Hedging in an international environment *

MSc essay for
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In this essay I discuss several aspects of hedging risks evolving for individual and institutional investors acting in an international environment. Typically the best hedging scheme decision comprises a trade-off between the minimization of risk for the investors and the sometimes substantial costs of achieving a risk reduction.

I present different mathematical models on how to find the optimal hedge strategy in order to minimize portfolio risk as well as currency risk exposure. I also disuss their advantages as well as problems regarding real investment applications.

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1 Introduction

A variety of financial transactions or investments carry some sort of risk. Issuing a call option, for example, on the S&P 500 index creates the risk for the writer of the contract that the S&P 500 will rise before the expiration date. The owner of the option would then exercise his right to buy some of the underlying asset below market price and thus create a loss for the option writer. Owning a stock portfolio carries portfolio risk, the possibility of decreasing stock prices. It very likely also contains currency risk, the possibility that unexpected exchange rate changes affect the performance of the portfolio. Expecting capital inflows from overseas activities or investments exposes investors not only to the always present default risk but also to the risk of unfavorable changes in the exchange rate.

Reducing the risk of a portfolio or investment return is generally referred to as hedging. Portfolio Insurance is any financial activity undergone in order to ensure that the value of a portfolio will not fall below a certain level, whereas Portfolio Immunization describes the process of making a portfolio relatively insensitive towards interest rate changes. Under the so-called Value At Risk approach investors are prespecifying a loss that will not be exceeded at a certain confidence level. It is well-known in portfolio management that domestic diversification, investing amongst different industries, and international diversification, investing into different foreign assets or currencies, substantially reduces the risk for investors.

Several financial instruments can be used to hedge against all the different sorts of risk. Traditionally holding a purely domestic stock portfolio can be hedged by purchasing a certain number of put options on the specific stock owned. These would then guarantee that the payoff of the portfolio will not fall below a certain level. Depending on the certainty of the planned time horizon for holding the assets, plain vanilla European or American style options will be the easiest to use for pure hedging purposes. A planned stock purchase in the future can be hedged by buying call options. The short-selling of shares indicates the investor's strong expectations of falling stock prices; this transaction can also be hedged by purchasing call options on the underlying stock at the same time. The issuance of floating rate notes with periodically adjusting interest rates can be hedged by simultaneously selling a number of interest rate CAPS and vice versa.

In practice different combinations of short and long positions in different kinds of options on the same primary security can be used to guarantee a certain payoff, put a barrier on possible losses or limit the anticipated probability of a loss on the investment.

Similarly, the issuance of any kind of option can be hedged by purchasing or (short-) selling a certain amount of the underlying asset. A financial institution that sells call options on shares can typically hedge this transaction by purchasing a number of these shares. The issuance of, say, put options on currency futures can be hedged taking a short futures position in order to eliminate the negative effects from falling futures prices. Typically, the risks evolving from writing options of any kind can be hedged by trading a certain number the appropriate underlying asset.

In open economies, investors today are unavoidably confronted with currency risk, due to the effect of fluctuating exchange rates on the return on their investments or transactions. Changing exchange rates have a direct impact on the measured performance of an international investment by affecting the result of the translation of the foreign currency return into domestic currency values. Via reinforcement unexpected currency appreciation or depreciation also affects the foreign currency performance measures, often in the opposite direction, and can even have an important impact on the return for purely domestically operating firms.

In globalizing money, capital, and goods markets it is not always clear, to what extend investors are in fact exposed to currency risk. However, in order to hedge against negative effects from unexpected exchange rate changes, forward and futures contracts are probably the most common financial instruments used. Experienced market participants also exploit several advantages by using swaps, swaptions or collateral to optimize their hedging strategy.

In section 2, I present the most common mathematical model for hedging the risks evolving from issuing European and American style options as well as how to use the latter to hedge portfolios of primary assets. This section is mainly based on papers by Huang, Subrahmanyam, and Yu [10], Broadie and Detemple [4], and Carr, Jarrow, and Myneni [5]. Section 3 captures the problematic of immunizing investors' portfolios or transactions against currency risk. I present a curreny hedge model by Briys and Solnik [3]. As the developed theoretical solutions contain several difficulties for usage in practice a more pragmatic approach is proposed by Bos, Mahieu, and Van Dijk [2].

2 Hedging written options

A financial institution that sells options or other derivatives is faced with the question of how to manage the consequent risks. One possible strategy is to do nothing. If the institution is selling for instance a European call option on a non-dividend paying stock with exercise price K and maturity T for the price of C_0, its instantaneous gain will be just C_0. As it is assumed that C_0 will be reinvested at the risk-free interest rate, $r(t)$, let C_T denote the gain at T; $C_T = C_0 e^{\int_0^T r(t)dt}$. If the stock price follows the price process S_t, the accumulated payoff at maturity however will be the return from the risk-free reinvestment of the initial gain, C_T, plus $min(0, K - S_T)$. This is also called a naked position.

Alternatively, the institution could take a covered position: it purchases one share as soon as the option is sold. If $K < S_T$ and the option is exercised, the payoff for the institution at T will be $(K - S_0)$ plus the gain from the reinvested C_0. As here it is assumed that the cost for the hedge is smaller than the gain from selling the call option in the first place, the payoff at time T would be positive. However, if the price of the share drops, the position's value at time T will result in $C_T + (S_T - S_0)$. The initial gain from selling the option, C_T, does not necessarily compensate for the loss on the stock market, $S_T - S_0$.

Another hedging idea to be proposed is the so-called stop-loss strategy: This simply involves the purchase of shares as soon as its price rises above X and the sale of them as soon as it falls below. The cost of writing the call option at time zero would be $max(S_0 - K, 0)$. With this strategy one can produce a perfect hedge, such that the payoff at all times will equal the gain from selling the call minus the cost of its hedge, $min(C_T, C_T - (S_0 - K))$. One could earn a riskfree profit by writing options and hedging them. This is not true mainly for two reasons: markets in real life are not complete, specifically they are not frictionless and transaction costs exist.

A problem that will occur several times while analyzing the shortcomings of proposed analytical hedging possibilities is that the time measure is indeed not frictionless. We can only observe prices and other important financial measures at discrete times and we can only trade in discrete time. This creates the problem that analytically optimal solutions may not be feasible. In the above stop-loss strategy it is only possible to observe that the stock price is moving outside the boundary when it is slightly above or below X. Hence, the investor can only buy at $X + \delta$ and sell at $X - \delta$. The strategy's costs increase.

Another problem is the existence of transaction costs. Every transaction in the real world comes with costs: trading fees for the dealer, costs for the sourcing of information, taxes for the government... In the ideal world of the above strategy, transaction costs do not exist, hence there is no limit on how many trades may be undertaken. In the real world this strategy could very easily create a scenario in which due to unlimited stock trading the hedging costs exceed the gain from selling the option in first place.

In this whole section I will assume that markets are complete. $(\Omega, \mathcal{F}, \mathcal{P})$ is the probability space that captures all possible values to describe the economy. Let $\tilde{W}_t$ define a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$ and $\mathcal{F}_t$ its filtration. $\mathcal{F}_t$ hence reflects the information available at time t. The economy contains a riskless security with the well-known price process

$$dB_t = r_t B_t dt \quad \forall t \in [0, T]$$

where B_0 is given and r_t as the riskfree interest rate is bounded, strictly positive and $\mathcal{F}_t$ - measurable. For simplification of the notation define the discount factor $R_{s,t} \equiv exp(-\int_s^t r_v dv)$.

The time subscript t will mainly be omited for security and derivative prices or interest rates.

2.1 Δ - Hedging

One concern about hedging options with stocks is the involved cash to purchase those shares. If as before one buys or sells one share per option when writing a number of options, the required cash for the purchase to hedge calls can be substantial, as would be the number of shares to be shorted when hedging a set of put options.

The so-called Δ - hedging strategy now takes advantage of the fact that as prices of primary assets most usually exceed option prices, so do their absolute changes. To hedge a number of written options it might thus not be necessary to trade the same number or value of the underlying security, but instead only a fraction of the value of options to be hedged.

The delta of an option is defined as the rate of change of the option's price with respect to the price of the underlying asset. Hence, if the primary security has a spot price S and the value of the option follows the price process V, the delta of the option will be $\Delta = \partial V / \partial S$.

It is proposed that a hedge ratio of Δ is optimal: purchasing the primary security with a face value of $\$\Delta N$ to hedge the issuance of call options valuing $\$N$ or similarly (short-)selling $\$\Delta N$ in assets when writing put options in a value of $\$N$ will create an ideal hedging position in the sense that the risk is minimized.

To derive the optimality of the Δ-hedging strategy the following assumptions must be met: The spot price of the underlying asset follows a geometric Brownian motion process

$$dS_t = S_t \mu_t dt + S_t \sigma_t d\tilde{W}_t,$$

where μ_t is the trend. The risk-free rate, r_t, and the volatility, σ_t, of the underlying asset's price are non-stochastic, markets are frictionless and continuous trading is possible.

We can set up a portfolio by writing one option which will be sold at V_0 and buying or selling h units of the underlying security with the price process S. Π denotes the value of the portfolio, which shall be self-financing for all t.

The value of the portfolio will thus be

$$\Pi = V(S,t) + hS(t)$$

and using Ito calculus it will change according to

$$\begin{aligned} d\Pi &= dV + hdS \\ &= [V_t + (V_S + h)\mu S + \frac{1}{2} V_{SS} S^2 \sigma^2] dt + (V_S + h)\sigma S d\tilde{W} \\ &= r\Pi dt \end{aligned}$$

Since we want the portfolio to be risk-free the last equality has to hold, and thus $(V_S + h) = 0$.

Hence, by choosing $h = \Delta$ the portfolio becomes riskless, as desired.

A position in which the issuance of options is hedged by appropriately trading shares as described above is called Δ-neutralized. A positive delta suggests to take a short position of $\$\Delta$ in the underlying asset, if Δ is negative one should take a long position of $\$\Delta$ on the primary security's market.

Now that h is optimally determined to be equal to $\partial V / \partial S$ it is of interest to explicitly determine the delta of the option.

2.1.1 European Style Options

For a European call option on a non-dividend paying stock that follows the geometric brownian motion process we know that V has to equal the rational price C_t of the option which satisfies:

$$C_t = S_t N(d_1) - e^{-r(T-t)} K N(d_2),$$

where

$$d_1 = d_1(S,K,r,T-t) = \frac{ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{1}$$

and

$$d_2 d_2(S,K,r,T-t) = d_1(S,K,r,T-t) - \sigma\sqrt{T-t}. \tag{2}$$

K is the exercise price of the option, T characterizes the time of maturity and N(.) is the standard normal distribution function.

Hence the delta of a European call option on a non-dividend paying stock must equal

$$\begin{aligned}
\frac{\partial C}{\partial S} &= N(d_1) + \frac{Sn(\frac{ln(\frac{S_t}{K})+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}})}{S\sigma\sqrt{T-t}} - \frac{Ke^{-r(T-t)}n(\frac{ln(\frac{S_t}{K})+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}})}{S\sigma\sqrt{T-t}} \\
&= N(d_1) + \frac{Sn(d_1) - e^{-r(T-t)}K\frac{1}{2\pi}e^{-\frac{d_1^2 - 2d_1\sigma\sqrt{T-t}+\sigma^2(T-t)}{2}}}{S\sigma\sqrt{T-t}} \\
&= N(d_1) + \frac{Sn(d_1) - e^{-r(T-t)}K\frac{S}{K}e^{r(T-t)}}{S\sigma\sqrt{T-t}} \\
\Delta_c &= N(\frac{ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}) = N(d_1).
\end{aligned}$$

Using put-call-parity the delta of a European put on the same stock must equal

$$\Delta_p = \frac{\partial P}{\partial S} = N(d_1) - 1.$$

For the European style call and put options on a stock that pays a continuous dividend at a constant rate α the deltas are

$$
\begin{aligned}
\Delta_c &= e^{-\alpha(T-t)} N(d_1) \\
\Delta_p &= e^{-\alpha(T-t)} (N(d_1) - 1),
\end{aligned}
$$

respectively.

For European style currency options we can obtain similar results. If r denotes the risk-free interest rate on the foreign currency, the Δ - hedging strategy yields an optimal hedge ratio of

$$\Delta_c = e^{r^*(T-t)} N(d_1)$$

for the call option and

$$\Delta_p = e^{r^*(T-t)} (N(d_1) - 1)$$

for the European put.

2.1.2 American Style Options

In the environment of the GBMP-model, the General Brownian Motion Process model, the primary security under the equivalent martingale measure follows the price process:

$$dS_t = S_t(r_t - \alpha_t)dt + S_t \sigma_t d\tilde{Z}_t \tag{3}$$

where r_t being the riskfree rate, α_t a non-negative, non-decreasing ratio and σ_t the volatility of the asset's price are bounded and $\mathcal{F}_t$ - measruable processes. The volatility is almost surely bounded away from zero, $\tilde{Z}_t$ denotes the appropriate Brownian motion process under the equivalent martingale measure Q evolving from $\tilde{W}_t$ under the original measure $\mathcal{P}$.

This generel setting enables us to price a range of American options by considering for instance continuous dividends on stocks, transaction costs or opportunity costs involved in holding foreign currency options.

The standard American option on a non-dividend paying stock with no specific carrying costs is a special case of the GBMP-model where α is equal to zero. For a contingent claim on a dividend-paying stock, α is simply the constant dividend ratio. Holding an asset with a constant proportional cost of carry, b, is mathematically equivalent to receiving a constant proportional dividend $\alpha = r - b$. We can also apply this model to commodity options with a constant convenience yield γ. In this case $b = r - \gamma$. Analyzing options on foreign currencies, the cost of carrying the foreign currency is the domestic riskless interest rate and b is the domestic interest rate less the foreign one.

American options differ from European ones by allowing the option holder to exercise the claim prior to maturity. I will refer to the European option of the same kind as the American claim, with equal strike price and maturity as the European equivalent. As in the following the American option will be priced using the early exercise representation, the resulting deltas in this section shall only be applied to American options to which a closed form solution exists for the European equivalent. If V denotes the price for an American Option on a security that follows the above price process, the Black Scholes PDE becomes

$$\frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + bS\frac{\partial V}{\partial S} - rV + \frac{\partial V}{\partial t} = 0 \quad \forall (S,t) \in \mathcal{C}^{2,1}((B_t,\infty),[0,T])$$

with a set of boundary conditions

$$\begin{aligned} C(S,T) &= max\{0, S-K\} \\ \lim_{S\to 0} C(S,t) &= 0 \\ \lim_{S\to B_t} C(S,t) &= B_t - K \quad \text{value matching condition} \\ \lim_{S\to B_t} \frac{\partial C}{\partial S} &= 1 \quad \text{supercontact condition} \end{aligned} \tag{4}$$

for American call options, where $V \equiv C$ and similarly

$$\begin{aligned} P(S,T) &= max\{0, K-S\} \\ \lim_{S\to \infty} P(S,t) &= 0 \\ \lim_{S\to B_t} P(S,t) &= K - B_t \quad \text{value matching condition} \\ \lim_{S\to B_t} \frac{\partial P}{\partial S} &= -1 \quad \text{supercontact condition} \end{aligned} \tag{5}$$

for the American Style Put with $V \equiv P$, where B_t denotes the yet to be determined early exercise boundary of the option.

If τ represents the optimal exercise time and Y_t is the payoff of the American option if exercised at time t, its value is given by

$$V_t = \sup_{\tau \in S_{t,T}} \{E_Q[R_{t,T} Y_\tau) | \mathcal{F}_t]\}$$

The value of an American contingent claim, and thus its rational price, can be given in the early exercise representation as the price of the European equivalent plus an early exercise premium, the extra value for being able to exercise before T.

It is proven in the appendix that the American call and put option values have the early exercise representations

$$C_t = c_t + \int_t^T [\alpha S_t e^{-\alpha(s-t)} N(\frac{ln(\frac{S_t}{B_s}) + (b + \frac{\sigma^2}{2})(s-t)}{\sigma\sqrt{s-t}}) \\ - rKe^{-r(s-t)} N(\frac{ln(\frac{S_t}{B_s}) + (b - \frac{\sigma^2}{2})(s-t)}{\sigma\sqrt{s-t}})]ds \quad (6)$$

and

$$P_t = p_t + \int_t^T [rKe^{-r(s-t)} N(\frac{ln(\frac{B_s}{S_t}) - (b - \frac{\sigma^2}{2})(s-t)}{\sigma\sqrt{s-t}}) \\ - \alpha S_t e^{-\alpha(s-t)} N(\frac{ln(\frac{B_s}{S_t}) - (b + \frac{\sigma^2}{2})(s-t)}{\sigma\sqrt{s-t}})]ds \quad (7)$$

where C_t and P_t are the prices of the American call and put, respectively and c_t and p_t are the prices for the appropriate European equivalent.

From the boundary conditions in (4) and (5) it follows that the boundaries, B_t, of immediate exercise satisfy the integral equations

$$B_t - K = c_t + \int_t^T [\alpha B_t e^{-\alpha(s-t)} N(\frac{ln(\frac{B_t}{B_s}) + (b + \frac{\sigma^2}{2})(s-t)}{\sigma\sqrt{s-t}}) \\ - rKe^{-r(s-t)} N(\frac{ln(\frac{B_t}{B_s}) + (b - \frac{\sigma^2}{2})(s-t)}{\sigma\sqrt{s-t}})]ds$$

for the American call and

$$K - B_t = p_t + \int_t^T [rKe^{-r(s-t)}N(\frac{ln(\frac{B_s}{B_t}) - (b - \frac{\sigma^2}{2})(s-t)}{\sigma\sqrt{s-t}})$$
$$-\alpha B_t e^{-\alpha(s-t)}N(\frac{ln(\frac{B_s}{B_t}) - (b + \frac{\sigma^2}{2})(s-t)}{\sigma\sqrt{s-t}})]ds$$

for the American put option. It is generally not possible to find a closed-form solution for the early exercise boundaries. Numerical methods such as a Newto-type interior point method (Coleman, [6]), the method of projected successive over-relaxation (PSOR) (Dewynne, [8]), even the SIMPLEX (Dempster, [7]), can yield a fairly good approximation, which then enables us to find analytical solutions for the options' prices and hedge parameters.

Differentiating equations (6) and (7) yields the deltas of the American call and put options, respectively:

$$\Delta_C = \Delta_c + \int_t^T [\alpha e^{-\alpha(s-t)}N(\frac{ln(\frac{S_t}{B_s}) + (b + \frac{\sigma^2}{2})(s-t)}{\sigma\sqrt{s-t}})$$
$$+ \frac{\alpha B_t - rK}{B_t\sigma\sqrt{s-t}}e^{-\alpha(s-t)}n(\frac{ln(\frac{S_t}{B_s}) + (b + \frac{\sigma^2}{2})(s-t)}{\sigma\sqrt{s-t}})ds$$
$$\Delta_P = \Delta_p - \int_t^T [\alpha e^{-\alpha(s-t)}N(\frac{ln(\frac{B_s}{S_t}) - (b + \frac{\sigma^2}{2})(s-t)}{\sigma\sqrt{s-t}})$$
$$+ \frac{rK - \alpha B_s}{S_t\sigma\sqrt{s-t}}e^{-r(s-t)}n(\frac{ln(\frac{B_s}{S_t}) - (b - \frac{\sigma^2}{2})(s-t)}{\sigma\sqrt{s-t}})]ds$$

It is important to realize that the delta of an option is not constant: with a change in the price of the underlying security or the option price itself, Δ changes as well. Thus, to Δ - neutralize an option position, continuous adjustments in the security holdings are necessary; the hedging of a portfolio requires continuous re-balancing. This strategy is also referred to as a dynamic hedging scheme.

A still remaining problem for this continuous time strategy is that in the real world one can trade only a discrete number of assets at discrete times and the observation of any financial measures usually involves at least a slight time delay.

The Δ of a portfolio of options on the same primary security can be calculated as $\Delta = \sum w_i \Delta_i$ where Δ_i is the Delta of the option i and w_i is the $ amount of option i in the portfolio.

It is clear that the Δ of a Δ - neutralized portfolio is equal to zero.

To hedge a whole portfolio of options using the delta hedging strategy is more feasible than applying the Δ - hedging strategy to a set of options of only one kind. Different kinds of options on the same underlying asset require different hedging actions, which can partly offset one another and thus reduce the accumulated number of needed transactions and hence the transaction costs.

2.2 Hedging Γ

As mentioned before the Δ of an option is changing with the price of the underlying asset. The Gamma of this option is defined as the rate of change of Δ with respect to a change of the price of the underlying asset: $\Gamma = \partial^2 V / \partial S^2$. Γ reflects the elasticity of Δ (and the optimal hedge ratio) on price changes of the underlying asset.

A delta-hedged portfolio requires continuous re-balancing in order to minimize risk. Continuous adjustments are impossible and very frequent operations can cause unjustifiably high transaction costs. The hedging strategy can easily become too expensive to be worthwhile for the investor. If he could take a Δ - neutralized position, $\Delta = 0$, while the gamma of the portfolio is small he could reduce the hedging costs. If the investor does not readjust the portfolio as soon as S and hence V and Δ are changing, he is no longer in an optimal position. However, as Γ is small, so is the change in Δ and the resulting deviation from the optimal hedge ratio; a hold and wait strategy can be less costly and more efficient than new re-balancing of the portfolio.

Assuming the general stock market model as in equation (3) we can calculate the Gamma of the option as the derivative of its Delta. Hence the Gamma of a European call and a European put option on the same underlying asset will be equal and follow

$$\Gamma_c = \Gamma_p = \frac{e^{-\alpha(T-t)}}{S_t \sigma \sqrt{T-t}} n\left(\frac{ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right).$$

For American options we get the following representations for Gamma:

$$\Gamma_C = \Gamma_c + \int_t^T [\frac{\alpha e^{-\alpha(s-t)}}{S_t \sigma \sqrt{s-t}} n(d_1(S_t, B_S, b, s-t))$$
$$- \frac{\alpha B_t - rK}{B_t \sigma^2 (s-t)} d_1(S_t, B_S, b, s-t) \frac{e^{-\alpha(s-t)}}{S_t} n(d_1(S_t, B_S, b, s-t))] ds$$

and

$$\Gamma_P = \Gamma_p + \int_t^T [\frac{\alpha e^{-\alpha(s-t)}}{S_t \sigma \sqrt{s-t}} n(d_2(B_S, S_t, -b, s-t))$$
$$+ \frac{rK - \alpha B_s}{S_t^2 \sigma^2 (s-t)} e^{r(s-t)} n(d_1(B_S, S_t, -b, s-t)) \{ln(\frac{B_s}{S_t}) - (b + \frac{\sigma^2}{2})(s-t))\}] ds,$$

using equations (1) and (2).

In fact, it is clear, that ideally the investor will desire $\Delta = 0$ and $\Gamma = 0$. Then the hedge ratio will stay constant while S changes and his position will remain optimal even if changes in the price of the underlying asset occur, making adjustments unnecessary. In practice it is important to optimize the Delta and the Gamma of the portfolio simultaneously, as both values clearly affect one another.

2.3 Vega

Vega is a measure for the volatility sensitivity of the option. It is defined as the rate of change of the option price with respect to the volatility of the underlying asset's price: $vega = \partial V / \partial \sigma$.

One of the underlying assumptions in the usual Black-Scholes model is that σ is constant; theoretically the definition of vega does not seem justified. The concept of implied volatility of an option suggests a somewhat different result. The implied volatility is defined as the volatility σ_i that satisfies the Black-Scholes PDE, taking all other measures of the option as given. In practice it can be observed that options on the same underlying asset that agree in the time to maturity but differ in the strike price have different estimated volatilities. This observation contradicts the assumption of the Black-Scholes model that σ is constant. However, it could be explained by the fact that investors might have a specific knowledge of the firm and its strategy that exceeds historically measurable data and one could hence interpret a change in σ as a re-adjustment due to new information.

Gamma - neutrality protects against jumps in the price of the underlying asset between hedge re-balancing, as continuous adjustments are impossible. Vega -

neutrality is a hedge against jumps in the volatility of the underlying asset's price. The model does not provide a possibility to optimize both Γ and vega of the option portfolio. Hence it will depend on the features of the underlying asset and the actual hedging costs as well as investors' preferences and risk aversion how to combine the strategies.

2.4 Theta

The theta of an option or portfolio of options is the rate of change of the option value or portfolio value with respect to time: $\Theta = \partial V/\partial t$. It is sometimes referred to as the time decay of the option or portfolio. Theta is usually negative - as time approaches maturity any option tends to loose value. Because of its non-stochastic nature theta is not a parameter to be taken care of in order to hedge risk. However, in practice it is often used as a proxy for Γ.

In a Δ - neutralized portfolio Θ can be connected to Γ via the equation

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi.$$

2.5 Rho

Rho measures the interest rate sensitivity of an option or portfolio: $\rho = \partial V/\partial r$.

The Rho of the European call and put on continuous dividend as well as non-dividend paying assets for instance are

$$\rho_c = (T-t)Ke^{-r(T-t)}N(d_2)$$

and

$$\rho_c = -(T-t)Ke^{-r(T-t)}N(-d_2).$$

Currency options have a rho with respect to the domestic interest rate and one with respect to the foreign interest rate.

2.6 Hedging a portfolio of assets

The issuance of any kind of options can be hedged by appropriately trading a certain amount of the underlying primary security as described above. In analogy to using the underlying asset in order to optimize the risk from writing options, so can options be used for Portfolio Insurance.

The purchase of put options can guarantee that the value of the assets held does not fall below a certain level. Call options can be bought to hedge a shortselling transaction or to ensure a certain low price for planned future asset purchases.

Mathematically, the problem of using options in order to hedge a portfolio of stocks or other primary assets is equivalent to using the underlying security for hedging against unfavorable changes in option prices. The same hedging rules as explained above apply to insuring the value of a portfolio of primary assets. To hedge a portfolio of assets with a \$ amount of \$$A$ we can purchase options on those assets for a value of \$$\Delta A$ to ensure or lock the value of the portfolio. Plain vanilla European or American style options are often the easiest to use for hedging purposes when simply aiming to minimize risk. However, exotic or Asian options, CAPS or a combination of all can provide a valueable alternative while eventually being cheaper but also slightly changing the hedging perspective.

3 Currency hedges

Any investment into foreign assets or derivatives carries currency risk. This risk can be divided into two components: The asset or derivative price in the foreign currency can be affected by the change in the exchange rate via reinforcement and the portfolio value can be changed via the translation into domestic currency, referred to as exchange risk.

Until the early 1990s, most of the largest investors in the world, such as U.S. and UK pension funds, did not hedge their international portfolios against currency fluctuations. Several explanations can account for this policy.

International assets, mostly stocks, represented only a tiny proportion of the global portfolio. The impact of currency risk was hence very limited and even beneficial as it provided an element of diversification for domestic monetary risk. In terms of return, the US-$ and the GBP were comparably weak currencies in the 1970s and '80s (Solnik, [15]), hence holding assets of stronger currencies provided an additional return. Also, many investors did not feel at ease using financial derivatives, necessary to hedge currency risk.

This picture has changed dramatically over the last decade. Bigger and bigger proportions of global portfolios are invested in foreign assets. Both, the U.S. and the UK have substantially gained economical power in terms of average asset return. The recent developments in international financial markets are leading to the immediate quest for an optimal currency hedging policy.

If the domestic currency strengthens, the exchange risk for the international investor increases: the translation of the ceteris paribus constant local portfolio return yields a lower domestic portfolio value. However, the stronger domestic currency negatively affects the competitiveness of domestic firms operating in an international business. It can weaken domestic investment returns and improve the performance of foreign competitors, hence increase the portfolio return on foreign investments. The latter process is called reinforcement.

In another scenario the reverse can happen: if the domestic currency weakens relatively to the foreign currencies in question, exchange risk for the investor decreases, but reinforcement can lead to a lower local (i.e. non-domestic) value of the foreign investments.

Thus, the cumulated effect of a domestic currency de- or appreciation on the overall currency risk is not always clear. It is however often assumed that pure exchange risk dominates the effect from reinforcement.

The above analysis has been made under the implicit assumption that local portfolio returns are positive. It shall only be mentioned that where exchange rate movements can increase or decrease anticipated local portfolio gains, the same is true for losses.

The time horizon of the portfolio setup makes currency risk more or less significant for the investor. Purchasing Power Parity suggests that exchange rate adjustments exactly equal inflation rate differentials. Thus, exchange risk would be a purely nominal measure and as investors are only concerned with real variables it should not affect investment or hedging decisions. In the long run mean-reversion of exchange rates leads to a relatively well established PPP such that currency risk exposure can sometimes become insignificant for practical hedging decisions. However, it is a well-known fact that especially in the short term PPP does not hold. Hence, in short and medium term decisions exchange rates become real variables and as such influence the choice on investments and require active consideration in the planning of the optimal hedging strategy.

It is intuitively clear that via reinforcement unexpected changes in the interest rate create an uncertainty about the investor's foreign portfolio value, which in a risk minimizing world should be hedged if possible. We should take a closer look at the pure exchange risk contained in the translation of the portfolio's local value into domestic currency.

Proponents of hedging exchange risk content that a no-hedge policy violates one of the major laws of finance, only to accept risk for which an adequate compensation is to be expected. With currencies being seen as having zero return and exchange rates fluctuating, they view exchange risk as uncompensated and argue that in a risk-averse world it should be minimized. They also point out that the reduction in portfolio variability gained from hedging exchange risk can be substantial.

Opponents of currency hedging believe that market frictions cause the costs of the hedging outweigh the risk reduction benefits. Sharpe, Lindner and Gordon [14] argue that assuming zero correlation between domestic and foreign returns, i.e. no reinforcement, the total risk of a portfolio containing foreign securities increases with the square of the amount of the unhedged currency risk, whereas the cost of hedging increases linearly. Thus, they claim the existence of a so-called foreign allocation threshold below which there should be no currency hedging and above which currency risk should be completely hedged.

Considering the rapid globalization of money, capital, and goods markets it is of crucial importance for individual as well as institutional investors to identify whether or not and to what extend they actually are exposed to exchange risk. Many investors nowadays spend a high percentage of their income on foreign goods, services or investments. In a sense the investors' own consumption baskets or investment intentions serve as an exchange risk hedge for the portfolio.

It should also be mentioned that even purely domestically operating firms or domestic portfolios are somewhat exposed to currency risk; in global markets with open economies unexpected exchange rate changes can affect domestic consumption behavior via reinforcement. Also, nowadays domestic portfolios are unavoidably partially made up of truly internationally competing companies, which creates a certain amount of currency risk even for those portfolios.

Briys and Solnik [3] developed a model to determine the optimal currency hedge strategy for international investments. Their assumptions can be summarized as follows:

- Markets are complete
- Utility maximization of the investors as measured in domestic currency
- Investment decision is given
- All necessary local hedges are already undertaken
- Existing asset position is temporarily untradable
- Single forward currency contract can be used to implement hedging decision
- Fixed and known planning horizon

The approach where the currency hedging decision is to be made independently of the underlying investment decision is called currency overlay management. Applying a decision made using this approach can lead to a sub-optimal portfolio setup. The currency overlay strategy ignores the benefits from portfolio diversification and thus neglects diversification as a valuable hedging instrument. It also ignores the fact that large unfavorable movements in the exchange rate may lead investors to pull out of the foreign market. However, the major reason to seperate the hedging decision from the investment decision at this stage is to obtain an increased transparency of the investment strategy.

Assuming the usage of only one single forward currency contract implies that investors have the ability and power to match the actual hedging plan perfectly in time and size with the theoretically optimal hedging solution. In this sense it makes the market almost frictionless from the investor's point of view.

The temporary non-tradability assumption for the portfolio supports the overlay management approach. If all asset positions would in fact be freely tradable, one would always have to consider that an asset trade may outperform the best hedge. This assumption probably also reflects upon possible temporary trade restrictions as often observed in emerging markets during extremely volatile time periods.

The advantage of Briys' and Solnik's [3] model over other available hedge suggestions is that they explicitly model the forward exchange rate behaviour and the influence of the interest rate differential (basis).

Again let the entire state space be represented by $(\Omega, \mathcal{F}, \mathcal{P})$ with a Brownian motion $\tilde{W}_t$ and filtration $\mathcal{F}_t$ as above. $\tilde{W}_*$ will denominate the Brownian motion processes used to describe the stochastic influences on assets, portfolio values or exchange rates under the original probability measure.

Subscripts t for time dependence will be omitted to ease the notation. The investors wealth at time t will be denoted as W and W^* for the wealth measured in local and domestic currency, respectively. Let the entire wealth be for simplification represented by the single investment in the abroad portfolio. The value of this portfolio at time t in local (i.e. non-domestic) currency follows:

$$dV = \mu_V V dt + \sigma_V V d\tilde{W}_V$$

where μ_V and σ_V are assumed to be constant. The exchange rate at t is defined as the domestic value of one unit of the foreign currency and follows the process:

$$dE = \mu_E E dt + \sigma_E E d\tilde{W}_E$$

The local value of the portfolio and the exchange rate are assumed to be correlated in the manner $cov(d\tilde{W}_V, d\tilde{W}_E) = \rho_{EV} dt$ where ρ_{EV} is the correlation coefficient between E and V per unit time.

The value of the portfolio measured in domestic currency is given as $V^* = EV$. Hence, using Ito's lemma:

$$\begin{aligned} dV^* &= dVE + VdE + \rho_{EV}\sigma_E\sigma_V EV dt \\ &= (\mu_V V dt + \sigma_V V d\tilde{W}_V)E + V(\mu_E E dt + \sigma_E E d\tilde{W}_E) + \rho_{EV}\sigma_E\sigma_V EV dt \\ &= V^*([\mu_V + \mu_E + \sigma_{EV}]dt + \sigma_V d\tilde{W}_V + \sigma_E d\tilde{W}_E). \end{aligned}$$

The local (non-domestic) interest rate movement is assumed to be captured in the Vasicek model:

$$dr = k(\mu - r)dt + \sigma_r d\tilde{W}_B,$$

similarly the domestic risk-free rate shall follow:

$$dr^* = k^*(\mu^* - r^*)dt + \sigma_r^* d\tilde{W}_{B^*},$$

where $d\tilde{W}_B$ and $d\tilde{W}_{B^*}$ capture the stochastic character of the appropriate bond. A correlation of ρ_{BB^*} is assumed.

Assuming also that locally the expectations hypotheses prevail, the zero coupon bond prices with maturity T abroad and at home then follow the processes:

$$\begin{aligned}\frac{dB}{B} &= rdt - \frac{\sigma_r}{k}(1 - e^{k(T-t)})d\tilde{W}_B \\ &= rdt - \sigma_B d\tilde{W}_B\end{aligned}$$

$$\begin{aligned}\frac{dB^*}{B^*} &= r^*dt - \frac{\sigma_{r^*}}{k^*}(1 - e^{k^*(T-t)})d\tilde{W}_{B^*} \\ &= r^*dt - \sigma_{B^*} d\tilde{W}_{B^*}.\end{aligned}$$

Interest rate parity suggests that average expected returns in two different countries with open economies and free capital movements translated into one common currency should equal each other. Covered Interest Rate Parity thus yields

$$F(t)B^*(t) = E(t)B(t),$$

where $F(t)$ represents the price of the forward contract at time t.

So, $F(t) = E(t)\frac{B(t)}{B^*(t)}$ and using again Ito we get:

$$\begin{aligned}dF = &\ \tfrac{B}{B^*} \quad dE + \frac{E}{B^*}dB - \frac{EB}{(B^*)^2}dB^* + \frac{EB}{(B^*)^3}(B^*)^2\sigma^2_{B^*}dt \\ & - \quad \rho_{B^*E}\frac{B}{(B^*)^2}dEdB^* - \rho_{BB^*}\frac{E}{B^*}dBdB^* + \rho_{BE}\frac{1}{B^*}dEdB \\ = &\ F \quad \frac{dE}{E} + F\frac{dB}{B} - F\frac{dB^*}{B^*} + F\sigma^2_{B^*}dt \\ & - \quad \rho_{B^*E}F\frac{dE}{E}\frac{dB^*}{B^*} - \rho_{BB^*}F\frac{dB}{B}\frac{dB^*}{B^*} + \rho_{BE}F\frac{dE}{E}\frac{dB}{B}\end{aligned} \tag{8}$$

Hence,

$$\frac{dF}{F} = \mu_F dt + \sigma_E d\tilde{W}_E - \sigma_B d\tilde{W}_B + \sigma_{B^*} d\tilde{W}_{B^*}$$

where $\mu_F = \mu_E + r - r^* + \sigma^2_{B^*} + \sigma_{B^*E} - \sigma_{BB^*} - \sigma_{BE}$ is the instantaneous trend of the forward price. The hedge ratio in this model is defined as

$$h = \frac{H(t)F(t)}{W^*(t)},$$

where $H(t)$ is the foreign currency amount of the foreign investment to be hedged.

The wealth process in domestic currency is thus characterized by

$$dW^* = (\mu_V + \mu_E + \sigma_{EV})W^* dt - c^* dt + \sigma_V W^* d\tilde{W}_V + \sigma_E W^* d\tilde{W}_E - HdF$$

As one assumption is that investors only care about the domestic value of their investments, the consumption decision has to be restricted to domestic consumption. As now H can be substituted using the hedge ratio h and furthermore making use of equation (8) the domestic wealth process can now be described as

$$\begin{aligned} dW^* = & ((\mu_V + \mu_E + \sigma_{EV})W^* - c^* - h\mu_F W^*)dt + \sigma_V W^* d\tilde{W}_V + \\ & (1-h)\sigma_E W^* d\tilde{W}_E + hW^*(\sigma_B d\tilde{W}_B - \sigma_{B^*} d\tilde{W}_{B^*}) \end{aligned} \quad (9)$$

Now we can use the HJB approach to maximize the investors' expected utility. As we assume that investors are only concerned about the domestic value of his foreign investment this implies that they will only be interested in domestic consumption. Hence, the optimization problem becomes:

$$J(W^*(t), r, r^*, t) = \max_{c^*, h} E_t \int_t^T u(c^*(s), s) ds$$

subject to the initial wealth of $W^*(0) = W^*_0$ where u is a Von Neumann-Morgenstern utility function.

The indirect utility or value function takes the form

$$\begin{aligned} 0 = \quad \max_{c^*,h} \quad & \{u(c^*(t),t) + J_t + ([\mu_V + \mu_E + \sigma_{EV}]W^* - c^* - h\mu_F W^*)J_{W^*} \\ + \quad & k(\mu - r)J_r + k^*(\mu^* - r^*)J_{r^*} \\ + \quad & \frac{1}{2}[\sigma^2_V + (1-h)^2\sigma^2_E + h^2\sigma^2_B + h^2\sigma^2_{B^*} + 2(1-h)\sigma_{EV} \end{aligned}$$

$$
\begin{aligned}
+ & \quad 2h\sigma_{BV} - 2h\sigma_{VB^*} + 2(1-h)h\sigma_{BE} - 2(1-h)h\sigma_{B^*E} - 2h^2\sigma_{BB^*}](W^*)^2 J_{W^*W^*} \\
+ & \quad \frac{1}{2}\sigma_B^2 J_{rr} + \frac{1}{2}\sigma_{B^*}^2 J_{r^*r^*} \\
+ & \quad (\sigma_{B^*V}W^* + (1-h)\sigma_{B^*E}W^* + h\sigma_{BB^*}W^* - h\sigma_{B^*}^2 W^*)J_{W^*r^*} \\
+ & \quad (\sigma_{BV}W^* + (1-h)\sigma_{BE}W^* + h\sigma_B^2 W^* - h\sigma_{BB^*}W^*)J_{W^*r}\}.
\end{aligned}
$$

Hence, the two first order conditions become the usual envelope condition

$$u_{c^*} = J_{W^*}$$

and

$$
\begin{aligned}
0 &= -\mu_F W^* J_{W^*} \\
&+ (-(1-h)\sigma_E^2 + h\sigma_B^2 + h\sigma_{B^*}^2 - \sigma_{EV} - \sigma_{B^*V} + \sigma_{BV} \\
&+ \quad (1-2h)\sigma_{BE} - (1-2h)\sigma_{B^*E} - 2h\sigma_{BB^*})(W^*)^2 J_{W^*W^*} \\
&+ (-\sigma_{B^*E} + \sigma_{BB^*} - \sigma_{B^*}^2)W^* J_{W^*r^*} \\
&+ (-\sigma_{BE} + \sigma_B^2 - \sigma_{BB^*})W^* J_{W^*r}\}.
\end{aligned}
$$

Thus, if we define σ_F^2 to be the variance of the price of the forward currency contract, the optimal hedge ratio will be

$$
\begin{aligned}
h &= \frac{\sigma_E^2 + \sigma_{B^*E} - \sigma_{BE}}{\sigma_F^2} + \frac{\sigma_{EV} + \sigma_{B^*V} - \sigma_{BV}}{\sigma_F^2} + \frac{\mu_F}{\sigma_F^2}\frac{J_{W^*}}{W^* J_{W^*W^*}} \\
&+ \frac{J_{W^*r^*}}{W^* J_{W^*W^*}}\frac{\sigma_{B^*E} + \sigma_{B^*}^2 - \sigma_{BB^*}}{\sigma_F^2} + \frac{J_{W^*r}}{W^* J_{W^*W^*}}\frac{\sigma_{BE} - \sigma_B^2 + \sigma_{BB^*}}{\sigma_F^2}. \qquad (10)
\end{aligned}
$$

The first term of the solution can be interpreted as the macroeconomic hedging parameter, covering the exchange rate behaviour, the second is the asset specific component, hedging against the reinforcement in local portfolio value changes. The third term can be viewed as the speculative component being driven by the coefficient of relative risk tolerance $\frac{J_{W^*}}{W^* J_{W^*W^*}}$. The last two terms in the optimal hedge ratio are due to the interest rates both, domestic and abroad, moving stochastically.

Using the HJB approach we have found the optimal hedge ratio to maximize the expected utility for investors, given an underlying investment decision. This however, is not the minimum variance hedge ratio. By minimizing the variance of equation (9) we can see that investors who follow the minimum variance strategy would choose a hedge ratio that only consists of the first two terms of equation (10):

$$h_m = \frac{(\sigma_E^2 + \sigma_{B^*E} - \sigma_{BE}) + (\sigma_{EV} + \sigma_{B^*V} - \sigma_{BV})}{\sigma_F^2}.$$

Other investors will deviate from h_m. As shown in the optimal hedge decision formula above, the 3rd term takes advantage of a bias on the forward market weighed according to the investor's relative risk aversion. Additionally, they would take into account the effect of a change in the domestic and foreign interest rate on their wealth and consumption process. $J_{W^*r^*}/W^*J_{W^*W^*}$ and $J_{W^*r}/W^*J_{W^*W^*}$ are to be interpreted as the relative risk aversion coefficients with respect ot the interest rate along the optimal path.

The major criticisms against the suggested hedging scheme above are to be found in some of the underlying assumptions as well as in the actual result itself:

First, as usual, markets in reality are not frictionless. The real-world hedging decision is not a continuous-time, but a discrete-time problem. For individual investors it is also unrealistic to assume that the whole hedging decision can be covered using a single forward contract. Most of the smaller investors can usually only enter into the future's market. This being far more standardized and rigid than the forward market, additional considerations become necessary when matching the size and time-horizon of the hedge with the original investment.

Secondly, currency risk and local portfolio risk are not additive. The variance of the domestically measured return of the foreign investment includes the correlation between the local market risk and the variance of the exchange rate. By setting the hedging decision in an overlay management environment this non-additivity, i.e. actually present reinforcement, is ignored.

Furthermore, the model focuses on currency hedging only the non-domestic part of the global portfolio. In all other major investment management decisions the focus has moved from an individual asset perspective to a total portfolio perspective. By focussing on the foreign part of the portfolio only Briys and Solnik [3] neglect the well-known benefits from diversifying against domestic budgetary, fiscal, and monetary risk.

Probably the biggest shortcoming of the above model is to be seen in the solution itself: The optimal hedge ratio depends on a lot of practically unobservable parameters, such as the known but difficult to formalize preferences of the investors and their relative risk aversion. Thus, even if the theoretical solution produces results very close to actually optimal values these are difficult to determine as some heroic guesses and simplifications are involved estimating the unobservable parameters.

Lastly, it is important to be aware that the model is set in the environment of stable and developed economies. Trying to hedge against crises as observed in Mexico at the end of 1994 and in South East Asia in 1997 is a much more complex issue. It often involves a lack of confidence in the consistency of the governments' monetary policies, especially in terms of exchange rate policy and the freedom of capital flows. Investors often act in herds which in the case of a crisis disturbs the predictability of important market measures under an equilibrium approach.

A probably much more pragmatic approach to find the optimal currency hedge ratio is given by Bos, Mahieu and Dijk [2]. They look at discrete time intervals. By covered interest rate parity the value of a forward currency contract with maturity T at time t satisfies

$$F_{t,T} exp(r^f_{t,T}) = E_t exp(r^d_{t,T})$$

with $r^f_{t,T}$ and $r^d_{t,T}$ being the foreign and domestic interest rates on default free pure discount bonds with maturity date T, respectively, and E_t being the foreign value of one unit of the domestic currency. Given a hedge ratio of h_t over the time interval [t,t+1] the continuously compounded domestic rate of return on the portfolio, r_t, then satisfies

$$e^{r_{t+1}} E_t = (1-h_t)E_{t+1} + h_t F_{t,T}.$$

Hence,

$$e^{r_{t+1}} = (1-h_t)\frac{E_{t+1}}{E_t} + h_t(r^d_{t,T} - r^f_{t,T}).$$

As the domestic wealth is assumed to change only as a result of the choice of the hedge parameter the next period's wealth is given by

$$W^*_{t+1} = e^{r_{t+1}} W^*_t$$

Investors still try to maximize their expected utility, but only as of time t+1. Assuming a standard power utility function with constant relative risk aversion $u(W_t^*) = \frac{(W_t^*)^\gamma - 1}{\gamma}, \quad \gamma < 1$ investors face the maximization problem

$$\max_{0 \le h_t \le 1} E_t[u(W_{t+1})] = \max_{0 \le h_t \le 1} E_t[\frac{[exp(r_{t+1}(e_{t+1}, h_t, r^f, r^d))]^\gamma - 1}{\gamma}],$$

where e_{t+1} is the exchange rate return from t to $t+1$, defined as $e_{t+1} = ln(\frac{E_{t+1}}{E_t})$.

This approach to currency hedging certainly is more pragmatic in that it avoids the problem of some important financial measures being hard or impossible to observe or to formalize mathematically. However, the assumptions and simplifications made are even more daring than in the former model and as any movements in the local portfolio value are neglected this very pragmatic model should only be used to hedge a certain expected constant future capital payoff.

4 Conclusions

In the first part of this essay I have developed several hedging parameters for European and American options. Leaning mainly on papers by Huang, Subrahmanyam, and Yu [10], Broadie and Detemple [4], and Carr, Jarrow, and Myneni [5], and using the early exercise representation for American option prices I was able to determine several Greek letters in an environment that allows for transaction costs. The GBMP-model in its continuous-time setting in the spirit of Merton (1969) can be applied to options on different kinds of underlying assets and provides an opportunity to consider carrying costs on the primary security.

Although this is a fairly general setting, there is still plenty of room left for further research, for instance on how to improve the numerical solutions for the early exercise boundaries for simple American options or considering more complex payoff or exercise characteristics.

By relying upon a continuous-time setting and stochastic interest rate models for the foreign and domestic economies according to the Vasicek model, Briys and Solnik [3] showed that the non-deterministic behaviour of interest rates has a significant influence on the optimal hedge ratio. Although, an overlay management environment is likely to produce a sub-optimal hedging decision, this assumption enabled us to decompose the theoretically optimal hedge parameter into five components. As these could be given specific economic interpretations, this provides us with valuable insight into the sources of currency risk. By using an individual asset perspective Briys and Solnik [3] neglect the contribution diversification makes towards portfolio hedging.

A more pragmatic currency hedging approach is offered by Bos, Mahieu, and Dijk [2]. The discrete-time setting enables users to employ several different time series models. In empirical tests they are able to show that for moderate risk-averse investors with an overlay management approach their model produces very successful results in comparison with other modelling methods.

How to optimally combine the analyzed local hedging strategies with possible currency hedges is a question that still deserves a lot more detailed investigation and can be material for further research.

5 Appendix

Derivation of equations (6) and (7)

It is intuitively clear that the rational price of the American option at time t is the expected discounted payoff of the option, conditional on the information at time t. If τ denotes the yet unknown optimal exercise time of this contingent claim, its price at time t is uniquely given by

$$V_t(Y) = E_Q[R_{t,\tau} Y_\tau | \mathcal{F}_t] \quad \forall t \in [0, \tau] \tag{11}$$

assuming that the given expression is integrable, which is guaranteed if the claim is attainable. Y_τ is the terminal payoff of the option when it is actually exercised. $R_{t,s}$ is the discount factor given as $exp(-\int_t^s r_v dv)$ and assumed to be $\mathcal{F}_t$ - measurable, Q represents the equivalent martingale measure (EMM). As τ was said to be the optimal exercise time $V_t(Y)$ can also be represented as

$$\sup_{\tau \in S_{t,T}} E_Q[R_{t,\tau} Y_\tau | \mathcal{F}_t] \quad \forall t \in [0, T] \tag{12}$$

where $S_{t,T}$ is the class of stopping times in $[t, T]$.

As this expression is intuitively clear I only state the result. A proof can be found in Broadie and Detemple [4] and in Karatzas and Shreve [12].

Now applying the Doob-Meyer decomposition to the terminal payoff of the option, the contingent claim under the EMM satisfies:

$$Y_t = Y_0 + A_t(Y) + M_t(Y) \quad \forall t \in [0, T]$$

where $M_t(Y)$ is a Q-martingale and $A_t(Y)$ is a non-decreasing process with initial value $A_0(Y) = 0$. Both processes are $\mathcal{F}_t$ - measurable. Since in the immediate exercise region the payoff of the option only depends on the current price of the underlying asset and the constant strike price, Y_t satisifes

$$dY_t = S_t(\mu - \alpha)dt + S_t \sigma d\tilde{Z}_t$$

and thus $dA_t(Y) = S_t(\mu - \alpha)dt$ for a not yet exercised option.

Define $D_t \equiv R_{0,t} Y_t$ and $Z_t \equiv sup_{\tau \in S_{t,T}} E_Q[D_\tau | \mathcal{F}_t]$; Z_t is called the Snell envelope of D. Broadie and Detemple [4] give a reference for the proof of why

$$Z_t + \int_0^t I_{\tau_s = s} R_{0,s}[r_s Y_s ds - dA_s(Y)] \;, \quad t \in [0,T]$$

is a Q-martingale.

Using the defining characteristics of a martingale it follows that

$$E_Q[Z_t + \int_0^t I_{\tau_s = s} R_{0,s}(r_s Y_s ds - dA_s(Y))] = E_Q[Z_0]. \tag{13}$$

By definition

$$Z_T = \sup_{\tau \in S_{T,T}} E_Q[D_\tau | F_T] = E_Q[D_T | F_T] = D_T.$$

Hence equation (13) becomes

$$E_Q[D_T] + E_Q[\int_0^T I_{\tau_s = s} R_{0,s}(r_s Y_s ds - dA_s(Y))] = E_Q[Z_0]$$

Using expression (12) this is equal to $V_0(Y)$. Generalizing over time and making use of equation (11) it follows that

$$V_t(Y) = E_Q[R_{t,T} Y_T | F_t] + E_Q[\int_t^T I_{\tau_s = s} R_{\tau_s,s}(r_s Y_s ds - dA_s(Y)) | F_t]$$

Explicitely calculating these expectations yields equations (6) and (7).

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