier which could be affected by its default. By virtue of this classification, the joint and marginal default probabilities can be derived without difficulty.

4.2 Kusuoka's example

Since (4.1) involves with looping definition of default times and default intensities, a natural question is whether or not they exit. In other words, we are interested in the existence of such default times and intensities that solve (4.1). Kusuoka (1999) gives an example of a two-firm case where the default times and default intensities satisfying (4.1) are constructed by means of change of probability measure. Before introducing Kusuoka's example, we first present its building blocks. We prefer to introduce the following propositions without getting into the detailed proofs, which can be found in Bielecki and Rutkowski (2003).

Kusuoka (1999) follows the standard setup of chapter 2 except that he chooses a fixed time horizon T and makes further assumptions:

Assumption 4.1. *The filtration $\{\mathcal{G}_t\}_{t\leq T}$ is generated by a Brownian motion W.*

Assumption 4.2. *Under $\mathbb{P}$, any $\mathcal{G}_t$-martingale is also an $\mathcal{H}_t$-martingale.*

Assumption 4.3. *For $i = 1, ..., I$, there exists an $\mathcal{H}_t$-adapted positive process λ_i, such that $M_i(t) = N_i(t) - \int_0^{t\wedge\tau_i} \lambda_i(s)ds$ is an $\mathcal{H}_t$-martingale, i.e., each default time τ_i admits an $\mathcal{H}_t$-intensity λ_i.*

Under these assumptions we have:

Proposition 4.1. *For any $\mathcal{H}_t$-martingale η we have:*

$$\eta(t) = \eta(0) + \int_0^t \xi(s)dW(s) + \sum_{i=1}^{I} \int_0^t \zeta_i(s)dM_i(s), \tag{4.2}$$

where ξ and ζ are $\mathcal{H}_t$-predictable processes.

Suppose we have a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ with density $\frac{d\mathbb{Q}}{d\mathbb{P}} = \tilde{\eta}$, which is an a.s. positive and $\tilde{\mathcal{F}}$-measurable random variable. Then we introduce the Radon-Nikodým density process $\eta(t)$ by setting:

$$\eta(t) := \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{H}_t} = \mathbf{E}_{\mathbb{P}}(\tilde{\eta} \mid \mathcal{H}_t), \quad \mathbb{P} - a.s. \tag{4.3}$$

By proposition 4.1 we have

$$\eta(t) = 1 + \int_0^t \xi(s)dW(s) + \sum_{i=1}^{I} \int_0^t \zeta_i(s)dM_i(s), \tag{4.4}$$

where ξ and ζ are $\mathcal{H}_t$-predictable processes. Since η is a positive martingale, we rewrite (4.4) as:

$$\eta(t) = 1 + \int_0^t \eta(s-)[\beta(s)dW(s) + \sum_{i=1}^{I} \kappa_i(s)dM_i(s)], \tag{4.5}$$

where $\beta(t) := \xi(t)\eta(t-)^{-1}$ and $\kappa_i(t) := \zeta_i(t)\eta(t-)^{-1}$, $i = 1, ..., I$, are $\mathcal{H}_t$-predictable processes.

Proposition 4.2. *Under probability measure $\mathbb{Q}$,*

$$\tilde{W}_t := W_t - \int_0^t \beta_s ds \tag{4.6}$$

is an $\mathcal{H}_t$-Brownian motion, and

$$\tilde{M}_i(t) = N_i(t) - \int_0^{t\wedge\tau_i} (1+\kappa_i(s))\lambda_i(s)ds, \qquad i = 1, ..., I \tag{4.7}$$

are $\mathcal{H}_t$-martingales.

Now we are ready to introduce Kusuoka's example. Our goal is to find two default times τ_A and τ_B such that their $\mathcal{H}_t$-intensities satisfy

$$\lambda_1(t) = a_1 + b_1 1_{\{t\geq\tau_2\}}, \tag{4.8}$$

and

$$\lambda_2(t) = a_2 + b_2 1_{\{t\geq\tau_1\}}, \tag{4.9}$$

where a_1, a_2, b_1, b_2 are positive constants. We start from a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$. Suppose we have density $\eta_T = \frac{d\mathbb{P}}{d\mathbb{Q}}$ and Radon-Nikodým density process $\eta_t := \frac{d\mathbb{P}}{d\mathbb{Q}} \mid_{\mathcal{H}_t}$, which will be specified later. Consider a two-firm economy, where two default times τ_1 and τ_2 live on the filtered probability space $(\Omega, \widetilde{\mathcal{F}}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$. Assume under $\mathbb{Q}$, τ_1 and τ_2 follow independent exponential distributions with intensities a_1 and a_2 respectively. Then the joint probability law of τ_1 and τ_2 has density function

$$f(x,y) = a_1 a_2 e^{-(a_1x+a_2y)}, \quad \forall (x,y) \in \mathbb{R}_+^2.$$

Notice that under the present setting the filtration $\{\mathcal{G}_t\}_{t\leq T}$ is trivial. Thus by virtue of (4.5), the Radon-Nikodým density process η_t, $t \in [0,T]$, satisfies

$$\eta(t) = 1 + \sum_{i=1}^{2} \int_0^t \eta(s-)\kappa(s)dM(s). \tag{4.10}$$

Kusuoka (1999) defines $\kappa_i(t)$, $i = 1, 2$, in the following way:

$$\kappa_1(t) := 1_{\{\tau_2<t\}}\frac{b_1}{a_1}, \quad \kappa_2(t) := 1_{\{\tau_1<t\}}\frac{b_2}{a_2}. \tag{4.11}$$

Bielecki and Rutkowski (2002) show that $\eta_i(t)$ can be explicitly represented by

$$\eta_1(t) = 1_{\{\tau_1\leq\tau_2\}} + 1_{\{\tau_2\leq t\leq\tau_1\}}e^{-b_1(t-\tau_2)} + 1_{\{\tau_2<\tau_1<t\}}\frac{a_1+b_1}{a_1}e^{-b_1(\tau_1-\tau_2)}, \tag{4.12}$$

and

$$\eta_2(t) = 1_{\{\tau_2 \le \tau_1\}} + 1_{\{\tau_1 \le t \le \tau_2\}} e^{-b_2(t-\tau_1)} + 1_{\{\tau_1 < \tau_2 < t\}} \frac{a_2 + b_2}{a_2} e^{-b_2(\tau_2 - \tau_1)}. \quad (4.13)$$

By virtue of proposition 4.2, under the probability measure $\mathbb{P}$, the compensated processes

$$\tilde{M}_1(t) = N_1(t) - \int_0^{t \wedge \tau_1} (a_1 + b_1 1_{\{t \ge \tau_2\}}) ds, \quad (4.14)$$

and

$$\tilde{M}_2(t) = N_2(t) - \int_0^{t \wedge \tau_2} (a_2 + b_2 1_{\{t \ge \tau_1\}}) ds \quad (4.15)$$

are $\mathcal{H}_t$-martingales. This means under $\mathbb{P}$, $\lambda_1(t) = a_1 + b_1 1_{\{t \ge \tau_2\}}$ and $\lambda_2(t) = a_2 + b_2 1_{\{t \ge \tau_1\}}$ are $\mathcal{H}_t$-intensities of τ_1 and τ_2 respectively.

Kusuoka (1999) also gives the marginal distributions of τ_1 and τ_2 explicitly under $\mathbb{P}$. Bielecki and Rutkowski (2003) further derive the joint law of τ_1 and τ_2. These results are shown in the following proposition.

Proposition 4.3. *Assume $a_1 \neq b_2$ and $a_2 \neq b_1$. Then for every $t > 0$,*

$$\mathbb{P}(\tau_1 > t) = \frac{a_2 e^{-(a_1+b_1)t} - b_1 e^{-(a_1+a_2)t}}{a_2 - b_1}, \quad (4.16)$$

and

$$\mathbb{P}(\tau_2 > t) = \frac{a_1 e^{-(a_2+b_2)t} - b_2 e^{-(a_2+a_1)t}}{a_1 - b_2}. \quad (4.17)$$

For $s > t > 0$, the joint law of τ_1 and τ_2 are given by

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = \frac{b_1 e^{-(a_1+a_2)s} - a_2 e^{-(a_1+b_1)s + (b_1 - a_2)t}}{b_1 - a_2}. \quad (4.18)$$

Yu (2007) obtains the same results as proposition 4.3 by using the so called "total hazard construction" algorithm, which is introduced in the next chapter.

Chapter 5

Yu's "total hazard construction"

Yu (2007) uses an algorithm called "total hazard construction" from Norros (1987) and Shaked and Shanthikumar (1987)[1], to construct an arbitrary number of default times that admit pre-specified default intensities. The model is introduced below.

Suppose we have a probability space $(\Omega, \widetilde{\mathcal{F}}, \mathbb{P})$, where there are a background process X and stopping times τ_i associated with the single jump processes N_i, $i = 1, ..., I$. Now we consider the filtered probability space $(\Omega, \widetilde{\mathcal{F}}, (\mathcal{H}_t)_{t \geq 0}, \mathbb{P})$, where $\mathcal{H}_t$ is defined as in chapter 2. Assume $\mathbb{P}(0 < \tau_i < \infty) = 1$, for each i and $\mathbb{P}(\tau_i = \tau_j) = 0$ for $i \neq j$. Suppose we have specified a certain type of default intensities λ_i, which may or may not be the same type as the ones in Jarrow and Yu (2001), but they are assumed to be deterministic functions of time t and the scenario of X between defaults. Then we focus on constructing the corresponding stopping times that have such intensities. First, consider the case without the background process X. Now the filtration $\mathcal{H}_t$ is reduced to $\mathcal{F}_t := \sigma\left(N_i(s), 1 \leq i \leq I, 0 \leq s \leq t\right)$. Notice that the information set $\mathcal{F}_t$ provides information of the statuses (default or survival) of each obligor and

[1]The algorithm traces its root to Meyer (1971)

the default times of the defaulted obligors. Suppose until time t, n obligors have defaulted and let k_j be the index of the jth defaulter and t_j be the corresponding default time, $j = 1, ..., n$. Define $I_n = (k_1, ..., k_n)$ and $T_n = (t_1, ..., t_n)$, then we can use (I_n, T_n) to represent the information of $\mathcal{F}_t$. Now write $\lambda_i(t) = \lambda_i(t \mid I_n, T_n)$ when t is between the nth default time t_n and the $(n+1)th$ default time t_{n+1}. Notice that, by assumption it is a deterministic function of time t.

Then we truncate time horizon by the default times t_j, $j = 1, ..., n$. Define the total hazard $\phi_i(t \mid I_n, T_n)$ accumulated by obligor i up to time t as:

$$\phi_i(t \mid I_n, T_n) := \sum_{j=1}^{n} \Lambda_i(t_j - t_{j-1} \mid I_{j-1}, T_{j-1}) + \Lambda_i(t - t_n \mid I_n, T_n), \tag{5.1}$$

where

$$\Lambda_i(s \mid I_j, T_j) := \int_{t_j}^{t_j+s} \lambda_i(u \mid I_j, T_j)du, \quad 0 < s \leq t_{j+1} - t_j \tag{5.2}$$

is the total hazard accumulated by obligor i from the period of $[\ t_j, t_j + s\ ]$.

Let ψ_i be the total hazard accumulated by obligor i until its default, then Norros (1987) shows that $\psi_i, i = 1, ..., I$ are i.i.d. unit exponential random variables. In light of Norros, Shaked and Shanthikumar (1987) start from a collection of independent unit exponential random variables and generate a collection of stopping times.

Since λ_i, $i = 1, ...I$ are non-negative processes, we can define the inverse of $\Lambda_i(s \mid I_n, T_n)$ as:

$$\Lambda_i^{-1}(x \mid I_n, T_n) := \inf\{s : \ \Lambda_i(s \mid I_n, T_n) \geq x\}, \quad x \geq 0. \tag{5.3}$$

Then $\hat{\tau} = (\hat{\tau}_1, ..., \hat{\tau}_I)$ are generated recursively by the following procedures as stated in Yu (2007):

1. Draw a collection of i.i.d. unit exponential random variables E_i, $i = 1, ..., N$.

2. The index of first defaulter k_1 is determined by

$$k_1 = \arg\min_{i=1,..,I}\{\Lambda_i^{-1}(E_i)\}, \tag{5.4}$$

and the corresponding default time $\hat{\tau}_{k_1}$ is defined as:

$$\hat{\tau}_{k_1} = \Lambda_{k_1}^{-1}(E_{k_1}), \tag{5.5}$$

and let $t_1 = \hat{\tau}_{k_1}$, then consequently we construct $(I_1, T_1) := (k_1, t_1)$.

3. Assume we have constructed $(\hat{\tau}_{k_1}, ..., \hat{\tau}_{k_{j-1}}) = (t_1, ..., t_{j-1})$ and the corresponding I_{j-1} and T_{j-1}, where $j \geq 2$. Then define the survival set as the compliment of I_{j-1}, $\bar{I}_{j-1} := (1, ..., I)\backslash I_{j-1}$. Remember $\phi_i(t_{j-1} \mid I_{j-1}, T_{j-1})$ is the total hazard accumulated by obligor i up to t_{j-1}. Then the index of the next defaulter k_j is determined by:

$$k_j = \arg\min_{i\in\bar{I}_{j-1}}\{\Lambda_i^{-1}(E_i - \phi_i(t_{j-1} \mid I_{j-1}, T_{j-1}) \mid I_{j-1}, T_{j-1})\}, \tag{5.6}$$

and the corresponding default time $\hat{\tau}_{k_j}$ is defined as:

$$\hat{\tau}_{k_j} := t_{j-1} + \Lambda_{k_j}^{-1}(E_{k_j} - \phi_{k_j}(t_{j-1}|I_{j-1}, T_{j-1}) \mid I_{j-1}, T_{j-1}). \tag{5.7}$$

Let $t_j = \hat{\tau}_{k_j}$, then consequently we construct $I_j = (k_1, ..., k_j)$ and $T_j = (t_1, ..., t_j)$.

4. Stop, if $j = I$. Otherwise increase j by 1 and go to step 3.

Norros (1987) and Shaked and Shanthikumar(1987) show that $\hat{\tau} \stackrel{d}{=} \tau$, where "$\stackrel{d}{=}$" means equality in law. This means we can use intensities $\lambda_i(t)$, $i = 1, ..., I$, to recover the law of τ. If we specify the probability space where $\hat{\tau}$ lives as $(\hat{\Omega}, (\hat{\mathcal{F}}_t)_{t\geq 0}, \hat{\mathbb{P}})$, where $\hat{\mathcal{F}}_t$ is the internal history of the single jump processes $\hat{N}_i$ associated with $\hat{\tau}_i$, $i = 1, ..., I$, then it can be shown that the $\hat{\mathcal{F}}_t$-intensity of $\hat{N}_i$ has the same form of intensity as that of N_i, which we have specified in the beginning, i.e., $\hat{\lambda}_i(t) =$

$\lambda_i(t, | I_n, T_n)$. This follows from:

$$\begin{aligned}
\hat{\lambda}_i(t) &= \lim_{\Delta t \searrow 0} -\frac{1}{\Delta t}\mathbb{P}(\hat{\tau}_i > t + \Delta t \mid \hat{\mathcal{F}}_t) \\
&= \lim_{\Delta t \searrow 0} -\frac{1}{\Delta t}\mathbb{P}(\hat{\tau}_i > t + \Delta t \mid \hat{\tau}_{k_1} = t_1, ..., \hat{\tau}_{k_n} = t_n) \\
&= \lim_{\Delta t \searrow 0} -\frac{1}{\Delta t}\mathbb{P}(\tau_i > t + \Delta t \mid \tau_{k_1} = t_1, ..., \tau_{k_n} = t_n) \\
&= \lambda_i(t, | I_n, T_n) \qquad (5.8)
\end{aligned}$$

Next, we take into account the background process X. Now, the filtration $\mathcal{H}_t$ contains not only $\mathcal{F}_t$ but also $\mathcal{G}_t$, i.e., $\mathcal{H}_t = \sigma(\mathcal{F}_t \bigcup \mathcal{G}_t)$. The pre-specified default intensity λ_i is represented as $\lambda_i(t) = \lambda_i(t, | I_n, T_n, \omega)$, where $\omega \in \mathcal{G}_t$. In order to construct $(\hat{\tau}_1, ..., \hat{\tau}_I)$ in this case, we need another assumption. We assume the σ-fields $\mathcal{G}_\infty$ and $\mathcal{F}_t$ are independent given $\mathcal{G}_t$. Under this assumption, it is straightforward to show that the $\mathcal{H}_t$-intensity λ_i is also the $\tilde{\mathcal{H}}_t = \sigma(\mathcal{F}_t \bigcup \mathcal{G}_\infty)$-intensity. The strategy, then, is to fix an element ω of $\mathcal{G}_\infty$ and follow the total hazard construction procedures above in the sense conditional on ω, i.e.:

1. Draw a complete sample path ω of X and i.i.d. unit exponential random variables E_i, $i = 1, ..., N$, independent of $\mathcal{G}_\infty$.
2. Following the procedures 2-4 above.

Following above, we can generate a set of stopping times $\hat{\tau} = (\hat{\tau}_1, ..., \hat{\tau}_I)$, which not only depend on the unit exponential random variables but also depend on $\omega \in \mathcal{G}_t$. Similarly, we can specify the probability space where $\hat{\tau}$ lives by $(\hat{\Omega}, (\hat{\mathcal{H}}_t)_{t \geq 0}, \hat{\mathbb{P}})$ as before, where $\hat{\mathcal{H}}_t := \sigma\left(\hat{\mathcal{F}}_t \bigcup \mathcal{G}_t\right)$. Let $\hat{\mathcal{H}}'_t := \sigma\left(\hat{\mathcal{F}}_t \bigcup \mathcal{G}_\infty\right)$. Yu (2007) shows that the $\hat{\mathcal{H}}'_t$-intensity $\hat{\lambda}_i$ of $\hat{\tau}_i$ is just $\lambda_i(t, | I_n, T_n, \omega)$. To see this, we have:

$$\begin{aligned}
\hat{\lambda}_i(t) &= \lim_{\Delta t \searrow 0} -\frac{1}{\Delta t}\mathbb{P}(\hat{\tau}_i > t + \Delta t \mid \hat{\mathcal{H}}'_t) \\
&= \lim_{\Delta t \searrow 0} -\frac{1}{\Delta t}\mathbb{P}(\hat{\tau}_i > t + \Delta t \mid \hat{\tau}_{k_1} = t_1, ..., \hat{\tau}_{k_n} = t_n, \omega) \\
&= \lim_{\Delta t \searrow 0} -\frac{1}{\Delta t}\mathbb{P}(\tau_i > t + \Delta t \mid \tau_{k_1} = t_1, ..., \tau_{k_n} = t_n, \omega) \\
&= \lambda_i(t, | I_n, T_n, \omega).
\end{aligned}$$

Now we still need to show that the $\hat{\mathcal{H}}_t$-intensity of $\hat{\tau}_i$ has the specified form. This is straightforward, since from the assumption, the $\mathcal{H}_t$-intensity λ_i is also the $\tilde{\mathcal{H}}_t = \sigma\left(\mathcal{F}_t \bigcup \mathcal{G}_\infty\right)$-intensity and consequently $\lambda_i = \lambda_i(t, | I_n, T_n, \omega)$ is actually $\mathcal{H}_t$-measurable. Thus, the $\hat{\mathcal{H}}_t$-intensity of $\hat{\tau}_i$ is also $\lambda_i = \lambda_i(t, | I_n, T_n, \omega)$.

Using the above algorithm, Yu (2007) constructs the default times modeled in Kusuoka's two-firm example. Recall in Kusuoka's example, the default intensities are

$$\lambda_1(t) = a_1 + b_1 1_{\{t \geq \tau_2\}}, \tag{5.9}$$

and

$$\lambda_2(t) = a_2 + b_2 1_{\{t \geq \tau_1\}}, \tag{5.10}$$

where a_1, a_2, b_1, b_2 are positive constants. Notice that in this case the background process X is trivial. Following the "total hazard construction", we first generate two independent unit exponential random variables E_1 and E_2. If both obligors survive up to time t, the total hazards accumulated from 0 to t are $\Lambda_1(t) = a_1 t$ and $\Lambda_2(t) = a_2 t$. If one of obligors has defaulted at time t, w.l.o.g. assuming obligor 2 defaulted at $t_2 < t$, then the total hazard accumulated by obligor 1 up to time t is $\phi_1(t\,|2, t_2) = a_1 t + b_1(t - t_2)$. By (5.5) and (5.7), we construct default times $\hat{\tau}_1$ and $\hat{\tau}_2$ as:

$$\hat{\tau}_1 = \begin{cases} \dfrac{E_1}{a_1}, & \dfrac{E_1}{a_1} \leq \dfrac{E_2}{a_2}, \\[2ex] \dfrac{E_2}{a_2} + \dfrac{1}{a_1 + b_1}\left(E_1 - \dfrac{a_1}{a_2}E_2\right), & \dfrac{E_1}{a_1} > \dfrac{E_2}{a_2}, \end{cases} \tag{5.11}$$

$$\hat{\tau}_2 = \begin{cases} \dfrac{E_2}{a_2}, & \dfrac{E_2}{a_2} \leq \dfrac{E_1}{a_1}, \\[2ex] \dfrac{E_1}{a_1} + \dfrac{1}{a_2 + b_2}\left(E_2 - \dfrac{a_2}{a_1}E_1\right), & \dfrac{E_2}{a_2} > \dfrac{E_1}{a_1}. \end{cases} \tag{5.12}$$

Since E_1 and E_2 are independent unit exponential random variables, it follows that the joint law of $\hat{\tau}_1$ and $\hat{\tau}_2$ is:

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = \frac{b_1 e^{-(a_1+a_2)s} - a_2 e^{-(a_1+b_1)s+(b_1-a_2)t}}{b_1 - a_2}, \tag{5.13}$$

and marginal distributions $\hat{\tau}_1$ and $\hat{\tau}_2$ are:

$$\mathbb{P}(\tau_1 \leq t) = 1 - \frac{a_2 e^{-(a_1+b_1)t} - b_1 e^{-(a_1+a_2)t}}{a_2 - b_1}, \tag{5.14}$$

and

$$\mathbb{P}(\tau_2 \leq t) = 1 - \frac{a_1 e^{-(a_2+b_2)t} - b_2 e^{-(a_2+a_1)t}}{a_1 - b_2}, \tag{5.15}$$

where $s > t > 0$. These results agree with proposition 4.3. These are also obtained in the Markov jump process approach by Herbertsson and Rootzén (2007), which basically is a special case of the "total hazard construction" approach but provides a different way of achieving the results. We introduce this approach in the next chapter.

Chapter 6

Markov jump process approach by Herbertsson and Rootzén

Herbertsson and Rootzén (2007) use a Markov jump process to define default times, and then use Markov process techniques and matrix-analytic methods to derive the price of basket default swap (BDS) [1]. Herbertsson and Rootzén (2007) start with a certain type of default intensities (constant between defaults) and define the default times in a way that they have the specified intensities. In this sense, this model is similar to the "total hazard construction approach", while different from Schönbucher and Schubert's copula model, which starts from the definition of default times. Here we only focus on introducing the model and the methods to derive the joint default probabilities and skip the application to pricing the specific credit derivatives. The model is introduced below.

6.1 Model setup and conclusion

As before, assume there are I obligors in the economy, with default times τ_i and $\mathcal{H}_t$-intensity λ_i. Now we introduce some new notations:

[1] BDS is a credit derivative like CDS but with multiple underlying assets and the spread is contingent on the kth default of the underlying assets.

1. Let $\mathbf{j} = \{j_1, ..., j_k\}, 1 \leq k \leq I$ be the set of defaulted obligors, i.e. $j_1, ..., j_k$ have defaulted.

2. Let $\mathbf{D}_k$ be the set of $\mathbf{j}$ consisting of exactly k defaulted obligors, $1 \leq k \leq I$,

$$\mathbf{D}_k = \{\mathbf{j} : \mid \mathbf{j} \mid = k\},$$

and let $\mathbf{D}_0$ be the state that no obligor has defaulted.

3. Let $\mathbf{D}$ be the union of all possible states,

$$\mathbf{D} = \bigcup_{k=0}^{I} \mathbf{D}_k.$$

Now we have constructed a state space $\mathbf{D}$, with states $\mathbf{j}$ and $\mathbf{D}_0$.

4. Let Δ_i be the set of states in which obligor i has defaulted,

$$\Delta_i = \{\mathbf{j} \in \mathbf{D} : \ j_n = i \ for \ some \ j_n \in \mathbf{j}\}.$$

We are interested in the type of default intensities as (4.1). That is the $\mathcal{H}_t$-intensity λ_i of τ_i has the following form:

$$\lambda_i = (a_i + \sum_{j \neq i} b_{i,j} \mathbf{1}_{\{\tau_j \leq t\}}) \mathbf{1}_{\{\tau_i \geq t\}},$$

where a_i and $b_{i,j}$ are constants and $a_i > 0$. Our goal is to define default times τ_i in a proper way such that they have the specified default intensities (4.1). To achieve this, Herbertsson and Rootzén (2007) make the following assumptions:

Assumption 6.1. *There exits a Markov jump process $(Y_t)_{t \geq 0}$ on the state space $\mathbf{D}$, and the stopping times τ_i are defined as:*

$$\tau_i := \inf \{ \ t > 0 \ : \ Y_t \ \in \ \Delta_i \ \}, \quad i = 1, ..., I. \tag{6.1}$$

Assumption 6.2. *The Markov process Y starts at state $\mathbf{D}_0$ and $\{1, ..., I\}$ is an absorbing state.*

Assumption 6.3. *Y can only jump forward, i.e. once at some state $\mathbf{j} = \{j_1, ..., j_k\} \in \mathbf{D}_k$, Y will either stay at $\mathbf{j}$ or jump forward to a state $\tilde{\mathbf{j}} = (\mathbf{j}, j_{k+1}) \in \mathbf{D}_{k+1}$.*

Assumption 6.4. *Let $\mathbf{Q}$ be the intensity matrix of Y. $\mathbf{Q}$ is specified by:*

$$\mathbf{Q}_{\mathbf{j}\tilde{\mathbf{j}}} = a_{j_{k+1}} + \sum_{i=1}^{k} b_{j_{k+1}, j_i}, \tag{6.2}$$

and $\mathbf{Q}_{\mathbf{j},\mathbf{j}}$ is determined by the requirement that any row sum of an intensity matrix is zero.

Assumption 6.2 and 6.3 are natural, since we presume that at most one obligor can default at a time and once defaulted, an obligor will never recover. Assumption 6.4 guarantees that the default times defined as (6.1) have intensities as (4.1). In fact, we have the following proposition.

Proposition 6.1. *Under Assumption 6.1-6.4, the default times τ_i defined as (6.1) have $\mathcal{H}_t$-intensities (4.1)*

The authors refer to Jacobsen (2006) for the proof, which seems complicated. A simple proof can be found in appendix D.

Notice that the space $\mathbf{D}$ contains 2^I states, i.e. $|D| = 2^I$. If indexing and ordering all the states in $\mathbf{D}$ from 1 to 2^I, we may define a map $\mathcal{M} : \mathbf{D} \longrightarrow \mathbb{R}^{|D|}$ such that for a state $\mathbf{j} \in \mathbf{D}$ with index k_j, $\mathcal{M}(\mathbf{j}) = \mathbf{m_j}$, where $\mathbf{m_j} \in \mathbb{R}^{|D|}$ is a column vector whose entry at k_j is 1 and the other entries are zero. Let $\mathbf{p}(t) = (\mathbb{P}(Y_t = \mathbf{j}))_{\mathbf{j}\in\mathbf{D}}$ be the probability distribution of Y_t. Given an initial distribution $\boldsymbol{\alpha} = (\mathcal{M}(\mathbf{D}_0))'$ from Markov theory we have:

$$\mathbf{p}(t) = \boldsymbol{\alpha} e^{\mathbf{Q}t}, \tag{6.3}$$

and

$$\mathbb{P}(Y_t = \mathbf{j}) = \boldsymbol{\alpha} e^{\mathbf{Q}t}\mathbf{m_j}, \tag{6.4}$$

where $e^{\mathbf{Q}t}$ is the matrix exponential which has a closed form expression in terms of the eigenvalue decomposition of $\mathbf{Q}$, but may need numerical computation when the eigenvalue matrices are ill-conditioned.

From Assumption 6.1, each state $\mathbf{j} \in \mathbf{D}$ represents the default statuses of all obligors at a time t. Thus the joint law of default times can be derived through (6.3) and (6.4).

6.2 Comparison with the "total hazard construction" approach

Since both the "total hazard construction" approach and the Markov jump process approach can deal with the same type of default intensities as in (4.1), we are interested in their relationship. In fact, the "total hazard construction" approach implies the Markov jump process approach. This is illustrated below.

Recall that, in section 6.1 we assume the existence of the Markov process $(Y_t)_{t\geq 0}$ and define the default times τ_i, $i = 1, ..., I$ through it. Consequently, the joint law of the default times is determined by the law of Y. However, in the "total hazard construction" approach, the default times are internally determined through the pre-specified default intensities. If we start from the default times that have $\mathcal{H}_t$-intensities (4.1), is it possible to define a Markov process such that it has the same law as Y? The answer is positive. In fact, we can proceed as follows. Define a Markov process $(Y'_t)_{t\geq 0}$ such that it satisfies assumption 6.1-6.3. Notice that assumption 6.1-6.3 actually allow Y' and $\boldsymbol{\tau} = (\tau_1, ..., \tau_I)$ to define each other mutually. By virtue of proposition 6.1, the transition matrix of Y' must be the same as $\mathbf{Q}$, since τ_i has intensity (4.1). Thus, Y' has the same law as Y. This means the law of $\boldsymbol{\tau} = (\tau_1, ..., \tau_I)$ derived from the Markov jump approach must be the same as that derived from the "total hazard construction" approach. To make it more clear, it is useful to recall the way in which a Markov chain can be simulated from its intensity matrix.

Consider a Markov chain on a space with three states and the intensity matrix given by:

$$\mathbf{Q} = \begin{pmatrix} -a_1 & a_{12} & a_{13} \\ a_{21} & -a_2 & a_{23} \\ a_{31} & a_{32} & -a_3 \end{pmatrix}. \tag{6.5}$$

Then the markov chain can be simulated as follows:

1. Fix an initial state of the Markov chain, say state 1.
2. Draw a unit exponential random variable η_1 and let η_1/a_1 be the amount of time that the chain stay in state 1.
3. The chain jumps to either state 2 with probability a_{12}/a_1 or state 3 with a_{13}/a_1.
4. If the new state is 3, draw another unit exponential random variable η_2 and let η_2/a_3 be the amount of time that the chain stay in state 3.
5. The chain jumps to either state 2 with probability a_{32}/a_3 or state 1 with probability a_{31}/a_3.
6. Continue the simulation until the chain hits an absorbing state or the elapsed time exceeds the desired time horizon.

We can simulate default times $\boldsymbol{\tau} = (\tau_1, ..., \tau_I)$ by simulating the Markov chain Y_t. In the example with two firms, the state space consists of 4 states: $\mathbf{D_0}$, $\{1\}$, $\{2\}$, $\{1,2\}$, standing for the defaulted firms. The intensity matrix $\mathbf{Q}$ is given by:

$$\mathbf{Q} = \begin{pmatrix} -(a_1+a_2) & a_1 & a_2 & 0 \\ 0 & -(a_2+b_2) & 0 & a_2+b_2 \\ 0 & 0 & -(a_1+b_1) & a_1+b_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.6}$$

To simulate the default times τ_1 and τ_2, we proceed as follows.

1. Draw a unit exponential random variable η_1. Let $\eta_1/(a_1+a_2)$ be the first default time, i.e., $\tau_{min} := \tau_1 \wedge \tau_2 = \eta_1/(a_1+a_2)$.

2. Draw a Bernoulli random variable X_1 independent of η_1, with $\mathbb{P}(X_1 = 0) = a_1/(a_1+a_2)$ and $\mathbb{P}(X_1 = 1) = a_2/(a_1+a_2)$, where $X_1 = 0$ means $\tau_{min} = \tau_1$ and $X_1 = 1$ means $\tau_{min} = \tau_2$.

3. Draw another unit exponential random variable η_2. If $\tau_{min} = \tau_1$, then let $\tau_2 = \eta_1/(a_1+a_2)+\eta_2/(a_2+b_2)$; if $\tau_{min} = \tau_2$, then let $\tau_1 = \eta_1/(a_1+a_2)+\eta_2/(a_2+b_2)$.

Equivalently, τ_1 and τ_2 can be represented by:

$$\begin{cases} \tau_1 = \dfrac{\eta_1}{a_1+a_2} + X_1 \dfrac{\eta_2}{a_1+b_1} & , \\ \\ \tau_2 = \dfrac{\eta_1}{a_1+a_2} + (1-X_1)\dfrac{\eta_2}{a_2+b_2} & . \end{cases} \tag{6.7}$$

By basic probability arguments, we can find the joint law of τ_1 and τ_2.

Proposition 6.2. *For $s > t > 0$, the joint law of τ_1 and τ_2 is given by:*

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = \frac{b_1 e^{-(a_1+a_2)s} - a_2 e^{-(a_1+b_1)s+(b_1-a_2)t}}{b_1 - a_2}. \tag{6.8}$$

No surprise, this result is the same as (5.13), which is derived by the "total hazard construction" approach. As stated before, the "total hazard construction" approach implies the Markov jump process approach. However, in the case of more firms ($I > 2$), the explicit joint default probability given by the "total hazard construction" approach is likely to be inconveniently long, therefore it can only be evaluated numerically by Monte-Carlo simulation. On the other hand, the Markov jump process approach gives explicit joint default probability through matrix exponential $e^{\mathbf{Q}t}$, but it may need some numerical computation effort as well. It should also be noticed that the Markov jump process approach only deals with default intensities that are deterministic and homogeneous in time like ones in (4.1), while the "total hazard construction" approach are still available for more general intensity forms. If the default intensities in (4.1) are of the interest, one can choose the approach that involves in less numerical effort but higher accuracy.

Chapter 7

Discussion and Conclusion

We have presented three approaches of the reduced form models that can capture the contagion effect: the copula approach, the "total hazard construction" approach and the Markov jump approach.

The copula approach allows one to separate modeling of default correlations via copula function from modeling the dynamics of individual pseudo default intensities. However, using copula function means one gives up the freedom of choosing a type of favorite default intensities like in (4.1), since the default intensities now depend on the choice of copula function. Another difficulty lies in the choice of the copula function, which may be based on empirical results.

The "total hazard construction" approach uses an algorithm to construct a set of default times that admit the specified default intensities. It allows a large varieties of default intensities and perfectly solves the looping default problem (4.1). The Monte-Carlo simulation is natural to this approach, since the "total hazard construction" algorithm appeals to it. The drawback of this approach is that it is limited to Monte-Carlo simulation even in the case of three obligors.

The Markov jump process approach is implied by the "total hazard construction" approach. This approach defines default times through a Markov jump process. The joint law of default times is derived from the law of the the Markov jump process. The intensity matrix of the Markov jump process and the default intensities have certain link, which is illustrated in proposition 6.1. This link allows us to specify the intensity matrix such that the desired default intensities as in (4.1) are obtained. This approach provides analytic joint law of default times, which, although it may also need some computation effort, provides another way of numerical implementation rather than Monte-Carlo method. Appendix A includes notations and abbreviations that are used in this essay. Appendix B,C and D provide proofs for some of the propositions in this essay.

Appendix A

Abbreviations and Notations

- a.s.: almost surely.
- i.i.d.: identically independent distributed.
- w.l.o.g.: without loss of generality.
- If B is a matrix, then B' means B transpose.
- If E is a countable set, then $|E|$ means the number of elements in it.
- If $\mathbf{x} = (x_1, ..., x_I)$ and $f(\mathbf{x}) : \mathbb{R}^I \to \mathbb{R}$, then we use the following notation if we replace the $i-$th component of $\mathbf{x}$ with z:

$$f(\mathbf{x}_{-i}, z) := f(x_1, ..., x_{i-1}, z, x_{i+1}, ..., x_I).$$

Appendix B

Proof of Theorem 3.1

First given the filtration $\tilde{\mathcal{H}}_t^i$,

$$\begin{aligned}
\mathbb{P}(\tau_i > T \mid \tilde{\mathcal{H}}_t^i) &= \mathbb{P}(\gamma_i(T) > U_i \mid \tilde{\mathcal{H}}_t^i) \\
&= \mathbb{P}(\gamma_i(T) > U_i \mid \mathcal{G}_\infty \vee \{\tau_i > t\}) \\
&= \frac{\mathbb{P}(\gamma_i(T) > U_i \mid \mathcal{G}_\infty)}{\mathbb{P}(\gamma_i(t) > U_i \mid \mathcal{G}_\infty)} \\
&= \frac{\gamma_i(T)}{\gamma_i(t)},
\end{aligned}$$

where the last equality comes from the fact that U_i is independent of $\mathcal{G}_\infty$ and $\gamma_i(t)$ is $\mathcal{G}_\infty$ measurable.

Next, given the filtration $\mathcal{H}_t^i$ and by the law of iterated expectation, we have:

$$\begin{aligned}
\mathbb{P}(\tau_i > T \mid \mathcal{H}_t^i) &= \mathbf{E}\left[\mathbf{1}_{\{\tau_i > T\}} \mid \mathcal{H}_t^i\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[\mathbf{1}_{\{\tau_i > T\}} \mid \tilde{\mathcal{H}}_t^i\right] \mid \mathcal{H}_t^i\right] \\
&= \mathbf{E}\left[\mathbb{P}(\tau_i > T \mid \tilde{\mathcal{H}}_t^i) \mid \mathcal{H}_t^i\right] \\
&= \mathbf{E}\left[\frac{\gamma_i(T)}{\gamma_i(t)} \mid \mathcal{H}_t^i\right],
\end{aligned}$$

where the second equality is from $\mathcal{H}_t^i \subset \tilde{\mathcal{H}}_t^i$.

Given the filtration $\mathcal{H}_t^i$ and from (2.4) we have:

$$\begin{aligned}
h_i(t) &= \lim_{\Delta t \searrow 0} -\frac{1}{\Delta t}\mathbb{P}(\tau_i > t + \Delta t \mid \mathcal{H}_t^i) \\
&= -\frac{\partial}{\partial T}\mathbb{P}(\tau_i > T \mid \mathcal{H}_t^i) \mid_{T=t} \\
&= -\frac{\partial}{\partial T}\mathbf{E}\left[\frac{\gamma_i(T)}{\gamma_i(t)} \mid \mathcal{H}_t^i\right] \mid_{T=t} \\
&= \mathbf{E}\left[-\frac{\partial}{\partial T}\frac{\gamma_i(T)}{\gamma_i(t)} \mid \mathcal{H}_t^i\right] \mid_{T=t} \\
&= \mathbf{E}[\lambda_i(t) \mid \mathcal{H}_t^i] \\
&= \lambda_i(t),
\end{aligned}$$

where the last equality holds because $\lambda_i(t)$ is adapted to $\mathcal{H}_t^i$. End of proof.

Appendix C

Proof of Theorem 3.2

As above, we first consider the cases under filtration $\tilde{\mathcal{H}}_t$. Suppose no obligors have defaulted at time t, then we have:

$$\begin{aligned}\mathbb{P}(\boldsymbol{\tau} > \mathbf{T} \mid \tilde{\mathcal{H}}_t) &= \mathbb{P}(\gamma_i(T_i) > U_i, i = 1, ..., I \mid \tilde{\mathcal{H}}_t) \\ &= \mathbb{P}(\gamma_i(T) > U_i, i = 1, ..., I \mid \mathcal{G}_\infty \vee \{\tau_i > t, i = 1, ..., I\}).\end{aligned}$$

Applying Bayes' rule, we get:

$$\mathbb{P}(\boldsymbol{\tau} > \mathbf{T} \mid \tilde{\mathcal{H}}_t) = \frac{\mathbb{P}(\gamma_i(T_i) > U_i, i = 1, ..., I \mid \mathcal{G}_\infty)}{\mathbb{P}(\gamma_i(t) > U_i, i = 1, ...I \mid \mathcal{G}_\infty)}.$$

Since $\gamma_i(t)$ are $\mathcal{G}_\infty$ measurable and U_i are independent of $\mathcal{G}_\infty$, $i = 1, ..., I$, by plugging in the copula function C we get:

$$\mathbb{P}(\boldsymbol{\tau} > \mathbf{T} \mid \tilde{\mathcal{H}}_t) = \frac{C(\gamma_1(T_1), ..., \gamma_I(T_I))}{C(\gamma_1(t), ..., \gamma_I(t))}.$$

From $\mathcal{H}_t \subset \tilde{\mathcal{H}}_t$ and the law of iterated expectation, we have:

$$\begin{aligned}\mathbb{P}(\boldsymbol{\tau} > \mathbf{T} \mid \mathcal{H}_t) &= \mathbf{E}\left[\mathbf{1}_{\{\boldsymbol{\tau} > \mathbf{T}\}} \mid \mathcal{H}_t\right] \\ &= \mathbf{E}\left[\mathbf{E}\left[\mathbf{1}_{\{\boldsymbol{\tau} > \mathbf{T}\}} \mid \tilde{\mathcal{H}}_t\right] \mid \mathcal{H}_t\right] \\ &= \mathbf{E}\left[\mathbb{P}(\boldsymbol{\tau} > \mathbf{T} \mid \tilde{\mathcal{H}}_t) \mid \mathcal{H}_t\right] \\ &= \frac{\mathbf{E}\left[C(\gamma_1(T_1), ..., \gamma_I(T_I)) \mid \mathcal{H}_t\right]}{C(\gamma_1(t), ..., \gamma_I(t))},\end{aligned}$$

which gives (3.7). If we let $T_i = T > t$ and $T_j = t, for j \neq i$ and apply (3.7) we get:

$$\mathbb{P}(\tau_i > T \mid \tilde{\mathcal{H}}_t) = \frac{\mathbf{E}\left[C(\boldsymbol{\gamma}_{-i}(t), \gamma_i(T)) \mid \mathcal{H}_t\right]}{C(\gamma_1(t), ..., \gamma_I(t))}.$$

Consequently the default intensity $h_i(t)$ given $\mathcal{H}_t$ is:

$$\begin{aligned} h_i(t) &= -\frac{\partial}{\partial T}\mathbb{P}(\tau_i > T \mid \mathcal{H}_t) \mid_{T=t} \\ &= -\frac{\mathbf{E}\left[\frac{\partial}{\partial T}C(\boldsymbol{\gamma}_{-i}(t), \gamma_i(T)) \mid \mathcal{H}_t\right]}{C(\gamma_1(t), ..., \gamma_I(t))} \mid_{T=t} \\ &= \lambda_i(t)\gamma_i(t)\frac{\frac{\partial}{\partial x_i}C(\boldsymbol{\gamma}(t))}{C(\boldsymbol{\gamma}(t))} \\ &= \lambda_i(t)\gamma_i(t)\frac{\partial}{\partial x_i}\ln C(\boldsymbol{\gamma}(t)), \end{aligned}$$

which gives (3.8).

Recall:

$$\gamma_i(t) = \exp(-\int_0^t \lambda_i(s)ds),$$

then applying *Itô*'s Lemma to $\gamma_i(t)$ and $\frac{\frac{\partial}{\partial x_i}C(\boldsymbol{\gamma}(t))}{C(\boldsymbol{\gamma}(t))}$ gives:

$$d\gamma_i(t) = -\gamma_i\lambda_i dt,$$

and

$$-d\frac{C_{x_i}(\boldsymbol{\gamma}(t))}{C(\boldsymbol{\gamma}(t))} = \sum_{j=1}^{I}\frac{C_{x_i}Cx_j - C_{x_i x_j}C}{C^2}\gamma_j\lambda_j dt.$$

Consequently,

$$\begin{aligned} dh_i &= \gamma_i\frac{C_{x_i}}{C}d\lambda_i + \lambda_i\frac{C_{x_i}}{C}d\gamma_i + \lambda_i\gamma_i d\frac{C_{x_i}}{C} \\ &= \frac{C_{x_i}}{C}\gamma_i\lambda_i\left[\left(\frac{d\lambda_i}{\lambda_i} - \lambda_i dt\right) - \sum_{j=1}^{I}\left(\frac{C_{x_i x_j}}{C_{x_i}} - \frac{C_{x_j}}{C}\right)\gamma_j\lambda_j dt\right], \end{aligned}$$

which gives (3.9).

Now let us consider the case of $k, (1 \leq k \leq I)$ defaulted obligors at time t, and w.l.o.g assuming they are the first k obligors with default times $t_n, 1 \leq n \leq k$ respectively. Let $\boldsymbol{\tau} = (\tau_{k+1}, ..., \tau_I)$ and $\boldsymbol{T} = (T_{k+1}, ...T_I)$, where $T_i \geq t$ for $k < i \leq I$. Then given filtration $\tilde{\mathcal{H}}_t$ we have:

$$\begin{aligned}
\mathbb{P}(\boldsymbol{\tau} > \mathbf{T} \mid \tilde{\mathcal{H}}_t) &= \mathbb{P}(\gamma_i(T_i) > U_i, i = k+1, ..., I \mid \tilde{\mathcal{H}}_t) \\
&= \mathbb{P}(\gamma_i(T) > U_i, i = k+1, ..., I \mid \mathcal{G}_\infty \vee \\
&\quad \{\tau_n = t_n, n = 1, ..., k;\ \tau_i > t, i = k+1, ..., I\}) \\
&= \frac{\mathbb{P}(\gamma_n(t_n) = U_n, n = 1, ..., k, \gamma_i(T_i) > U_i, i = k+1, ..., I \mid \mathcal{G}_\infty)}{\mathbb{P}(\gamma_n(t_n) = U_n, n = 1, ..., k, \gamma_i(t) > U_i, i = k+1, ...I \mid \mathcal{G}_\infty)} \\
&= \frac{C_{x_1 ... x_k}(\gamma_1(t_1), ..., \gamma_k(t_k), \gamma_{k+1}(T_{k+1}), ..., \gamma_I(T_I))}{C_{x_1 ... x_k}(\gamma_1(t_1), ..., \gamma_k(t_k), \gamma_{k+1}(t), ..., \gamma_I(t))},
\end{aligned}$$

and again applying the law of iterated expectation we get:

$$\begin{aligned}
&\mathbb{P}(\boldsymbol{\tau} > \mathbf{T} \mid \mathcal{H}_t) \\
&= \frac{\mathbf{E}\left[\, C_{x_1 ... x_k}(\gamma_1(t_1), ..., \gamma_k(t_k), \gamma_{k+1}(T_{k+1}), ..., \gamma_I(T_I)) \mid \mathcal{H}_t\right]}{C_{x_1 ... x_k}(\gamma_1(t_1), ..., \gamma_k(t_k), \gamma_{k+1}(t), ..., \gamma_I(t))},
\end{aligned}$$

which gives (3.10).

we can obtain (3.11) and (3.12) by almost the same arguments as in the proof of (3.8) and (3.9), except replacing the function $C(\gamma_1(t), ..., \gamma_I(t))$ with $C_{x_1 ... x_k}(\gamma_1(t_1), ..., \gamma_k(t_k), \gamma_{k+1}(t), ..., \gamma_I(t))$.

Appendix D

Proof of Proposition 6.1

First, it is convenient to make $\mathbf{Q}$ an upper-triangle matrix. This can be done by ordering the states in $\mathbf{D}$ in the following way: start from $\mathbf{D}_0$, then take states in $\mathbf{D}_1$ in an arbitrary order, then take states in $\mathbf{D}_2$ in an arbitrary order, and so on. Suppose at time t, the process Y is at state $\mathbf{j} \in \mathbf{D}_k$. Let $\boldsymbol{\alpha}_\mathbf{j} \in \mathbb{R}^{|\mathbf{D}|}$ be a column vector where the the entry is 1 at position $\mathbf{j}$ and 0 anywhere else. Here, $\boldsymbol{\alpha}_\mathbf{j}$ stands for $\mathbf{j}$ in $\mathbb{R}^{|\mathbf{D}|}$. From the property of Markov jump process, the probability distribution of Y at some later time $T > t$ is given by:

$$\boldsymbol{\alpha}_\mathbf{j}' e^{\mathbf{Q}(T-t)}.$$

Suppose obligor i is still survival at time t, then the survival probability of obligor i at T conditional on the filtration $\mathcal{H}_t$ is given by:

$$\mathbb{P}(\tau_i > T \mid \mathcal{H}_t) = \boldsymbol{\alpha}_\mathbf{j}' e^{\mathbf{Q}(T-t)} \boldsymbol{\beta}_i,$$

where $\boldsymbol{\beta}_i \in \mathbb{R}^{|\mathbf{D}|}$ is a column vector and its entries at positions that contain obligor i are zero and the others are 1. This means we sum up all the probabilities at states where obligor i are survival. Hence, the $\mathcal{H}_t$-intensity λ_i of τ_i is given by:

$$\begin{aligned} h_i(t) &= -\frac{\partial}{\partial T}\mathbb{P}(\tau_i > T \mid \mathcal{H}_t) \mid_{T=t} \\ &= -\frac{\partial}{\partial T}\boldsymbol{\alpha}_{\mathbf{j}}^{'} e^{\mathbf{Q}(T-t)}\boldsymbol{\beta}_i \mid_{T=t} \\ &= -\boldsymbol{\alpha}_{\mathbf{j}}^{'}\mathbf{Q}\boldsymbol{\beta}_i \\ &= \mathbf{Q}_{\mathbf{j},\tilde{\mathbf{j}}} \\ &= a_i + \sum_{j\in\mathbf{j}} b_{i,j}, \end{aligned}$$

where $\tilde{\mathbf{j}} = (\mathbf{j}, i)$ and we use the fact that the sum of entries of $\mathbf{Q}$ within each row is zero. This agrees with (4.1).

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