

# **Portfolio management with index insurance**

by

Peter Newton Bell

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## ABSTRACT

This graduating essay uses techniques from mathematical finance to investigate a basic question: how much insurance should a farmer buy? The farmer, an economic agent, uses index insurance products to manage risk in their agricultural business. The portfolio allocation problem is stated in terms of the Black-Scholes model for option pricing. The problem is considered in discrete and continuous time with one or two assets. Numerical results are used to characterize the solution in discrete time. In continuous time, a replicating argument allows the put option to be expressed as a portfolio of the index and cash. The portfolio allocation problem in continuous time with two assets captures a key feature of index products - basis risk. This provides new results based on a graduate course in math finance (Ekeland, 2010). In an afterword, I consider why a government may want to insure the domestic farm industry.

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## 1 INTRODUCTION

There has been extensive analysis, practice, and publication on index products in the agricultural insurance market in the USA. Most research uses the actuarial framework, rather than mathematical finance. In the USA, agricultural insurance is provided by public-private partnership and this academic, business, and government involvement makes it a rich area for research.

This paper explores index insurance products with the math finance toolbox. An index product is where indemnity is not based on subscriber's losses, but is based on an index of losses experienced by similar farmers. The Group Risk Plan (GRP) uses average crop yield at the county level as the index and group criteria. The GRP is the index product that is the focus of my attention.

The farmers using index insurance face a portfolio allocation problem because index products "are essentially put options" (Barnett, 2006, p.5). I pose this portfolio problem for one agent with discrete and continuous time, with one or two assets. I show the agent intends to over insure their crop in discrete time with no basis risk. In continuous time, I present a replication argument based on the Black-Scholes equations for vanilla options. This replicating argument allows me to restate the portfolio problem as a portfolio problem with the underlying asset, which is answered by the math finance results I use (Ekeland 2010). Analytic and numerical results are presented to characterize the optimal amount of index insurance a farmer should purchase.

## 2 LITERATURE REVIEW

Insurance is a risk management tool. Index insurance products have been used in the USA since 1993 (Barnett, 2004, p. 3) and there is practical understanding of their use. There is a good peer review literature centered, in my opinion, around Professor Barry Barnett; a conference paper and a testimony before government give primary motivation for my work. Although other papers may be more refined, these two are candid. The perspective of the insurance providers and purchasers comes through these works, which I think is a crucial to understand the situation.

There is a microeconomic literature (Rees and Wambach, 2008) that may harbour useful ideas on index insurance, but it is mostly disjoint from the math finance literature I use.

### 2.1 *Scope of index products*

The index insurance market is a growing area in the USA where the percentage of total liabilities was 3% in 2002 and 14% in 2006 (Barnett et al., 2008, p. 223). As of 2006, “almost all crops of economic significance are either currently insurable or soon will be insurable” (Barnett, 2006, p.1). Exotic indices, such as weather and livestock profit margins, are available (Barnett, 2004, p.11). In his 2004 paper, Barnett asks: how can we insure commodities like ‘hay’, ‘pasture, and rangeland’ (Barnett, 2006, p.1)? The answer is index insurance products (further reading Halcrow, 1949). The reason is simply that county level estimates for these commodities are known, but farm level estimates are not (Barnett, 2004, p. 8); insurers do not want to insure what they cannot measure. Therefore, index products allow insurance to be offered to farmers for parts of their operation that were previously uninsurable.

Although situations where index products are used in agricultural insurance are limited, the GRP is used for primary crops in major production areas (Barnett, 2004, p.2). The definition of the GRP index products (Barnett, 2004, p.3) show that indemnity is equal to yield-loss times price:  $(Y^K - Y)^+P$  where  $Y^K$  is trigger yield,  $Y$  is realized yield,  $P$  is price for commodity. Therefore, I will specify dynamics of  $I = PY$ ; the index insurance product is a put option on the value of the harvest, which means the index is the value of the harvest. This is a subtle point that allows me to use the stock from the Black-Scholes world, Geometric Brownian Motions (GBM), to describe the index.

### 2.2 *Basis risk*

The farmers using index insurance face a portfolio allocation problem. When an agent is endowed with a risky asset and allowed to trade put options based on an index, the possible difference between indemnity and losses is called basis risk. Here, basis refers to the difference between loss and indemnity. The basis can be positive or negative, there can be: no difference, loss but no indemnity, indemnity but no loss. I will create basis risk in the Black-Scholes world by specifying GBM equations for both the index and endowment, and forcing the noise in the index and the endowment to be correlated.

### 2.3 *Information asymmetry*

When Barnett writes that “index insurance products are not susceptible to the common insurance problems of moral hazard and adverse selection” (Barnett, 2004, p. 6), I pose two explanations to justify this claim: since county level yields of the insured commodity are known, there is symmetry of information; by calculating commodity yield at county level, farmers share a risk factor that will align their incentives. Both these ideas are not fully developed, and Barnett provides a different justification:

Index insurance products, however, are not susceptible to moral hazard and adverse selection because indemnities are based on an index over which the policyholder has no control. ... Further, there is no reason to believe that growers have any better information about expected county average yields than does the insurer. Because there is no information asymmetry, there should be no problems with moral hazard or adverse selection. (Barnett, 2004, p.8)

The claim from Barnett is effective. If policyholders do not impact the index, then they are motivated to achieve the best results at their farm. Further, there is no asymmetric information regarding index levels. This argument should be explored in a more formalized setting, however, the result is useful since symmetric information is a requirement for the Black-Scholes model.

### 2.4 *Further topics in index insurance*

Index products were first proposed to provide insurance to forage (further reading Halcrow, 1949) but index products are popular today because they have low transactions costs: no historic farm-level yield data are required, and no farm level loss adjustments are required (Barnett, 2004). These transaction costs are important for purchasers and sellers of insurance; this paper considers how much insurance a farmer should buy, but an important question that remains is whether an agent will sell this type of insurance. There are exciting developments happening today, centered around index product development and private-public partnerships to provide insurance. I think it is interesting to consider if an index product could be developed to offset risk of catastrophic failure of genetically modified seeds.

### 3 ANALYSIS I: DISCRETE PROBLEM

First I consider models with two time steps. The problem has one objective function, one constraint, and two choice variables. The objective is to maximize expected utility of terminal wealth, where terminal time is equal to the expiry of a put option. The constraint is a self financing condition for trading in put options. The choice variables are the allocation to put options and cash. There is a fixed exposure to a risky asset and the agent decides their exposure to insurance through the put option. This approach can cover one or two assets, which provides basis risk. For numerical results, I use the binomial distribution.

#### 3.1 General results

$$\max_{\alpha, \beta} E(U(W_T)) \quad (3.1)$$

$$s.t. \quad W_0 = S_0 + X_0 \quad (3.2)$$

$$X_0 = \alpha B_0 + \beta P_0 \quad (3.3)$$

$$W_T = S_T + \alpha B_T + \beta P_T \quad (3.4)$$

This terminal wealth problem does not yet specify the dynamics of  $S, P$  because this allows us to derive results for the model with basis risk, or without it. The choice variable  $\beta$ , the number of put options held, is unconstrained because I want to determine how farmers want to trade insurance: do they over insure their crop, under insure their crop, or try to sell insurance? The problem can be rewritten as a maximization of one variable:  $\alpha B_0 = X_0 - \beta P_0$ , or  $\alpha = X_0/B_0 - (P_0/B_0)\beta$ . This allows us to rewrite the terminal wealth problem:

$$\max_{\beta} E(U(S_T + (X_0/B_0 - \beta(P_0/B_0))B_T + \beta P_T)) \quad (3.5)$$

With standard theory of utility functions (bounded, continuously differentiable, increasing, and strictly concave where terminal wealth is positive) the first order condition for this problem gives a solution. With the derivative inside the expectation, the first order condition is:

$$0 = E(U'(W_T) \frac{dW_T}{d\beta}) \quad (3.6)$$

where  $\frac{dW_T}{d\beta} = -1P_0(B_T/B_0) + P_T$ . I use the log utility function for numerical results. A numerical procedure can generate results with these steps: specify a full model, calculate  $E(U'(W_T) \frac{dW_T}{d\beta})$  for many values of  $\beta$ , then search for  $\beta$  that satisfy the first order condition. The interval of  $\beta$  values to search should be based on physical interpretation of the problem. The sign of  $\beta$  determines if the agent want to buy, or sell insurance. The size of  $\beta$  determines if the agent want to trade more insurance than their endowment of stock.



### 3.2 Results with one asset

$$\max_{a,b} E(U(W_T)) \quad (3.7)$$

$$s.t. \quad X_0 = \alpha B_0 + \beta P_0 \quad (3.8)$$

$$W_T = S_T + \alpha B_T + \beta P_T \quad (3.9)$$

$$S_T = uS_0 \text{ w.p. } P(u), \quad S_T = dS_0 \text{ w.p. } P(d), \quad P(u) + P(d) = 1 \quad (3.10)$$

$$S_T = uS_0 \rightarrow P_T = 0, \quad S_T = dS_0 \rightarrow P_T = K - dS_0 \quad (3.11)$$

$$P_0 = (K - dS_0)P(d) \quad (3.12)$$

This is like the standard microeconomic insurance problem (Rees and Wambach, 2008), except I do not force  $\beta > 0$  because I want to see the optimal incentive for an agent. There is no basis risk since the put option is based on the endowment (the index is farm-level loss). I use a numerical procedure to assess the optimal amount of put options in this binomial model. The results show the agent seeks to buy more insurance than their endowments, which is over-insurance.

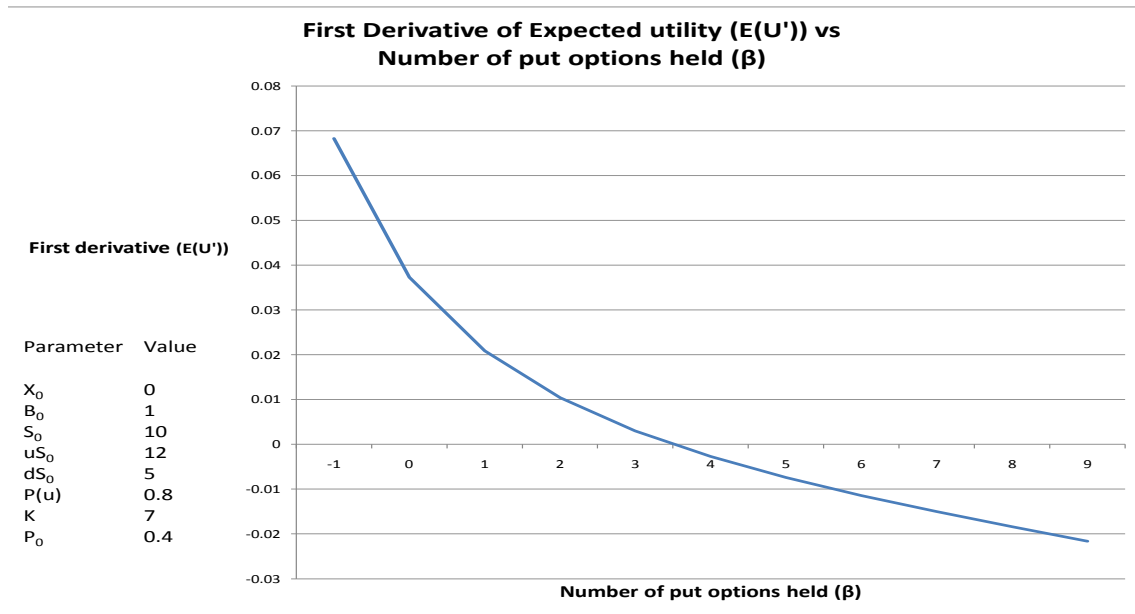


FIGURE 1. Search for first order condition

The parameter values reported in Figure 1, are the exogenous parameters ( $X_0, B_0, S_0, u, d, P(u), P(d), K, r, T$ ) and one endogenous parameter ( $P_0$ ). Three parameters appear to be crucial:  $d, P(d), K$ ; the downward movement in the stock in the binomial model, the probability of this movement, and the strike price for the put. The reason these parameters are important is they determine the downside risk in the agent's portfolio, which determines their need for insurance.

The  $dS_0 = 5, P(d) = 0.2$  means the farmer faces a drastic loss (50%) with large probability (20%). The exact solution I calculate is  $\beta = 3.5$ , which is the number of puts the agent should buy when they are forced to hold one unit of stock. When the agent is forced to hold two units, *ceteris paribus*, the optimal is  $\beta = 7$ . These results reflect over-insurance, since the agent buys more puts than he has exposure to the underlying.

#### 4 ANALYSIS II: CONTINUOUS PROBLEM WITH ONE ASSET

Second I consider a model with one asset and continuous time. The problem still has the same one objective function, one constraint, and two choice variables. There is still no basis risk because the index insurance product is a put option on the asset. The continuous time model allows us to use standard results for optimal terminal wealth (Ekeland, 2010). I introduce a replication argument that is crucial for the calculations. The assets are defined by:

$$dS/S = \mu_1 dt + \sigma_1 dW_1 \quad (4.1)$$

$$dB/B = r dt \quad (4.2)$$

##### 4.1 Replication argument

$$S'_t = aB_t + bP_t \quad (4.3)$$

To replicate one unit of stock,  $S'$ , requires a mix of bonds and puts. I require that fractional amounts of bonds and puts can be traded. There are particular values for  $a, b$  that can be found by substituting the Black-Scholes bond and put price into the equation:

$$S'_t = aB_t + bKe^{-rT}(1 - N(d_2)) - bS_t(1 - N(d_1)) \quad (4.4)$$

$$aB_t + bKe^{-rT}(1 - N(d_2)) = 0 \quad (4.5)$$

$$-bS_t(1 - N(d_1)) = S_t \quad (4.6)$$

$$b = -1/(1 - N(d_1)) \quad (4.7)$$

$$a = (K/B_t)e^{-rT}(1 - N(d_2))/(1 - N(d_1)) \quad (4.8)$$

It is useful to sign these parameters:  $a > 0, b < 0$  for all situations. In other words, to replicate a stock you have to sell puts and buy bonds. This will become more complicated when the agent has a nonzero amount of stock at time zero. The result 4.7 shows the unique way to replicate stock with puts and bonds. This is an important result because the optimal amount of risky asset ( $S$ ) is known and the replication argument allows this optimal amount to determine a unique trade in put options. This simple derivation, matching terms in the definition of prices, is equivalent to creating a delta-neutral replicating portfolio.

To assess the accuracy of the replicating strategy, I calculate it for particular parameter values. For  $r = 0.01, T = 0.8, \mu = 0.2, \sigma = 0.3, K = 50$  it is straightforward to calculate  $a, b$ . I report the calculation of the synthetic stock for several values of  $S$ . Table 1 is calculated at initial time,  $t = 0$ .

TABLE 1. Calculation of replication portfolio

S	b	a	P	B	S'
30	-1.0	50.16	19.72	1.01	30
36	-1.2	52.23	14.24	1.01	36
42	-1.5	55.53	9.59	1.01	42
48	-2.0	59.73	6.05	1.01	48
54	-3.1	64.53	3.60	1.01	54
60	-5.0	69.71	2.05	1.01	60

The Table 1 shows the replicating strategy works, since  $S = S'$ . However, this replicating strategy requires dynamic adjustment and it breaks down at terminal time. As expected,  $b < 0$  throughout (this occurs because a put option provides negative exposure to the underlying) and  $b$  can become very large for put options that are deep out of the money. This may be a problem for the practicality of the results because insurance contracts are generally signed out of the money.

#### 4.2 Standard Results

The terminal wealth problem is stated as:

$$\max_{X_T} E(\ln(X_T)) \quad (4.9)$$

$$s.t. \quad B_T X_0 = E(Z_T X_T) \quad (4.10)$$

For log utility, the optimal wealth ( $X_t$ ) is known by Ekeland's notes [4] to be:

$$X_t = X_0 (S_t/S_0)^{(\mu-r)/\sigma^2} \exp\left[\frac{\mu-r}{2\sigma^2} (\mu-r+\sigma^2)t\right] \quad (4.11)$$

The optimal amount of stock is known ( $h_t = \frac{\partial X_t}{\partial S_t} = \frac{\mu-r}{\sigma^2} \frac{X_t}{S_t}$ ) and referred to as Merton's rule. I know how many units of cash and put option are required to replicate one unit of stock, so I also know how many units are required to replicate  $h_t$  units. I suppose the agent has zero endowment of stock, and trades the replicating portfolio to maximize utility. The optimal position in puts is:

$$h_t b P_t = \frac{(\mu-r)X_t}{\sigma^2 S_t} \frac{-1}{(1-N(d_1))} P_t \quad (4.12)$$

To determine the optimal amount of cash to hold, recognize that cash should be held to achieve the replicating portfolio and Merton's rule. The Merton rule gives  $X_t = h_t S_t + g_t B_t$  where everything is known except  $g_t$ . I get  $g_t B_t = X_t (1 - \frac{\mu-r}{\sigma^2})$ . The replicating portfolio calls for  $h_t a B_t$  position in cash. I have  $h_t a = \frac{(\mu-r)X_t}{\sigma^2 S_t} \frac{e^{-r(T-t)} K}{B_t} \frac{1-N(d_2)}{1-N(d_1)}$ . So, the optimal amount of cash when the agent has zero endowment is:

$$g_t B_t + h_t a B_t = X_t \left[ 1 - \frac{\mu-r}{\sigma^2} + \frac{\mu-r}{\sigma^2} \frac{K e^{-r(T-t)}}{S_t} \frac{1-N(d_1)}{1-N(d_2)} \right] \quad (4.13)$$

### 4.3 Portfolio problem results

Although the standard problem can determine optimal behaviour when the agent has an endowment of risky asset (it changes the  $X_0$ ), I consider what the agent will do when they cannot trade  $S$  but can trade a replicating asset composed of bonds and puts  $S''$ . Here, I use  $S''$  to refer to the amount of synthetic stock the agent should hold to move from endowment to optimal amount,  $\eta_0$  denotes the endowment of stock,  $h_t$  is the optimal amount of money in stock from before. Therefore,  $\eta_0 S_t + S'' = h_t S_t$ .

The agent does not trade  $S$ , but the replicating asset composed of bonds and puts  $S''$ . This replicating portfolio represents the trade required to move from endowment to optimal; this is different from the  $a, b$  calculations where I replicate one unit of stock. New notation for the replicating portfolio is:  $S'' = eB_t + fP_t$ . Calculation of the replicating portfolio follow:

$$S''_t = (h_t - \eta_0)S_t \quad (4.14)$$

$$(eB_t + fP_t) = (h_t - \eta_0)S_t \quad (4.15)$$

$$(eB_t + fKe^{-rT}(1 - N(d_2)) - fS_t(1 - N(d_1))) = (h_t - \eta_0)S_t \quad (4.16)$$

By matching terms with and without  $S$ , I have the conditions and results:

$$eB_t + fKe^{-rT}(1 - N(d_2)) = 0 \quad (4.17)$$

$$-fS_t(1 - N(d_1)) = (h_t - \eta_0)S_t \quad (4.18)$$

$$f = (h_t - \eta_0) \frac{-1}{(1 - N(d_1))} = \left( \frac{\mu - r}{\sigma^2} \frac{X_t}{S_t} - \eta_0 \right) \frac{-1}{(1 - N(d_1))} \quad (4.19)$$

This result determines the allocation to the put option. Next I calculate the allocation to cash in the replicating portfolio, which is used to move the agent from endowment  $\eta_0$  to optimal  $h_t$ .

$$e = -f(K/B_t)e^{-rT}(1 - N(d_2)) \quad (4.20)$$

$$e = (h_t - \eta_0)(K/B_t)e^{-rT} \frac{1 - N(d_2)}{1 - N(d_1)} \quad (4.21)$$

Again, the optimal amount of cash is determined by the replicating portfolio plus the amount by Merton's rule. The Merton rule gives  $g_t B_t = X_t(1 - \frac{\mu - r}{\sigma^2})$  again. The replicating portfolio gives  $eB_t = (h_t - \eta_0)Ke^{-rT} \frac{1 - N(d_2)}{1 - N(d_1)}$  this time. The  $X_t$  does not appear in the  $eB_t$  because this replicating portfolio represents the trade from endowment to optimal. The total amount of cash to be held is when the agent has nonzero endowment of stock is:

$$g_t B_t + h_t a B_t = X_t \left[ 1 - \frac{\mu - r}{\sigma^2} \right] + (h_t - \eta_0) K e^{-rT} \frac{1 - N(d_2)}{1 - N(d_1)} \quad (4.22)$$

#### 4.4 Use of insurance

To understand how farmers use index insurance, it is crucial to understand  $f$  in terms of  $S$ . The parameter  $f$  represents position in the put and  $sign(f) = sign(\eta_0 - h_t)$ , however, this approach is limited because  $h_t$  is a moving target. To determine how the agent will trade the put, I use the formula for  $h_t, X_t$  to revise the expression for  $f$ :

$$f = b(h_t - \eta_0) \quad (4.23)$$

$$f = \frac{-1}{(1 - N(d_1))} \frac{\mu - r}{\sigma^2} \left[ S_0 \left( \frac{S_t}{S_0} \right)^{(\mu-r)/\sigma^2 - 1} e^{\frac{\mu-r}{2\sigma^2}(\mu-r+\sigma^2)t} - \eta_0 \frac{\sigma^2}{\mu - r} \right] \quad (4.24)$$

There is a critical value of stock that determines when the agent will change from buying to selling. The condition is derived from  $f = 0$ :

$$h_t = \eta_0 \rightarrow S_t^{(\mu-r)/\sigma^2 - 1} = (\eta_0/X_0) \frac{\sigma^2}{\mu - r} S_0^{(\mu-r)/\sigma^2} e^{-\frac{\mu-r}{2\sigma^2}(\mu-r+\sigma^2)t} \quad (4.25)$$

$$S_t = \left( \frac{X_0}{\eta_0} \frac{\mu - r}{\sigma^2} S_0^{-(\mu-r)/\sigma^2} e^{\frac{\mu-r}{2\sigma^2}(\mu-r+\sigma^2)t} \right)^{\sigma^2/(\sigma^2 - \mu + r)} \quad (4.26)$$

I expect a farmer to buy insurance when the value of their asset is low. For  $\frac{\mu-r}{\sigma^2} - 1 > 0$ , the  $f$  will be negatively related to  $S$ ; the agent will buy insurance when their asset value is low. This quantity is Merton's rule,  $\frac{\mu-r}{\sigma^2}$  is equal to the proportion of wealth that should be held in the risky asset (Ekeland, 2010). In order for the farmer to buy insurance when their asset value is low, the optimal proportion of wealth in risky asset is greater than one. The interval of values for which the agent buys puts,  $f > 0$ , is defined by  $h_t < \eta_0$ :

$$S_t < \left( \frac{X_0}{\eta_0} \frac{\mu - r}{\sigma^2} S_0^{-(\mu-r)/\sigma^2} e^{\frac{\mu-r}{2\sigma^2}(\mu-r+\sigma^2)t} \right)^{\sigma^2/(\sigma^2 - \mu + r)} \quad (4.27)$$

The replicating argument determines a number of puts the agent should trade. The short formula is  $f = (h_t - \eta_0)b$  where  $b$  is determined by the replicating argument,  $h_t$  is determined by standard theory, and  $\eta_0$  is the exogenous initial wealth. I calculate the  $f$  for a range of values of  $S$  with fixed parameters. The physical interpretation of the parameters is  $\eta_0$  is the number of units in the endowment of risky asset  $S$ ,  $X_0$  is the size of the portfolio available for risk management.

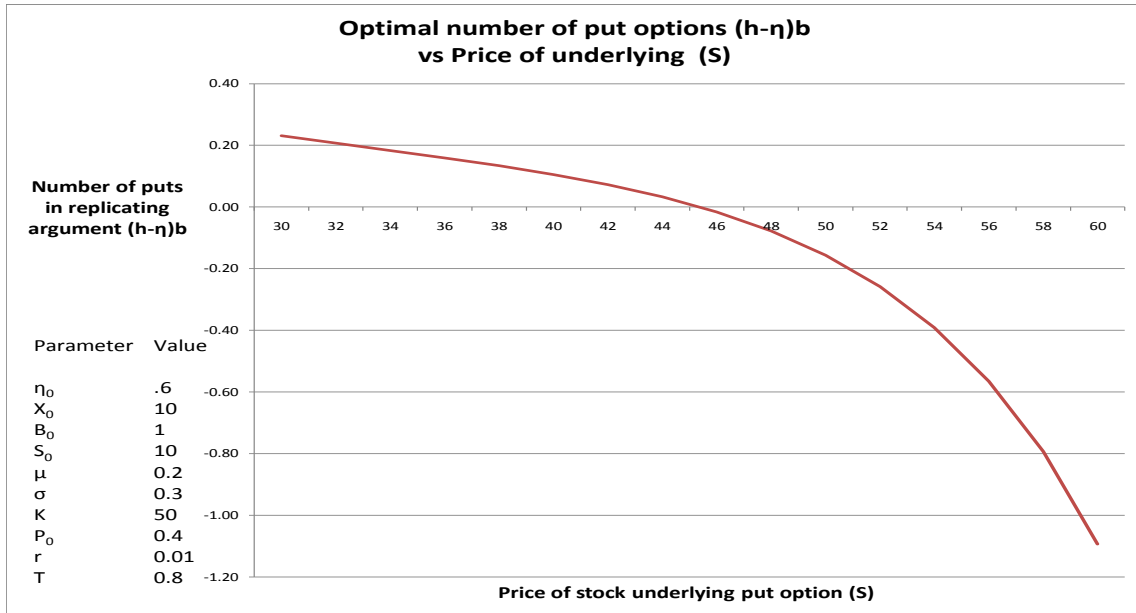


FIGURE 2. Optimal position in put options against underlying price

Recall that  $(\mu - r)/\sigma^2 - 1 > 0$  suggest the agent should buy puts for low  $S$ . In Figure 2  $(\mu - r)/\sigma^2 = 2.11$ , therefore, the calculations verify the analytic predictions. An interesting figure to compare against the graph is the number of units in the endowment,  $\eta_0$ . For  $S \in [30, 54]$  the optimal number of put options is less than the size of the endowment of risky asset. This means the agent is not buying more insurance than they have exposure to risk. For  $S > 56$  we see the number of puts required becomes a large negative number, the agent will benefit by selling put options. This does not match intuition. To further study the diagram it is useful to compare the sign of  $(\mu - r)/\sigma^2 - 1$  and the size of  $\eta_0$ ; the sign of the optimal number of puts and the size relative to  $\eta_0$  are the key predictions from the model.

## 5 ANALYSIS III: CONTINUOUS PROBLEM WITH TWO ASSETS

To create basis risk in the portfolio problem, I use the Black-Scholes model with two assets. The agent is endowed with one risky asset ( $S$ ) and allowed to trade put options ( $P$ ) that are based on the index ( $I$ ); I assume the  $S$  asset is not traded and  $I$  is traded because the index  $I$  is a widely available government statistic. The agent will be endowed with one unit of stock and  $x$  units of cash, the agent trades puts in a self financing portfolio to maximize expected utility.

### 5.1 Asset dynamics

$$dS/S = \mu_1 dt + \sigma_1 dW_1 \quad (5.1)$$

$$dI/I = \mu_2 dt + \sigma_2 dW_2 \quad (5.2)$$

$$\text{corr}(dW_1, dW_2) = E(dW_1 dW_2) = \rho \neq 0 \quad (5.3)$$

To create basis risk in this market, I introduce correlation in the noise of the index and farm value. A first question from these definitions is: what is the correlation between the put and stock? I have  $\text{corr}(dS/S, dI/I) = E(\sigma_1 dW_1 \sigma_2 dW_2) = \sigma_1 \sigma_2 \rho$ . The more intuitive correlation,  $\text{corr}(dS/S, dP) = \sigma_1 \sigma_2 \rho E(I \frac{dP}{dI} dW_1 dW_2)$ , does not have such a convenient expression.

### 5.2 Portfolio problem

$$\max_{X_T} E(\ln(S_T + X_T)) \quad (5.4)$$

$$\text{s.t. } X_0 = E(Z_T X_T), \quad X_T \in F_2, \quad Z_T \in F_2 \quad (5.5)$$

$$S_T = S_0 \exp[(\mu_1 - \sigma_1^2/2)t + \sigma_1 W_1] \in F_1 \quad (5.6)$$

Here  $X_T$  is the amount of money the agent should allocate to the asset  $I$ . I assume that  $I$  trades and  $S$  does not trade; this means  $Z_T = \exp[(-1/2)(\frac{\mu-r}{\sigma})^2 T - \frac{\mu-r}{\sigma} w]$ . Also, I assume the agent knows the value of the index  $I$  when they determine the allocation to the put option at terminal time (to calculate the optimal portfolio before terminal time, it is possible to use a martingale condition on the portfolio). The Lagrange equation for the portfolio problem can be written:

$$\max \left[ \int \ln(S_T + X_T) dP - \lambda \left( X_0 - \int Z_T \frac{X_T}{B_T} dP \right) \right] \quad (5.7)$$

There are two equations given by the first order condition for this problem:

$$E\left(\frac{1}{S_T + X_T} | W_2\right) = \lambda \frac{Z_T}{B_T} \quad (5.8)$$

$$X_0 = E(Z_T X_T) \quad (5.9)$$



There is not an analytic solution to this problem. However, it is possible to propose a numerical scheme. The scheme will use the first order condition, equation (5.8), to solve the optimal distribution of terminal wealth for many values of the index ( $W_2$ ). Then the distribution must be scaled so that it satisfies the budget constraint, equation (5.9).

To start, suppose  $\lambda = 1$ . Draw a value from the random variable called noise in the index,  $W_2 = w$ . Calculate equation (5.8) for many values of  $X_T$  and find the value of  $X_T$  that satisfies equation (5.8). Draw another value for  $W_2$  to calculate the optimal terminal wealth for many values of the index,  $X_T(W_2)$ . In standard model, this distribution of optimal terminal wealth can be solved analytically; here it must be solved numerically.

The budget constraint, equation (5.9), can be rewritten as  $X_0 = \int Z_T(w)X_T(w)f(w)dw$  where  $f(w)$  is the probability density function for the random variable  $W_2$  and  $Z$  is the probability transform for the  $I$  market. The distribution  $X_T(w)$  I calculate by the method described above, will either satisfy equation (5.9) or it will not. If it does, then this is the solution. If not, then I believe it is necessary to repeat the process for different value of  $\lambda$ . This represents the optimal position in the  $I$  asset. The corresponding position in put options can be determined by the replicating argument used before.

## 6 CONCLUSIONS

The classical math finance framework can address a very basic question: how much insurance should an agent buy. I have used techniques from a course with Professor Ekeland (Ekeland, 2010) to address this question in portfolio allocation with derivatives. In a discrete model with binomial stock distribution, I showed the agent intends to over-insure their production; this result is different from the standard result where a risk averse agent pays to have all risk removed from their portfolio (Rees and Wambach, 2008). With the Black-Scholes framework, I presented a replication strategy that allows us to use standard results to determine the optimal investment in insurance. I showed that Merton's rule has a great effect on the agent's use of insurance and the agent under-insures in this setting. I also presented a numerical procedure to calculate optimal allocation to insurance in the continuous model with basis risk.

In reality, insurance contracts are signed out of the money; the agent buys a put option when the value of their asset is high. This is not predicted by the continuous model with one asset when  $\frac{\mu-r}{\sigma^2} > 1$ ; however, it is predicted when  $\frac{\mu-r}{\sigma^2} < 1$ . It is interesting to note that  $\frac{\mu-r}{\sigma^2}$  is the optimal proportion of wealth to hold in the risky asset, Merton's rule (Ekeland, 2010, p. 26). When the agent tries to hold a leveraged portfolio, they sell out of the money puts. When the agent does not have a position in the risky asset that is larger than their initial portfolio (unleveraged), they will buy out of the money put options. This explanation is based on Figure 2 and it is a crucial prediction of this essay.

There are several avenues for future work. It is possible to compare the discrete model presented here with a discrete model from microeconomics of insurance. It is possible to extend the continuous model with two assets, with basis risk, to a situation where  $S$  and  $I$  are both assumed to trade. Another way to build basis risk in this approach is:  $dW_1 = dW_2 + dV$  where  $dV$  represents idiosyncratic farm risk; this form fits in standard theory and can be generalized to several agents. Finally, the replicating argument can be avoided by using the dynamic programming framework, where the put option appears directly in the utility function. It is possible to compare these models based on two criteria: does the agent buy or sell insurance, does the agent trade more or less insurance than their endowment of risky assets.

A strength of this work is the ability to stay within the framework of the course notes (Ekeland, 2010), however, a possible limitation is obscure notation and techniques particular to myself or the Ekeland course. There are other widely discussed limitations to the Black-Scholes framework. I do not anticipate commercial applications of these research findings, however, there are opportunities for business development and research in the areas of agricultural insurance and index insurance. The mathematical finance framework is a large, flexible area that should be used further to provide insight into these into agricultural index insurance.

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## FURTHER READING

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## A AFTERTHOUGHT

### A.1 *Why should a government insure domestic farming?*

Agricultural insurance is a rich area of theory and practice today. In the USA, the Risk Management Agency (RMA) of the Department of Agriculture designs crop insurance and private businesses offer it to agricultural producers (roughly). An effect of government insurance on domestic farm industry is increased profitability in the industry. The statistical distribution of profits within an industry can be depicted, as Figure 3. The graph compares percentage of farm business and amount of operating gains. This distribution may encourage new entrants to enter the industry, or discourage them. I suspect that profitability of an industry positively affects size of the industry.

To increase the profitability of an industry, a government could target the losers. Insurance for farms that incur losses would shift density from losses towards profit. This is shown in Figure 4 with a simple uniform distribution. The figure shows that insurance shifts the distribution of profit in an industry upwards: the profitability of the industry increases.

Suppose a government had goals to increase food security, or reliability of food supply. A good way to achieve this is with a strong domestic food production industry. If increased profitability causes an industry's size to increase, then insurance would be a great way for government to increase the size of an industry. Thus, farm insurance may reflect goals of a government for a large domestic farm industry. Although this discussion is theoretical, the fact is that insurance for farms in the USA exists and this may reflect real thoughts of policy makers.

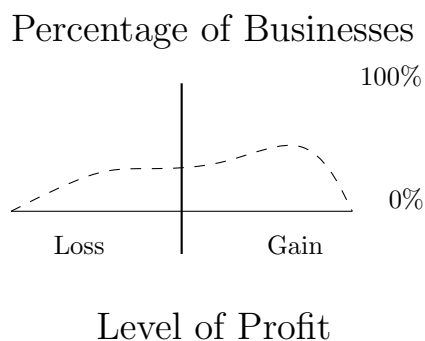


FIGURE 3. Example of distribution of profit across farms

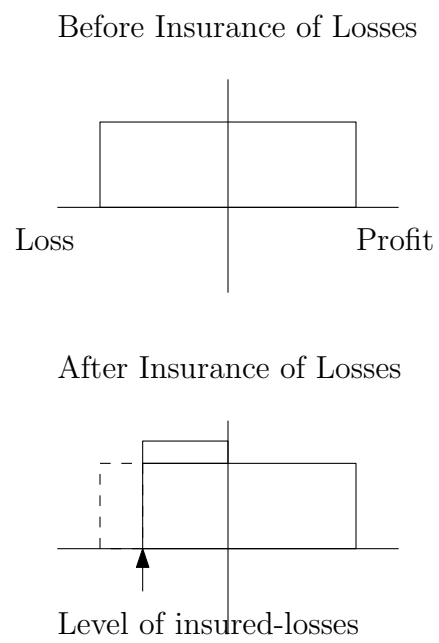


FIGURE 4. Effect of insurance on the distribution of profit in an industry.