STABILIZATION VIA SMOOTH PARTITIONS, TRANSVERSALITY AND GRAPHS

By

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Abstract

With the aim of circumventing the difficulty in constructing Liapunov functions, a strategy for the design of static stabilizing feedback control laws of nonlinear systems is proposed.

The basic method is to partition the state space and to find all controls so that the closed-loop dynamics are transverse or coincident to the partition edges. Stability is analyzed through the directed graph whose vertices are the subsets of the partition and whose arcs are consistent with the transversality. A strategy for the choice of partition is proposed using computable Pfaffian systems.

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Chapter 1

Introduction

Advances in engineering, science and technology have had a profound effect on Canadian society. Popular examples are the tangible advances such as the invention of plastics or the use of computers.

A more pervasive but inconspicuous advance, however, was the invention of the idea of **active control**, for example, feedback. Indeed, feedback is used in walkmans, in telecommunications, for cruise-control and anti-lock brakes in automobiles, in Tokomak reactors, and even in our own physiological processes.

The necessary choice of **mathematical** model and control strategy, however, has a crucial effect on the performance of systems using active control. The classical linear models and control strategies suffice in many instances, but for advanced systems with inherent nonlinearity, or where performance is essential, the more realistic but difficult nonlinear models and strategies are necessary.

In this thesis we propose a control strategy for engineering systems modelled by nonlinear differential equations.

1.1 Systems, Feedback and Design

A typical example of a nonlinear system is a **robot arm**. Such a system must contain **actuator** and **sensor** components to control and observe itself, as in Figure 1.1. In the example of a robot arm, typical actuator and sensor hardware include motors and potentiometers respectively, placed at the joints of the arm. A motor accepts a voltage **input**

Figure 1.1: Components within the plant.

signal to influence the arm, and a potentiometer **outputs** a voltage signal representing some observation of the arm.

As stated, the arm may be influenced, e.g. made to perform a task, by selecting an input signal to relay to the actuator. By augmenting the arm with a **controller** to automatically make this selection, the robot arm may then be made to "control itself". A controller's choice of instantaneous input is often based on the fed back output observed from the sensors. The controller may then be summarized in a **feedback law** which associates a control with an observed output.

Using such a strategy, the **control problem** within system design may be isolated as the choice of the feedback law and the analysis of its influence on the "closed-loop" system. The choice of feedback law is based on a **mathematical input/output** model, called the **plant** P, representing the effect of a control signal u on the output signal y.

Controllers are often implemented using digital computers so that the controller may be reprogrammed easily. It is important to note that it is the mathematically designed software executing on the computer which actually "controls" the system.

An input/output model for systems such as robot arms may be derived via Newtonian, Lagrangian or Hamiltonian mechanics. In the simplest situation, the model is realized by nonlinear but smooth deterministic differential equations in a finite number of variables,

Figure 1.2: A closed-loop interconnection of plant and controller.

Figure 1.3: A compensation interconnection between plant and controller.

that is, a class accommodated within the sequel.

A basic requirement of the closed-loop system is global **stability**: intuitively, that a prescribed orbit, e.g. an equilibrium point, attracts all other orbits, or bounded reference inputs r produce bounded outputs y, respective of Figures 1.2 and 1.3. Other design specifications implicitly assume stability or are essentially implied by stability, for example tracking.

In conclusion, we have isolated within engineering system design the mathematical problem of choosing a feedback law for a system given an Input/Output model realized by a system of nonlinear differential equations so that the closed-loop is stable, i.e. the **Nominal Stabilization Problem**.

Figure 1.4: Orbits converging on level curves of a Liapunov function.

1.2 Liapunov's Method

While there is no general method for the construction of stabilizing feedback control laws of nonlinear systems, the most viable is **Liapunov's method** [AV]. This method is contingent upon the designer being able to find a function satisfying algebraic hypotheses relating the geometry and dynamics of the flow. The deficiency of this method is that there is no algorithm for constructing such a function.

In this thesis, we re-interpret the algebraic "definiteness" hypotheses of Liapunov's method in terms of the geometric notion of transversality to a submanifold. The concepts of partition, transversality and graphs are then used to give a geometric version of Liapunov's method.

Intuitively, the stability relationship between geometry and dynamics in Liapunov's classical method is implied as in Figure 1.4, while for the method of this thesis, by the graph implied by a direction field superimposed over a partition of the state space, as in Figure 1.5.

Differential geometry gives a convenient, if not necessary, language to formulate the above concepts.

Figure 1.5: A partition of the state space and a superimposed direction field together implying a graph.

1.3 Overview

In chapters two and three we review the background for modelling mechanical systems, specifically, control theory fundamentals and geometric mechanics respectively. Some of the mechanical examples of chapter three appear to give a new perspective of exhibiting phenomena discussed only within the modern theory of differential geometry.

Chapter four outlines the stated nonlinear design strategy. The author is not aware of the method appearing anywhere in the literature, but there is some evidence that similarities exist with the theory of hybrid systems [LS]. The aim of this thesis is not to present a definitive algorithm but to initiate the use of certain geometric methods.

In chapter five, strategies are presented for constructing partitions and Liapunov functions. Some mathematical prerequisites are reviewed in the appendices.

Part I

Modelling

Chapter 2

Control Theory Fundamentals

2.1 Engineering Motivation

Consider a programmable robot arm as in Figure 2.6. Such a robot must have **actuators**, for example, simplified motor components as in Figure 2.7.

Question: 2.1.1 *In figure 2.7, what voltage do we apply to the motor, say, to have the arm 2 track a predetermined trajectory?*

This engineering tracking problem may be taken as one motivation for the mathematics of control and system theory.

2.2 States, Signals and Systems

In this section we will contrast the "dynamical system" and "input/output" models discussed in this thesis, using the arm in the previous section as an example.

Figure 2.6: A typical industrial robot.

Figure 2.7: A simplified motor component.

2.2.1 Dynamical Systems

Once programmed, the complete arm is an example of a **mechanical system**, with measurable idealized physical properties such as instantaneous relative angular displacement, velocity and acceleration, mass, etc. The arm is, furthermore, **dynamic**, in that some of the measurements may evolve over time. Depending on the implementation, the arm may be **deterministic**, meaning that if all the physical properties of the arm were measured at one instant, there exist models which could predict all future measurements very well. Associated with such models is the concept of **state**, specifically, the least amount of instantaneous information x which uniquely determines all other measurable physical properties. The set M of all attainable states is called the **state space**.

For the arm above, the set $M := R^2$ of all paired attainable relative angular displacements and velocities at any instant may be taken as the state space, depending upon the implementation of the arm and controller. The evolution of the system is then a path $x_\xi : R \to R^2(= M)$ in the state space, identified by the initial state $\xi \in R^2$ at time 0. We summarize this information by defining the map $\phi : R \times M \to M$ by

$$\phi(t, \xi) := x_\xi(t) \tag{2.1}$$

for all $t \in R$. The map ϕ is called a **dynamical system** or **flow**. The geometric meaning

Figure 2.8: A dynamical system or flow on a plane and its geometric meaning.

of ϕ is, on one hand, that $t \mapsto \phi(t, x_0)$ models the evolution of the arm when initialized at a state $x_0 \in M$, and on the other hand, that $x \mapsto \phi(t_0, x)$ summarizes the evolution of the arm after time t_0 over all initial states $x \in M$. See Figure 2.8.

The dual interpretation of a flow is useful for defining **stability**, asymptotic stability, ω-limits, invariant sets, etc. The reader is asked to refer to [HS] for a discussion of these and further concepts which occur frequently in this thesis.

2.2.2 Input/Output Systems

Before the arm is programmed, however, we must discuss how the external concepts of input, output and causality effect on the internal dynamics of the arm. The voltage applied to the motor, the **input**, and the measured angular displacement and velocity of the arm, the **output**, as vector-valued functions of time, are examples of **signals**. The transformation from input signal to output signal, as in the arm, is called an **input/output system**, or I/O system for short.

In figures, signals and input/output systems are represented by directed lines and black-boxes respectively, as in Figure 2.9, and are often given names like P or C for

Figure 2.9: Graphic representation of an input/output system.

plant (the arm in our case) or controller.

This suggests we should model the algebraic structure of the (initialized) I/O motor-arm-potentiometer system by an operator

$$P : C(R, R) \to C(R, R^2) : v(\cdot) \mapsto \begin{pmatrix} \theta(\cdot) \\ \dot{\theta}(\cdot) \end{pmatrix} \tag{2.2}$$

mapping a continuous voltage signal to a smooth angular displacement and velocity signal. These linear operators are frequently realized by systems of differential equations or, through Laplace transforms, by matrices of complex functions. We will consistently use the former representation in this thesis.

The reader is asked to refer to [Vid] for a discussion of (UBIBO) **I/O stability**, the I/O version of dynamical systems stability: intuitively, that a bounded control signal produces a bounded output signal.

2.3 Feedback

One strategy for implementing control is through the use of **feedback**, that is, basing our choice of instantaneous control on the past and present observed output. Under this

Figure 2.10: A feedback loop interconnecting the plant and controller.

scheme the control is often effected on the plant through a I/O system called the **controller** C connected to the plant as in Figure 2.10 resulting in an autonomous dynamical system exhibiting **closed-loop dynamics**.

Feedback control theory is concerned with the design of the controller C and the analysis of its influence on the closed-loop dynamics.

2.4 Example - The Simple Robot Arm

Feedback is particularly useful for stabilizing an unstable equilibrium of a mechanical system, using the controller to compensate for perturbations and the tendency to move away.

Following [So], consider the vertical robot arm modelled as a point mass m at the end of a rigid massless rod of length l with a motor at the pivot supplying a variable torque u and a potentiometer observing the angular displacement and velocity as in Figure 2.11. Suppose we wish to stabilize the arm against gravity and perturbations into a vertically upward motionless state (say while other robot arm components perform a task).

Assume that the potentiometer measures angular displacements in a continuous fashion. That is, if the arm rotates 2π radians clockwise, then 2π is added to the displacement.

Figure 2.11: A controlled-observed robot arm.

Figure 2.12: Free body diagram of the mass m.

As such, let $\theta \in R$ denote the clockwise angular displacement of the arm from vertically up.

From Figure 2.12 and Newton's second law, the governing equation is

$$ml\ddot{\theta} = mg\sin\theta + \frac{u}{l}, \tag{2.3}$$

or in terms of the state vector $(x_1, x_2) := (\theta, \dot{\theta}) \in R^2$,

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{g}{l}\sin x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix} u. \tag{2.4}$$

The equation (2.4) is a model for the open-loop robot arm and potentiometer: given a control $u(\cdot) : [0, \infty) \to R$ in L^1, (2.4) has a unique solution $x : [0, \infty) \to R^2$ as in

Figure 2.13: Open-loop robot arm.

Figure 2.13 (see [Ar1]).

Using a primitive version of the technique of feedback linearization [Is], we now change control variables to linearize (2.4). Specifically, let

$$\tilde{u} := mlg(-x_1 + \sin x_1) + u \tag{2.5}$$

so that on substitution, (2.4) becomes, in standard form,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix} \tilde{u}. \tag{2.6}$$

As an aside, note that it not always possible to linearize control systems. If we apply a linear feedback law, say

$$\tilde{u}(x_1, x_2) := (-\alpha - \beta) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.7}$$

via Figure 2.10, then the associated closed-loop dynamics is obtained by substituting (2.7) into (2.6). We calculate

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix} (-\alpha - \beta) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.8}$$

Figure 2.14: A robot arm with vertically up-motionless stable.

or

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} - \frac{\alpha}{ml^2} & -\frac{\beta}{ml^2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2.9}$$

We must choose α and β so that (2.9), the closed-loop dynamics, will have have $(0, 0)$ as a stable equilibrium point, i.e. the coefficient matrix must has eigenvalues with negative real part. Necessary and sufficient conditions are easily seen to be

$$\alpha > mgl, \; \beta > 0. \tag{2.10}$$

In conclusion, if we choose any α and β satisfying (2.10), the robot arm of the closed-loop dynamical system in Figure 2.14 will rise to the vertically upward motionless state and return there when perturbed.

2.5 Tracking

Recall the tracking problem of § 2.1. If we model the motor+arm+potentiometer by the plant in Figure 2.13, then the simplest strategy would be to connect P in tandem with an "inverse" Q as in figure 2.15 (derived, say, by substituting $x_1 = r, x_2 = \dot{r}$ into (2.4) and solving for u, where r is the reference to be tracked). There are fundamental math-

Figure 2.15: The open-loop tracking strategy of tandem connection with an inverse.

Figure 2.16: A compensator plant controller interconnection.

ematical and engineering difficulties with this strategy (causality, sensitivity, robustness) [Ka].

Alternatively, the stabilizer of the arm in the previous example may be used to solve the tracking problem motivated in the question of § 2.1. Specifically, if P and C represent the plant and controller of Figures 2.13 and 2.14, then associated with the stable dynamical system in Figure 2.10 is the **compensated** I/O system in Figure 2.16 which takes the desired trajectory $r(\cdot)$ to be tracked as input, properly scaled by ml^2, and outputs the angular displacement and velocity $\theta(\cdot), \dot{\theta}(\cdot)$. It follows from (2.4) that these signals satisfy the differential equation

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} - \frac{\alpha}{ml^2} & -\frac{\beta}{ml^2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + r. \tag{2.11}$$

For example, with α and β chosen so that

$$\frac{g}{l} - \frac{\alpha}{ml^2} = -1, \quad -\frac{\beta}{ml^2} = -2 \tag{2.12}$$

Figure 2.17: A reference input $r(\cdot)$ and the compensated arm's response $\theta(\cdot)$ satisfying (2.11).

and for the reference trajectory

$$r(t) := \begin{cases} 0 & \text{for } t \leq 0 \\ \frac{1}{2}t^2 & \text{for } 0 \leq t \leq \frac{1}{2} \\ \frac{5}{2}t^2 - 2t + \frac{1}{2} & \text{for } \frac{1}{2} \leq t \leq 1 \\ 1 & \text{for } 1 \leq t \end{cases} \tag{2.13}$$

a simulation of the response of the arm is given in Figure 2.17. The stability of this I/O system is asserted in the next section.

2.6 Stabilization within Control Design

In most popular design strategies, for example H^∞ control [DFT] [BB], the assumption is often made that the plant is I/O stable. Thus, to apply these powerful methods to unstable plants we must first compensate them, that is, find a controller K_0 so that

Figure 2.18: A)Compensated I/O system and b)associated dynamical system.

the I/O system $P(I - K_0P)^{-1}$ in Figure 2.18a) is I/O stable. But this I/O system $P(I - K_0P)^{-1}$ would be stable if the closed-loop dynamical system in Figure 2.18b) is asymptotically stable, as argued by theorem B.7.2 of the appendix. Furthermore, **any** stabilizer would suffice as any undesired response could be undone through feedback so no regard need be paid towards the design specifications.

Note that controllers used in H^∞ are typically "observer-based", so we will assume an **input-to-state** model [So] for our plants in the sequel. For example, if the potentiometer in example 2.4 did not observe the complete state, then an **observer** would be used to asymptotically estimate the remaining state variables, and this estimated state would be used in the control law.

In summary, within most control design problems we may isolate the **nominal stabilization** problem: given a plant P find a controller K_0 so that the closed-loop system in figure 2.18b) is asymptotically stable.

In this thesis, we will propose a framework for constructing such a controller for nonlinear but smooth plants P. We will now discuss the plants accommodated within this framework.

Chapter 3

Differential Geometry in Control Theory

3.1 Overview

We now turn to an issue in the modelling of classical mechanical systems, specifically, the structure of the state and control space. This chapter is intended to familiarize the reader with the basics of classical mechanics and differential geometry. The majority of this chapter is not referred to in subsequent chapters, and thus may be skipped on a first reading.

The classical Lagrangian model [Go], for systems without control, is coordinate-based. One chooses n "generalized" coordinates $x_i \in R$, together with associated n "generalized" velocities $\dot{x}_i \in R$, and define a Lagrangian function $L : R^{2n} \to R$ via energy arguments based on physical laws or models. The evolution of the mechanical system is then modeled by the Euler-Lagrange equations

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{x}_i}\right\} - \frac{\partial L}{\partial x_i} = 0 \tag{3.14}$$

for $1 \leq i \leq n$. The form of these differential equations is, most generally,

$$F(x, \dot{x}, \ddot{x}) = 0 \tag{3.15}$$

for a function $F : R^{3n} \to R^n$. Unfortunately, for systems such as the spherical pendulum discussed below, there are some singular states $(x, \dot{x})$ for which (3.15) cannot be solved for $\ddot{x}$ uniquely. The initial-value problem (3.15) with initial condition at one of these points thus has no unique solution.

More importantly, in a punctured neighborhood of these singular states, that is, where we can solve for $\ddot{x}$, say as

$$\ddot{x} = f(x, \dot{x}), \tag{3.16}$$

the vector field f is typically discontinuous. In summary, flow computation is not a well-posed problem in the sense of Hadamard [Ga] for such an example. (Furthermore, numerical approximation of the differential equation in a neighborhood of these points is prone to error.) As this indeterminacy is not observed in physical reality, the classical model of Euler-Lagrange is inappropriate for some problems. The physical states at which this indeterminacy occurs depends, suspiciously, on the choice of coordinates. In fact, we may recover an adequate well-posed model by "combining" two overlapping coordinate charts, with determinacy being implied in at least one coordinate chart, at each physical state. Differential geometry is the conceptual framework for such a model.

It should be noted, however, that there are many other motivations for the use of differential geometry in mechanics. A few are illustrated in the following examples. Important motivations not mentioned here include nonholonomic systems [vW] and the theory of families of linear systems [HK].

In subsequent chapters we will speak of arbitrary subsets defining a partition of the state space and thus we will continue to use the language of differential geometry in the sequel.

Abortive discussions and hints of the following examples and formalism may be found throughout the existing literature. Example §3.3 was taken almost verbatim from [So], but the other examples appear to give an original perspective and motivation. Example §3.5 was especially contrived to exhibit a phenomena hinted at in [Ba]. The formalism §3.6 is intended to be tentative, following [Br].

Figure 3.19: The spherical pendulum

3.2 Example - The Spherical Pendulum

Consider the spherical pendulum as in Figure 3.19, where the joint J has "two-degrees of freedom".

The natural candidates for the generalized coordinates are the spherical coordinates $(\theta, \phi) \in R^2$ in equations (3.17) and in Figure 3.20.

$$\begin{aligned} x &= \cos\theta \sin\phi \\ y &= \sin\theta \sin\phi \\ z &= \cos\phi \end{aligned} \tag{3.17}$$

For a specific configuration (θ, ϕ), the potential energy of the mass m is $P(\theta, \phi) := mg\cos\phi$, up to an additive constant. Differentiating (3.17) with respect to time along a trajectory $(\theta(\cdot), \phi(\cdot))$, we have that the velocity v of the mass m in rectangular coordinates at a specific state $(\theta, \phi, \dot{\theta}, \dot{\phi})$ is

$$v = (-\sin\theta \sin\phi\, \dot{\theta} + \cos\theta\cos\phi\, \dot{\phi}, \cos\theta\sin\phi\, \dot{\theta} + \sin\theta\cos\phi\, \dot{\phi}, -\sin\phi\, \dot{\phi}), \tag{3.18}$$

using the notation $\sin\phi\, \dot{\theta} := (\sin\phi)\, \dot{\theta}$, so the kinetic energy function $\frac{1}{2}m\|v\|_2^2$ may be

Figure 3.20: The spherical coordinate system (3.17).

shown to be

$$K(\theta, \phi, \dot{\theta}, \dot{\phi}) = \frac{1}{2} m (\sin^2 \phi \, \dot{\theta}^2 + \dot{\phi}^2). \tag{3.19}$$

Therefore, the Lagrangian $L = K - P$ is

$$L(\theta, \phi, \dot{\theta}, \dot{\phi}) = \frac{1}{2} m (\sin^2 \phi \, \dot{\theta}^2 + \dot{\phi}^2) - mg \cos \phi. \tag{3.20}$$

Thus, the Euler-Lagrange equations are

$$\begin{aligned} \frac{d}{dt}[\sin^2 \phi \, \dot{\theta}] &= 0 \\ \frac{d}{dt}(\dot{\phi}) - \sin \phi \cos \phi \, \dot{\theta}^2 - g \sin \phi &= 0 \end{aligned}$$

or

$$\begin{aligned} \sin^2 \phi \, \ddot{\theta} &= -2 \sin \phi \cos \phi \, \dot{\theta} \dot{\phi} \\ \ddot{\phi} &= \sin \phi \cos \phi \, \dot{\theta}^2 + g \sin \phi. \end{aligned} \tag{3.21}$$

Notice that any initial-value problem (3.21), $\theta(0) = \theta_0, \dot{\theta}(0) = \dot{\theta}_0, \phi(0) = n\pi, \dot{\phi}(0) = \dot{\phi}_0, n \in Z$ does not have a unique solution as $\ddot{\theta}(0)$ is not well-defined. Indeed, when $\phi = n\pi, n \in Z$, θ has no physical meaning. Furthermore, deleting these hyperplanes

Figure 3.21: The second spherical coordinate system (3.23).

$\phi = n\pi, n \in Z$, we may write (3.21) as

$$\begin{aligned} \ddot{\theta} &= -2\cot\phi\,\dot{\theta}\dot{\phi} \\ \ddot{\phi} &= \frac{1}{2}\sin 2\phi\,\dot{\theta}^2 + g\sin\phi. \end{aligned} \tag{3.22}$$

The right-hand side vector field, through $\cot\phi$, is discontinuous. Thus any numerical solution based on these equations will be very prone to error near these hyperplanes.

Again, the physical points at which indeterminacy occurs depends on the choice of coordinates. In the coordinate system (3.17), these physical states are vertically up or down, of any velocity. By a choice of a second superimposed coordinate system (3.23) say in Figure 3.21,

$$\begin{aligned} x &= \cos\phi_2 \\ y &= \cos\theta_2\sin\phi_2 \\ z &= \sin\theta_2\sin\phi_2 \end{aligned} \tag{3.23}$$

we obtain a second model

$$\begin{aligned} \sin^2\phi_2\,\ddot{\theta}_2 &= -2\sin\phi_2\cos\phi_2\,\dot{\theta}_2\dot{\phi}_2 - g\cos\theta_2\sin\phi_2 \\ \ddot{\phi}_2 &= \sin\phi_2\cos\phi_2\,\dot{\theta}_2^2 - g\sin\theta_2\cos\phi_2. \end{aligned} \tag{3.24}$$

Figure 3.22: A robot arm before a) and after c) winding b).

Numerical simulation of the evolution of the mass m requires both (3.21) and (3.24). We use the former on $\{(\theta, \phi, \dot{\theta}, \dot{\phi}) : |\phi - n\pi| > \epsilon \text{ for all } n\}$ and the later on $\{(\theta_2, \phi_2, \dot{\theta}_2, \dot{\phi}_2) : |\phi_2 - n\pi| > \epsilon \text{ for all } n\}$, for some $\epsilon > 0$, converting between them when one chart ends, via (3.17), (3.23) and their derivatives. This example illustrates the need for a conceptual framework dealing with multiple coordinate systems. This theory exists as the differential geometry of manifolds and vector bundles.

The non-well posed phenomena occurred in this example because the natural state space TS^2, the tangent bundle to the sphere S^2 (see § B.4.1) is not diffeomorphic to the classical Euler-Lagrange state space R^4, with the discontinuity being reflected in equation (3.22). Indeed, now is an appropriate time to review the motivated differential geometric approach to mechanics described in [Ar2] and [Mar].

3.3 Example - The Simple Robot Arm

Following [So], recall the simple robot arm of example of § 2.4 for the scenario of Figure 3.22 where the arm is externally rotated by 2π radians clockwise, $n \in Z$, from equilibrium and released.

In implementations where there is no physical memory in the system of the number of complete revolutions from some reference, we would consider the wound state as returned to equilibrium. The unwinding by 2π radians counterclockwise would be unnecessary, so we seek a strategy different from that in § 2.4. In our current implementation, the natural **configuration** manifold of the arm is the circle S^1 rather than R.

It may be shown that the dynamics/control of the arm is again governed by the differential equations

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \frac{g}{l}\sin\theta + \frac{u}{ml^2} \end{aligned} \tag{3.25}$$

but rather on the state space manifold $TS^1 = S^1 \times R$, with coordinates $\theta \in S^1$, and $\omega \in R$.

As a consequence of theorem B.5.1, however, we have, following [So],

Proposition 3.3.1 *There does not exist a continuous static feedback law $k : S^1 \times R \to R$ stabilizing (3.25). That is, there is no continuous k for which the differential equation*

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \frac{g}{l}\sin\theta + \frac{k(\theta,\omega)}{ml^2} \end{aligned} \tag{3.26}$$

on the manifold $S^1 \times R$ has a point $(\theta_0, 0)$ as a global attractor.

Proof: If such a k did exist, by continuous dependence on initial conditions [Ar1], it would define a continuous flow on $S^1 \times R$. Indeed, the flow is maximally defined by the hypothesis the there is a global attractor. Furthermore, by that hypothesis, $S^1 \times R$ is a domain of attraction of the flow. Thus, by the theorem B.5.1, $S^1 \times R$ would be contractable, which is a contradiction. □

There is no agreement in the literature on how to construct a stabilizer K in view of the previous proposition. We must use discontinuous feedback laws and/or suffice to

Figure 3.23: Closed-loop dynamics (3.25) under (3.27) with three orbits $(\pi, 0), \Gamma_+, \Gamma_-$ not attracted to $(0, 0)$.

attract only a portion of $S^1 \times R$. In regards to the second alternative, as suggested by the damped vertical planar pendulum, any

$$K(\theta, \omega) := \alpha \sin\theta + \beta\omega \tag{3.27}$$

with $\alpha < -mlg$ and $\beta < 0$ will stabilize (3.25) to $(0, 0)$ except for three orbits, as in Figure 3.23.

3.4 Tentative Formalism

In summary, the first example § 3.2 demonstrates that for **some** mechanical systems, the differential geometric framework of [Mar] is necessary. The second example § 3.3 demonstrates that the topological structure of the state space is a **consequence** of the design specifications.

By analogy with the development of linear control theory from the theory of linear differential equations, it seems fitting at this time to speculate on the extension of the

Figure 3.24: A vector field $f \in \mathcal{X}(M)$ defined on a manifold M.

framework of [Mar] necessary to model the previous control examples, following [Br]:

(H1) we model the state and control spaces on differentiable manifolds, say M and U. Thus the state+control space (which consists of the attainable states and the admissible controls while at that state) is modeled on a topological product manifold $M \times U$. For later purposes, we note its trivial bundle structure $\pi_{sc} : M \times U \to M : (x, u) \mapsto x$.

(H2) the effect of the control $u(\cdot) : R \to U$ on the state $x(\cdot) : R \to M$ is modeled by a differential equation

$$\dot{x}(t) = f(x(t), u(t)) \tag{3.28}$$

for some function (section)$f : M \times U \to TM$ (i.e. satisfying $f(x, u) \in T_x M$). The geometric meaning of f (or rather, just a vector field with no controls) is given in Figure 3.24. As regularity hypotheses, we assume that f is continuous and the diagram in Figure 3.25 commutes, where the tangent bundle $\pi_T : TM \to M : X_x \mapsto x$ maps a tangent vector to the point at which it is based at.

The next example, however, does not fit this framework. Although it may appear contrived, it is a simplified idealization of a phenomena observed in the study of satellites.

Figure 3.25: A commutative diagram for the tentative definition.

Figure 3.26: Mass m constrained to move on a manifold N propelling itself by a force in the tangent space.

3.5 Example - Controlled Constrained Motion

Consider a mass m constrained to move on a frictionless embedded 2-submanifold N of R^3. While at $n \in N$, the mass is assumed to be able to apply to itself, i.e. by its own propulsion, any force in T_nN, the tangent plane to N at n. See Figure 3.26.

If we model m by a point, the configuration space is N. We might suspect that the control space is $U := R^2$. Indeed, at each configuration $n \in N$, we must map a control $(u_1, u_2) \in R^2$ to a vector in T_nN, the propulsion force, as in Figure 3.26.

Thus we must define a bijection $\Phi : N \times R^2 \to TN$, that is $\Phi(n, u) \in T_n N$ for all $n \in N, u \in R^2$. This map Φ, coordinating the effect of propulsion control, is used in deriving the governing equations $\dot{x} = f(x, u), x \in TN, u \in R^2$, for the mass m.

The point of this example is that via theorem B.4.1, for N diffeomorphic to say S^2, Φ and thus f cannot be continuous, thus violating hypothesis (H2) above. Thus the state+control space of this example does not have the topological product structure $(TN) \times U$ between state and control spaces. Instead,

Proposition 3.5.1 *The state+control space has the natural rank two non-trivial vector bundle structure $\tilde{\pi} : p^*TN \to TN$ given by the pullback of the natural configuration+control tangent bundle structure $\pi : TN \to N$ under the natural state space tangent bundle projection $p : TN \to N$ onto the configuration manifold.*

Proof: We stated earlier the hypothesis that the set of admissible controls while at a configuration $n \in N$ is $T_n N$. Therefore, the configuration+control manifold has a natural bundle structure $\pi : TN \to N$ where, for any configuration $n \in N$ in the base manifold, the fibre $\pi^{-1}(n) := T_n N$ consists of the admissible controls.

Likewise, the state space has a natural tangent bundle projection $p : TN \to N$ [Mar] onto the configuration manifold so the fact that the set of admissible controls while at a state x depends only on the configuration $p(x)$ implies that the state+control space has a natural pullback vector bundle structure $\tilde{\pi} : p^*TN \to TN$ given algebraically by

$$\tilde{\pi}^{-1}(x) := \pi^{-1}(p(x)) \tag{3.29}$$

$$= T_{p(x)} N, \tag{3.30}$$

which may be read as the set of admissible controls while at a state x is the set of admissible controls while at its configuration $p(x)$.

The non-triviality of the vector bundles follows from the non-triviality of $TN \to N$ for N diffeomorphic to S^2 §B.4.1. □

Thus, for mechanical systems where there does not exist a consistent admissible control space defined across the state space, due to the non-Euclidean topology of the later, one must resort to defining separate admissible control spaces for each coordinate chart of the state space, i.e. trivialize the state+control space.

3.6 Formalism

We now state the formalism and problems motivated by the examples of the preceding sections and chapters. We will only discuss **Input-to-State** models, i.e. where there is no output space, in view of the "observer-based" strategy described in § 2.6.

- The state space is modeled on a finite-dimensional C^∞- manifold M with corners. Typically M is a tangent or cotangent bundle.

- The set of conceivable controls $U(x)$ while at a state x is assumed to have the structure of a Lie group G, with the admissible controls a subset of $U(x)$. The state+control space

  $$P := \bigcup_{x \in M} \{x\} \times U(x) \tag{3.31}$$

  is assumed to have the structure of a principal G-bundle

  $$\pi : P \to M : \{x\} \times U(x) \mapsto x \tag{3.32}$$

  i.e. with typical fibres the (state in product with the) conceivable controls. The notation P should not be confused with the notation P for a plant.

- The law governing the dynamics/control is encoded in a C^∞ map

  $$f : P \to TM \tag{3.33}$$

Figure 3.27: Commutative diagram for $f : P \to TM$

for which the diagram in figure 3.27 commutes.

Geometrically, given a state+control, f specifies a tangent vector based at the state. Thus f defines, **locally**, the usual differential equation

$$\dot{x} = f(x, u). \tag{3.34}$$

We will not discuss the mathematics of this approach, but rather assign meaning to f only when used in a feedback loop as below.

- A (memoryless nonlinearity) feedback control law is a section $\Phi : M \to P$ of the state+control bundle P, i.e. such that $\pi \circ \Phi = 1_M$. Geometrically, at each state $x \in M$, $\Phi(x) \in \{x\} \times U(x)$ specifies a choice of control.

- The closed-loop dynamics associated with exerting a feedback law Φ is encoded in $f \circ \Phi$. Indeed, $f \circ \Phi : M \to TM$ is a section of the tangent bundle to M and thus defines an autonomous differential equation

$$\dot{x} = f \circ \Phi(x) \tag{3.35}$$

and thus the closed-loop dynamical system indirectly.

3.6.1 Nominal Stabilization of Nonlinear Control Systems

Within this context then, the nominal stabilization problem is to find a (memoryless nonlinearity) feedback law $\Phi : M \to P$ such that the vector field $f \circ \Phi \in \mathcal{X}(M)$ of the associated closed-loop dynamics has a globally asymptotically stable equilibrium.

This is the basic problem of this thesis. It will be abbreviated by the subsection number §3.6.1.

3.6.2 Example § 3.5 continued.

As argued before, the configuration space is S^2, the state space has a bundle structure $p : TS^2 \to S^2$, the configuration+control space has a bundle structure $\pi : TS^2 \to S^2$, and thus the state+control space has a pullback bundle structure $\tilde{\pi} : p^*TS^2 \to TS^2$.

Consider an arbitrary local coordinate chart $(U, (X_1, X_2))$ for S^2,

$$(X_1, X_2) : U \to R^2, \tag{3.36}$$

say stereographic projection. The state space has an induced coordinate patch

$$(TU, (X_1, X_2, \frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}))$$

denoted rather by

$$(X_1, X_2, \dot{X}_1, \dot{X}_2) : TU \to R^4. \tag{3.37}$$

Likewise the configuration+control space TS^2 has an induced coordinate patch

$$(TU, (X_1, X_2, \frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}))$$

denoted rather by

$$(X_1, X_2, U_1, U_2) : TU \to R^4. \tag{3.38}$$

The pullback bundle π^*TS^2 has thus an induced coordinate patch

$$(\pi^*TU, (X_1, X_2, \dot{X}_1, \dot{X}_2, U_1, U_2))$$

$$\phi := (X_1, X_2, \dot{X}_1, \dot{X}_2, U_1, U_2) : \pi^* TU \to R^6. \tag{3.39}$$

The double tangent bundle TTS^2 has an induced coordinate chart

$$(TTU, (X_1, X_2, \dot{X}_1, \dot{X}_2, \frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial \dot{X}_1}, \frac{\partial}{\partial \dot{X}_2})).$$

In terms of these local coordinates the evolution of the system is encoded via (H4) above with

$$f \circ \phi^{-1}(x_1, x_2, \dot{x_1}, \dot{x_2}, u_1, u_2) := \dot{x_1}\frac{\partial}{\partial X_1} + \dot{x_2}\frac{\partial}{\partial X_2} + \frac{u_1}{m}\frac{\partial}{\partial \dot{X}_1} + \frac{u_2}{m}\frac{\partial}{\partial \dot{X}_2}$$

as a tangent vector in $T_{\phi^{-1}(x_1, x_2, \dot{x_1}, \dot{x_2})}TS^2$.

Part II

Design

Chapter 4

Smooth Partitions, Transversality and Graphs

4.1 Motivation

In the classical Liapunov method §B.8 for the design of a memoryless stabilizing feedback control law of a nonlinear control system

$$\dot{x} = f(x, u),$$

in the local coordinate form of §3.6.1, we must find a differentiable function $V : M \to R$ with nested compact sub-level sets and a feedback control law $u : M \to U$ such that the closed-loop dynamics converge on the level-sets of V as in Figure 4.28. Intuitively then, the closed-loop dynamics is asymptotically stable.

Separately achieving each hypothesis is relatively easy, but together these hypotheses typically conflict. In the sequel, we will propose a method based on a different relation between geometry and closed-loop dynamics than that given in Figure 4.28. We will refer to geometric concepts, like submanifold and transversality, rather than the classical algebraic concepts such as positive-definiteness.

Admittedly, cleverness is needed to apply this method in practice, so simpler linear examples are used initially to illustrate the method. A genuinely nonlinear system for which standard nonlinear methods appear inconclusive is studied in the next chapter.

It **appears** that the design strategy proposed in this chapter is original, although the cited concepts, such as partitions, transversality, graphs, framing and Liapunov functions, are all based upon classical versions. A related analysis stressing combinatorial issues

Figure 4.28: Level sets of a Liapunov function and the orbits of a closed-loop dynamical system converging on them.

Figure 4.29: An example of a finite-state machine.

over the choice of partition is given in [Hsu].

4.2 Nonlinear Control Systems Modelled as Finite-State Machines

In [So] the claim is (implicitly) made that the finite-state machines (FSMs) of automata theory are instances of control systems. Instead of a differentiable manifold we have a finite set of points $\mathcal{S} := \{A, B, C, \ldots\}$ as state space and instead of a differential equation with controls we have a finite set of transitions, say, $\mathcal{T} := \{A \to B, B \to C, C \to A, \ldots\}$ brought about by inputs (controls) $\mathcal{U} := \{u_1, u_2, u_3, \ldots\}$ to the machine, as in Figure 4.29.

As a rough analogy of the subsequent strategy, we will attempt to model nonlinear

Figure 4.30: a) the direction field of a dynamical system across some triangles, and b) the associated finite state machine.

control systems by FSMs $(\mathcal{S}, \mathcal{T})$. Each state $A \in \mathcal{S}$ will represent a subset $\mathcal{F}(A)$ of the state space on the nonlinear system, and each transition $A \to B$ will be brought about by a control law which forces each state $x \in \mathcal{F}(A)$ into $\mathcal{F}(B)$, as in Figure 4.30.

In this way the sub-level sets of the classical Liapunov theorem are replaced by arbitrary geometric subsets of the state-space and the Liapunov functions are now used to assert the transition from one subset to another. The FSM merely codifies the relation between the geometry and the dynamics of the system. Such a relation is already present in the classical theorem, (as in Figure 4.28) so no conceptual change is proposed here.

One important distinction, however, is that **some** conclusions can be made by working with one (vs. a foliation of) level set(s) of a Liapunov function.

In the first section we will clarify the geometric concepts using definitions. For example, graphs rather than FSMs should be used to model the relation between geometry and dynamics. Furthermore, differential geometry will continue to be used in order to define **transversality**.

4.3 Definitions

In this section we propose definitions used to clarifying the subsequent strategy.

4.3.1 Smooth Partitions

Definition 4.3.1 *Given an n-manifold M with corners §A.2.2, such as the state space of a nonlinear control system, we mean by a* **smooth partition** $\mathcal{P} := \{F_i\}$ *a partition of M into n-manifolds F_i with corners, called* **faces**, *so that $F_i \cap F_j$ is either empty or an m-manifold with corners with $1 \leq m < n$, for each $i \neq j$.*

The **edge set** $\partial\mathcal{P}$ *of $\mathcal{P}$ is defined to be $\{F_i \cap F_j : F_i \cap F_j$ is a $(n-1)$-manifold with corners$\}$, and the elements of this set are called* **edges**.

The smooth partitions are assumed to be locally finite in that only a finite number of faces intersect any bounded subset of M. Each face $F \in \mathcal{P}$ is assumed to be a closed set in the topology of M.

For example, the real line R may be partitioned up into

$$\cdots \cup [-2,-1] \cup [-1,0] \cup [0,1] \cup [1,2] \cup \cdots . \tag{4.40}$$

Note that the neighboring faces $[n, n+1]$ and $[n+1, n+2]$ intersect at the edge $\{n+1\}$. More classical definitions of partition assume the topological or analytic structure of a simplex [Fl] or an analytic stratification [Su] respectively, for the component sets.

4.3.2 Transversality

Definition 4.3.2 *Following [AR], suppose that M is a smooth manifold, E is a codimension 1 embedded submanifold of M without boundary and $f \in \mathcal{X}(M)$ is a continuous vector field on M. We say f is* **transverse** *to E if*

$$span(f(x)) + T_x E = T_x M \tag{4.41}$$

for all $x \in E$. Likewise, we say f is **coincident** *with E if*

$$f(x) \in T_x E \tag{4.42}$$

Figure 4.31: Flows a) transverse to, b)coincident with, and c)neither transverse to nor coincident with an edge.

for all $x \in E$. *See Figure 4.31.*

Definition 4.3.3 *Suppose that* M *is a smooth manifold,* $\mathcal{P}$ *is a smooth partition of* M *and* $f \in \mathcal{X}(M)$ *is a continuous vector field on* M*. We say that the dynamical system defined by* f *is* **framed** *by* $\mathcal{P}$ *if, for every edge* $E \in \partial\mathcal{P}$*,* f *is either transverse to or coincident with the* $(n-1)$*-interior of* E*, exclusively.*

Associated with a smooth framed flow is a disjoint partition of M *into* $\text{int}\mathcal{P}, \partial_T\mathcal{P}, \partial_C\mathcal{P}$ *and* $\partial_{\partial T}\mathcal{P}$ *each consisting of the union of the interior of the faces of* $\mathcal{P}$*, the open edges for which the flow is transverse, the closed edges for which the flow is coincident, and the boundary of transverse edges not intersecting coincident edges.*

4.3.3 Example

Consider the linear dynamical system

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -2x_1 - 3x_2 \end{aligned} \tag{4.43}$$

which has phase portrait in Figure 4.32. As such consider the smooth partition $\mathcal{P} = \{F_1, \cdots, F_{10}\}$ in Figure 4.33.

Figure 4.32: Phase portrait of (4.43).

Figure 4.33: a) A smooth partition framing the flow (4.43) and b) the superposition of the direction field.

Figure 4.34: The graph associated with the flow (4.43) and the smooth partition in Figure 4.33.

Clearly the dynamical system is framed by the smooth partition.

4.3.4 Graphs of Framed Dynamical Systems

Definition 4.3.4 *A smooth partition $\mathcal{P}$ framing a dynamical system defines a graph $\mathcal{G} := (V, T)$. The faces of the smooth partition comprise the set V of the graph, $V := \mathcal{P}$, and two vertices $F_1, F_2 \in V = \mathcal{P}$ have a directed arc $F_1 \to F_2 \in T$ (respectively, an undirected arc $(F_1, F_2) \in T$) between them if*

- **(H1)** *the common edge $E := F_1 \cap F_2 \neq \emptyset$ is a non-empty $(n-1)$-submanifold and*
- **(H2)** *the flow is transverse to the $(n-1)$-interior of E from F_1 to F_2 (respectively is coincident with E).*

4.3.5 Example § 4.3.3 cont'd.

The graph associated with the linear system (4.43) and the smooth partition in Figure 4.33 is given in Figure 4.34. Note that it has both directed and undirected arcs.□

In contrast to § 4.1, the more general concept of a graph is used to analyze the relation between geometry and control. Indeed, in example § 4.3.3 above, the graph 4.34 could not be associated with the state and transitions of a FSM as described in § 4.1, since there exist distinct states in F_5 which evolve into F_6 and F_7 respectively, in violation of the determinism hypothesis of FSMs.

4.3.6 Liapunov Functions

Definition 4.3.5 *Following [La], a function $V : \bar{\Omega} \to R$ of class $C(\bar{\Omega}) \cap C^1(\Omega)$, $\emptyset \neq \Omega \subseteq M$ open, is said to be a* **Liapunov function** *of an autonomous dynamical system defined by a vector field $f \in \mathcal{X}(M)$, if the Lie-derivative $\dot{V} : \Omega \to R$ is nowhere zero, $\dot{V}(x) \neq 0$, for every $x \in \Omega$.*

Recall that the Lie derivative of V with respect to a flow ϕ is given by the derivative of V along the trajectories of ϕ, i.e.

$$\dot{V}(x) := \frac{\partial}{\partial t} V(\phi(t,x))|_{t=0} \tag{4.44}$$

or, using the chain rule, by

$$\dot{V}(x) := dV(x)f(x) \tag{4.45}$$

which does not require an explicit formula for the orbits. Thus $\dot{V} \neq 0$ may be interpreted as saying that V is consistently increasing or decreasing along the trajectories, or alternatively, as saying that the orbits intersect the level curves of V transversely.

The following lemma is basic to the proofs of most Liapunov theorems.

Lemma 4.3.6 *Consider a dynamical system $\phi(\cdot,\cdot) : R \times M \to M$ with forward orbit Γ_+ trapped by a compact set $F \subseteq M$, $\Gamma_+ \subseteq F$. Suppose further that there exists a class $C(F) \cap C^1(\text{int}F)$ Liapunov function $V : F \to R$ on the interior of F, satisfying $\dot{V}(x) < 0$ for all $x \in \text{int}F$. Then $\emptyset \neq \omega(\Gamma_+) \subseteq \partial F$.*

Proof taken from [HS] and [La]: As F is a compact trapping set for Γ_+, we have that $\emptyset \neq \omega(\Gamma_+) \subseteq F$, and that $V \circ \Gamma_+ : R_+ \to R$ is well-defined.

By definition 4.3.5, $V \circ \Gamma_+$ is non-increasing and bounded below (as V is continuous on the compact set F). Thus there exists a $c \in R$ such that

$$V(\Gamma_+(t)) \to c \text{ as } t \to \infty. \tag{4.46}$$

Now for any $y \in \omega(\Gamma_+)$ there exists a sequence $t_n \to \infty$ such that $\Gamma_+(t_n) \to y$ and thus $V(\Gamma_+(t_n)) \to V(y)$, all limits as $n \to \infty$, so by the uniqueness of limits of subsequences and by (4.46),

$$V(y) = c \text{ for every } y \in \omega(\Gamma_+). \tag{4.47}$$

Suppose that there exists a $y \in \omega(\Gamma_+) \cap \text{int}F$. Then as ω-limits are invariant, $\Gamma(y) \subseteq \omega(\Gamma_+)$ so by (4.47), $V(\phi(t, y)) = c$ for all t. Differentiating with respect to time at $t = 0$ we may conclude from (4.44) that

$$\frac{\partial}{\partial t} V(\phi(t, y))|_{t=0} = \dot{V}(y) = 0 \tag{4.48}$$

which contradicts the fact that $\dot{V}(y) < 0$, as $y \in \text{int}F$.

In conclusion $\omega(\Gamma_+) \cap \text{intF} = \emptyset$. □

4.4 Liapunov's Theorem

In this section we propose a version of Liapunov's theorem for asserting the asymptotic stability of a flow via the analysis of its graph associated with a smooth partition.

These results are used in the design of control systems but **not** for the analysis of specific dynamical systems due to the nature of the hypotheses. See § 4.6.

A flow ϕ framed by a smooth partition $\mathcal{P}$ is **face-monotone** on $\mathcal{P}$ if for every face $F \in \mathcal{P}$ there exists a Liapunov function on int$F$.

Theorem 4.4.1 (A version of Liapunov's theorem) *Suppose that $\{\phi_t\}_{t\in R}$ is a dynamical system framed by a smooth partition $\mathcal{P}$ with associated graph $\mathcal{G}$ satisfying the hypotheses*

- **(H1)** *ϕ has an equilibrium point $\bar{x}$,*
- **(H2)** *$\mathcal{G}$ is Q-stable as in definition A.4.1 where Q identifies the vertices associated with faces containing the equilibrium point $\bar{x}$,*

$$Q := \{F \in \mathcal{P} : \bar{x} \in F\},$$

- **(H3)** *ϕ is face-monotone on $\mathcal{P}$, and*
- **(H4)** *for every state $x \in \partial_C \mathcal{P} \cup \partial_{\partial T} \mathcal{P}$, that is on a coincident edge or only on the boundary of transverse edges, either x is attracted to the equilibrium point $\bar{x}$ along $\partial_C \mathcal{P} \cup \partial_{\partial T} \mathcal{P}$ or else there exists a face $F \in \mathcal{P}$ and a $T > 0$ such that $\phi(T, x) \in \mathrm{int} F$,*
- **(H5)** *the only ϕ_1-connected subset of $\partial_C \mathcal{P} \cup \partial_{\partial T} \mathcal{P}$ is the equilibrium point $\bar{x}$ itself, (referring to § B.2)*
- **(H6)** *for unbounded faces $F \in \mathcal{P}$, the associated Liapunov function $V : F \to R$ is unbounded in that $V(x) \to \infty$ as $|x| \to \infty$ in F, but $\dot{V}(x) < 0$ for all $x \in \mathrm{int} F$, and finally*
- **(H7)** *ϕ is smooth and autonomous.*

Then $\bar{x}$ is globally asymptotically stable for ϕ. □

Hypothesis (H6) may be replaced by analogues like

- **(H6)**$'$ for unbounded faces $F \in \mathcal{P}$ the associated Liapunov function $V : F \to R$ is bounded below but $\dot{V}(x) \to -\infty$ as $|x| \to \infty$ in F.

Figure 4.35: Conceivable situations that Liapunov's theorem excludes.

Such algebraic hypotheses could be avoided by partitioning M into an infinite number of bounded faces. The reader should realize that the proof is not immediate as we must exclude situations as in Figure 4.35.

Proof of theorem 4.4.1: Assume (H1)-(H7) all hold and let $x \in M$ be an arbitrary but fixed initial state. We seek to show that $\omega(x) = \{\bar{x}\}$ with the aid of some preliminary lemmas.

Lemma 4.4.2 *If the forward orbit $\Gamma_+(z)$ of an arbitrary state $z \in M$ is trapped by a face $F \in \mathcal{P}$ then $\omega(z) = \{\bar{x}\}$.*

Proof of lemma 4.4.2: By hypothesis (H3), there exists a Liapunov function $V : F \to R$ on int$F$. Without loss of generality, assume $\dot{V} < 0$ on int$F$. By definition 4.3.5 then, it follows that $t \mapsto V(\phi(t,z))$ is non-increasing and thus from hypothesis (H6) that for an arbitrary $\epsilon > 0$,

$$F_\epsilon := \{y \in F : V(y) \leq V(z) + \epsilon\}$$

is a compact trapping set for $\Gamma^+(z)$. Thus from lemma 4.3.6 we may conclude that

$$\emptyset \neq \omega(z) \subset \partial F_\epsilon \subset \partial F.$$

To prove otherwise, suppose that there exists a $y \in \partial_T F \cap \omega(z)$. Then $\Gamma(y) \subset \omega(z)$, as ω-limit sets are invariant sets. By Theorem B.1.1, however, $\Gamma(y) \not\subset F$ so $\omega(y) \not\subset F$, which contradicts $\Gamma^+(z) \subset F$. Thus $\emptyset \neq \omega(z) \subset \partial_C F \cup \partial_{\partial T} F$.

By Theorem B.2.2 of § B.2 and hypothesis (H5) we may thus conclude that $\omega(z) = \{\bar{x}\}$.□

Lemma 4.4.3 *Given a state $y \in M$ and a face $F \in \mathcal{P}$ to which y belongs, either*

A1) $\omega(y) = \{\bar{x}\}$ or

A2) there exists $\tau, \nu \in R$, $0 < \tau < \nu$ and a $F' \in \mathcal{P}$ such that $\phi(\tau, y) \notin F$ and $\phi(\nu, y) \in \mathrm{int}F'$

Proof of lemma 4.4.3: As $t \mapsto \phi(t, x)$ is continuous, F is closed and $\phi(0, y) \in F$ we have either that

i) $\phi(t, y) \in F$ for all $t \geq 0$ and thus (A1) $\omega(y) = \{\bar{x}\}$ by lemma 4.4.2, or

ii) there exists a $\tau > 0$ such that $z := \phi(\tau, y) \notin F$.

To prove otherwise, suppose that $\Gamma_+(z)$ is trapped by $\partial_T \mathcal{P}$. Then as $\partial_T \mathcal{P}$ is a locally finite union of open codimension 1 submanifolds, there exists a $T > 0$ and a unique open transverse edge $E \in \partial_T \mathcal{P}$ such that $\phi(t, z) \in E$ for all $0 \leq t \leq T$. Differentiating with respect to time at $t = 0$ we see that $f(z) \in T_z E$, where f is the vector field defining ϕ. This contradicts the fact that E is a transverse edge. In conclusion,

$$\Gamma_+(z) \cap [\partial_C \mathcal{P} \cup \partial_{\partial T} \mathcal{P} \cup \mathrm{int}\mathcal{P}] \neq \emptyset. \tag{4.49}$$

If $\Gamma_+(z) \cap \mathrm{int}\mathcal{P} \neq \emptyset$ then there exists a $T > 0$ and a $F' \in \mathcal{P}$ such that $\phi(T, z) \in \mathrm{int}F'$ and thus (A2) $\phi(\tau + T, y) \in \mathrm{int}F'$.

If instead

$$\Gamma_+(z) \cap \text{int}\mathcal{P} = \emptyset \tag{4.50}$$

then by (4.49), $\Gamma_+(z) \cap [\partial_C \mathcal{P} \cup \partial_{\partial T} \mathcal{P}] \neq \emptyset$ so there exists a $T \geq 0$ such that $\phi(T, z) \in \partial_C \mathcal{P} \cup \partial_{\partial T} \mathcal{P}$. Then by hypothesis (H4), alternative A1) holds as (4.50) excludes the second alternative of that hypothesis (H4). □

Lemma 4.4.4 *For the previously fixed but arbitrary state $x \in M$ either*

A1) $\omega(x) = \{\bar{x}\}$ or

A2) there exist sequences $(\nu_i)_{i \geq 1}, (\tau_i)_{i \geq 1}$ in R with $\nu_i < \tau_i < \nu_{i+1}$ for all $i \geq 1$ and a sequence $(F_i)_{i \geq 1}$ of faces in $\mathcal{P}$ such that $\phi(\nu_i, x) \in \text{int}F_i$ but $\phi(\tau_i, x) \notin F_i$ for all $i \geq 1$.

Proof of lemma 4.4.4: The proof is by induction on the index i of alternative A2). To start, there exists a face $F \in \mathcal{P}$ such that $x \in F$ as $\mathcal{P}$ is a partition. Lemma 4.4.3 applied at $y = x$, however, implies that either alternative A1) holds or that there exists a $\nu_1 > 0$ and a face $F_1 \in \mathcal{P}$ such that $\phi(\nu_1, x) \in \text{int}F_1$. Assume that we have concluded by induction that either A1) holds or that there exists finite, void or finite, and finite sequences $(\nu_i)_{1 \leq i \leq n}$, $(\tau_i)_{1 \leq i \leq n-1}$ and $(F_i)_{1 \leq i \leq n}$ in R, R and $\mathcal{P}$ respectively satisfying the conclusion A2) for only $1 \leq i \leq n$. Lemma 4.4.3 applied at $y = \phi(\nu_n, x)$, however, implies that either A1) $\omega(\phi(\nu_n, x)) = \omega(x) = \{\bar{x}\}$ or that there exists $0 < \tau < \nu$ and $F_{n+1} \in \mathcal{P}$ such that $\phi(\tau, \phi(\nu_n, x)) \notin F_n$ but $\phi(\nu, \phi(\nu_n, x)) \in F_{n+1}$. In the case of the second alternative then, there exists $\nu_{n+1} := \nu + \nu_n, \tau_n := \tau + \nu_n$ satisfying $\nu_n < \tau_n < \nu_{n+1}, \phi(\nu_{n+1}, x) \in \text{int}F_{n+1}$ and $\phi(\tau_n, x) \notin F_n$.

The proof follows by induction. □

Lemma 4.4.5 *Suppose that there exists an arc* $\gamma := \{\phi(t,z) : 0 \le t \le T\}$*, faces* $F_-, F_+ \in \mathcal{P}$ *and a* $\tau \in R$ *such that* γ *is not closed,* $z \in \mathrm{int}F_-, \phi(T,z) \in \mathrm{int}F_+, 0 < \tau < T$ *and* $\phi(\tau, z) \notin F_-$*. Then* $F_- \neq F_+$ *and there is a directed path in the associated graph* $\mathcal{G}$ *from* F_- *to* F_+*.*

Proof of lemma 4.4.5: By theorem B.1.1 there exists a tubular flow (Ω, Φ) of the flow ϕ with $\gamma \subset \Omega$. Then there exists $x_-, x_+ \in I, y_0 \in I^{n-1}$ such that $\Phi(z) = (x_-, y_0), \Phi(\phi(T,z)) = (x_+, y_0)$.

But the family of cubes $B_r^n(p) := \{q \in R^n : \|q-p\|_\infty < r\}$, $r > 0, p \in R^n$ is a base for the Euclidean topology on R^n. Thus, as Φ is a diffeomorphism, and $\mathrm{int}F_-$ and $\mathrm{int}F_+$ are neighborhoods for z and $\phi(T,z)$ respectively, there exists a $r > 0$ such that

$$B_r^n(\Phi(z)) \subseteq I^n \tag{4.51}$$

$$B_r^n(\Phi(\phi(T,z))) \subseteq I^n \tag{4.52}$$

$$\Phi^{-1}(B_r^n(\Phi(z))) \subseteq \mathrm{int}F_- \tag{4.53}$$

$$\Phi^{-1}(B_r^n(\Phi(\phi(T,z)))) \subseteq \mathrm{int}F_+, \tag{4.54}$$

all as in figure 4.36.

Consequently, define the rectified tube $\Sigma := [x_-, x_+] \times B_r^{n-1}(y_0)$, the tube base $\partial_-\Sigma := \{x_-\} \times B_r^{n-1}(y_0)$, the projection $\pi_- : \Sigma \to \partial_-\Sigma : (x,y) \mapsto (x_-, y)$, and the tubular flow $(\tilde{\Omega}, \tilde{\Phi})$ for ϕ by $\tilde{\Omega} := \Phi^{-1}(\Sigma)$ and $\tilde{\Phi} := \Phi|_{\tilde{\Omega}}$, all as in figures 4.37.

Now the orbits of the induced flow $\tilde{\Phi}_*\phi$ on Σ are parallel to those of the constant flow $(1,0)$. Thus for every $(n-1)$-dimensional coincident edge $E \in \partial_C\mathcal{P}$, we have that $\pi_-\tilde{\Phi}(E \cap \tilde{\Omega})$ is a $(n-2)$-dimensional submanifold with

Figure 4.36: Geometry of a tubular flow around γ.

Figure 4.37: Rectified flow in a subtube from F_- to F_+.

corners in $\partial_-\Sigma$. Likewise, $\partial_{\partial T}\mathcal{P}$ is a a locally finite union of at most $(n-2)$-dimensional submanifolds with corners in M. Thus it may be shown then that

$$D := \pi_-\tilde{\Phi}((\partial_C\mathcal{P} \cup \partial_{\partial T}\mathcal{P}) \cap \tilde{\Omega})$$

is a finite union of at most $(n-2)$-dimensional submanifolds with corners in $\partial_-\Sigma$. By dimensionality arguments, D is a proper subset of $\partial_-\Sigma$. Note that open transverse edges $E \in \partial_T\mathcal{P}$ induce open transverse edges $\tilde{\Phi}(E \cap \tilde{\Omega})$ of the induced flow $\tilde{\Phi}_*\phi$, as Φ is a diffeomorphism.

Let $q \in \partial_-\Sigma \setminus D$ be arbitrary. Then the induced arc

$$\{\tilde{\Phi}_*\phi(t,q) : 0 \leq t \leq T\}$$

intersects only induced interior faces and induced open transverse edges in Σ. Specifically, by the locally finite hypothesis of smooth partitions, there exists a finite sequence $(\tilde{F}_j)_{1\leq j\leq N}$ of faces in $\mathcal{P}$ and a finite sequence $(\mu_j)_{0\leq j\leq N}$ in R satisfying $0 = \mu_0 < \mu_1 < \cdots < \mu_N = T$ and

$$\tilde{\Phi}_*\phi(t,q) \in \tilde{\Phi}(\tilde{\Omega} \cap \mathrm{int}\tilde{F}_j)$$

for $\mu_{j-1} < t < \mu_j$, for all $1 \leq j \leq N$. By the original hypotheses of this lemma, however, we have that $N > 1, \tilde{F}_1 = F_-$, and $\tilde{F}_N = F_+$. To be consistent with the induced flow, we must have that $\tilde{F}_j \to \tilde{F}_{j+1}$ is a directed path in the associated graph $\mathcal{G}$, for each $1 \leq j < N$. In conclusion, there is a non-trivial directed path from F_- to F_+. If $F_- = F_+$, however, then there would be a cycle in the associated graph $\mathcal{G}$, which contradicts hypothesis (H2). □

Lemma 4.4.6 *If alternative A2) of lemma 4.4.4 holds, then $F_i \neq F_{i+1}$ and there is a directed path in the associated graph $\mathcal{G}$ from F_i to F_{i+1}, for each $i \geq 1$.*

Figure 4.38: Geometry to exclude closed trajectories.

Proof of lemma 4.4.6: Let $\gamma_i := \{\phi(t,x) : \nu_i \leq t \leq \nu_{i+1}\}$.

To prove otherwise, assume that γ_i is a closed trajectory, say with period $\sigma > 0$. Set $z := \phi(\nu_i, x)$ and $F_- := F_i$, so $z \in \text{int}F_-$. Choose a $\epsilon > 0$ sufficiently small so that $\phi(-\epsilon, z) \in \text{int}F_-$, and set $T := \sigma - \epsilon$, and $F_+ := F_-$ so we have that $\phi(T,z) \in \text{int}F_+$, and $\gamma := \{\phi(t,z) : 0 \leq t \leq T\}$ is a compact but not closed arc. Finally, $\phi(\tau_i - \nu_i, z) \notin F_-$, sa in figure 4.38. By lemma 4.4.5 applied to γ then, we conclude that $F_- \neq F_+$, which contradicts the construction $F_- = F_i = F_+$. Therefore, the original arc γ_i is not closed.

Thus applying lemma 4.4.5 to the original γ_i, with $z := \phi(\nu_i, x), F_- := F_i, F_+ := F_{i+1}, T := \nu_{i+1} - \nu_i, \tau := \tau_i - \nu_i$, we may conclude that $F_i \neq F_{i+1}$ and that there is a nontrivial directed path in $\mathcal{G}$ from F_i to F_{i+1}. □

Completion of the proof of theorem 4.4.1:

If alternative A2) of lemma 4.4.4 holds then by lemma 4.4.6, there is an infinite directed path in the associated graph $\mathcal{G}$, which contradicts hypothesis (H2). Thus alternative A1), $\omega(x) = \{\bar{x}\}$, must hold. □

4.4.1 Example § 4.3.3 cont'd

Recall the system (4.43) of example § 4.3.3 framed by the smooth partition $\mathcal{P} := \{F_1, \cdots, F_{10}\}$ of Figure 4.33a), and with associated graph $\mathcal{G}$ in Figure 4.34.

Here $Q := \{F_1, F_4, F_5, F_6, F_9, F_{10}\}$ and indeed $\mathcal{G}$ is Q-stable (H2). The flow is face-monotone (H3) as demonstrated by the choice of (linear!) Liapunov functions $V(x_1, x_2) := -x_1$ on $F_1 \cup F_2 \cup F_3 \cup F_4$, $V(x_1, x_2) := 2x_1 + 3x_2$ on F_5, $V(x_1, x_2) := x_1$ on $F_6 \cup F_7 \cup F_8 \cup F_9$, $V(x_1, x_2) := -2x_1 - 3x_2$ on F_{10}. Hypothesis (H6) is clearly satisfied. The flow on the union of coincident edges $(F_1 \cap F_4) \cup (F_2 \cap F_3) \cup (F_6 \cap F_9) \cup (F_7 \cap F_8)$ and $[(F_3 \cup F_4) \cap F_5] \cup [(F_8 \cup F_9) \cap F_{10}]$ is topologically equivalent to the flow $\dot{x} = -x$ on R, so hypotheses (H4) and (H5) are satisfied.

4.5 Liapunov Functions for Control Systems

Definition 4.5.1 *A function $V : \bar{\Omega} \to R$ of class $C(\bar{\Omega}) \cap C^1(\Omega)$, $\emptyset \neq \Omega \subseteq M$ open, is said to be a* **Liapunov function** *of a control system*

$$\dot{x} = f(x, u)$$

if the x-Lie-derivative $\dot{V} : \Omega \times U \to R$ defined by

$$\dot{V}(x, u) := dV(x) f(x, u)$$

has at every state $x \in \Omega$ an admissible control $u = u(x) \in U$ for which it is nonzero:

$$\forall x \in \Omega,\ \exists u \in U \text{ so that } \dot{V}(x, u) \neq 0.$$

4.6 Design of Coincident and Transverse Edges and Face-Monotone Dynamics of Control Systems

In contrast to dynamical systems, one may design the invariant manifolds of a control system and algebraically find the feedback law realizing them. Specifically, consider a nonlinear control problem

$$\dot{x} = f(x, u) \tag{4.55}$$

and a given submanifold E of the state space M defined by $E = \{x \in M : V(x) = V_0\}$ for some known function $V : M \to R$ of class C^1. If $x(\cdot)$ is an arbitrary closed-loop trajectory constrained to satisfy $V(x(t)) = V_0$ for $t \in R$ then differentiating we see by the chain rule and (4.55) that

$$\left.\frac{\partial V}{\partial x_1}\right|_{x(t)} f_1(x(t), u) + \cdots + \left.\frac{\partial V}{\partial x_n}\right|_{x(t)} f_n(x(t), u) = 0. \tag{4.56}$$

Thus if we can solve the system

$$\dot{V}(x, u) = \frac{\partial V}{\partial x_1}(x) f_1(x, u) + \cdots + \frac{\partial V}{\partial x_n}(x) f_n(x, u) = 0, \tag{4.57}$$

say via §B.3, for a $u = u(x) \in U$ for each $x \in E$, then by the implicit function theorem we can define a class C^1 control law $u : E \to U$ realizing the response with E invariant.

Likewise, we can try to find a control law so that the closed-loop is transverse to E by replacing the equality in (4.57) by an inequality.

Finally, suppose we have a Liapunov function $V : F \to R$ for a control system on a face F as in definition 4.3.5. If we can find a class C^1 function $u : F \to U$ such that

$$\dot{V}(x, u(x)) \neq 0$$

for each $x \in F$, then the closed-loop response of the control system under that feedback law u will be face-monotone on F with V as a Liapunov function.

4.7 Graphs of Control Systems

Definition 4.7.1 *Consider a nonlinear control system § 3.6.1 and a smooth partition $\mathcal{P}$ of the state space M. Again we define an associated graph $\mathcal{G} = (V, T)$. Specifically,* *the faces of $\mathcal{P}$ are the vertices V of the graph, and two vertices $F_1, F_2 \in V = \mathcal{P}$ have a directed arc $F_1 \to F_2 \in T$ (respectively an undirected arc $(F_1, F_2) \in T$) between them if*

(H1) *the common edge $E := F_1 \cap F_2 \neq \emptyset$ is a non-empty $(n-1)$-submanifold and*

(H2) *there exists a continuous control law $u|E : E \to U$ defined on E for which the flow is transverse across the (n-1)-interior of E from F_1 to F_2. (respectively is coincident with E).*

When (H2) is met for every edge in $\partial\mathcal{P}$ we say that $\mathcal{P}$ **frames** *the control system.*

Recall that hypothesis (H2) may be checked using the techniques of §4.6.

4.8 Example: the 2-integrator.

Consider the linear dynamical system

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= u \end{aligned} \tag{4.58}$$

and the smooth partition $\mathcal{P} = \{F_1, \cdots, F_{10}\}$ in Figure 4.33a) again. The associated graph is given in Figure 4.39.

For example, on the edge $F_1 \cap F_4$, we have that $W(x_1, x_2) := x_1 + x_2$ is constant. We may thus use this function to study hypothesis (H2) of the definition of 4.7 via §4.6. Specifically, $\dot{W}(x_1, x_2, u) = x_2 + u$, so if we choose $u = u(x) > -x_2$ (respectively $u = u(x) < -x_2$, respectively $u = u(x) = -x_2$) on $F_1 \cap F_4$, the resulting closed-loop dynamics will be transverse across that edge from F_1 to F_4 (respectively, transverse to that edge from F_4 to F_1, respectively coincident with that edge).

Furthermore, on the 1-interior of the edge $F_1 \cap F_2$, $H(x_1, x_2) := x_1$ is constant, and $\dot{H}(x, u) = x_2 > 0$ so any closed-loop dynamics is transverse to that edge from F_2 to F_1.

Again $V(x_1, x_2) := x_1$ is a Liapunov function of the control system for each face.

Figure 4.39: The graph associated with the control system (4.58) and the smooth partition in Figure 4.33.

4.9 A Proposed Method for the Design of Framed Dynamics

Given a nonlinear control system, consider the problem of selecting a feedback control law for which the closed-loop dynamics is stable. When modelled on the examples and graphs discussed in the previous sections, this selection is reflected in the transition from Figure 4.39 for the control system to the stable subgraph in Figure 4.34 for the closed-loop dynamical system.

The first part of the design strategy proposed in this chapter is then, loosely,

1. Select a smooth partition of the state space framing the control system (using the ideas of chapter 5).

2. Construct the associated graph.

3. Select a subgraph $\tilde{\mathcal{G}}$ with exactly one arc (directed or undirected) between faces with a common edge, so that hypothesis (H2) of theorem 4.4.1 is satisfied. If it cannot be satisfied, retry the previous steps.

4. Realize a class C^1 control $k|_{\partial\mathcal{P}}$ on $\partial\mathcal{P}$ as per §4.6 with the transversality/ coincidence of $\tilde{\mathcal{G}}$. For example, if $F_1 \to F_2$ is a directed arc in $\tilde{\mathcal{G}}$ then realize a law $k|_{F_1\cap F_2}$ on $F_1 \cap F_2$ with $F_1 \cap F_2$ a transverse edge of the closed-loop dynamics in the direction $F_1 \to F_2$.

5. Check hypotheses (H4), (H5). If they are not satisfied retry the previous steps.

6. Realize a control law $k|_{\text{int}\mathcal{P}}$ on int$\mathcal{P}$ with face-monotone closed-loop dynamics as per § 4.6. That is, for each face $F \in \mathcal{P}$,

   (a) choose a Liapunov function V of the controls system on int$F$ in the sense of definition 4.5.1 (say using the methods of chapter 5)

   (b) check that $\dot{V}(x, k|_{\partial\mathcal{P}}(x)) < 0$ for all $x \in \partial F$, referring to the control law constructed in step 4, (It is the consistency of parity, rather that the actual sign that is important, so > 0 would be just as good)

   (c) find a control $k|_{\text{int}\mathcal{P}}$ on int$F$ such that the closed-loop dynamics is face-monotone on F with V as a Liapunov function § 4.3.6, say via §4.5.1, and so that
$$k(x) := \begin{cases} k|_{\partial\mathcal{P}}(x) & x \in \partial F \\ k|_{\text{int}F}(x) & x \in \text{int}F \end{cases} \tag{4.59}$$
   is of class C^1, and

   (d) check hypotheses (H3) and (H6), retrying the previous steps as necessary.

7. Apply Theorem 4.4.1.

Admittedly, the choice of smooth partition in step 1 has a crucial effect on the success of later steps in the algorithm. In practice, then, the algorithm would have to be repeated for successive **refinements** of the partition. Note that the presence of redundant edges only complicates the combinatorial problem in step 3.

4.9.1 Example § 4.8 cont'd

We return to examples §4.3.3, §4.3.5, §4.4.1 and §4.8 to illustrate this design strategy. From equation (4.58) and the smooth partition 4.33a), we construct the associated graph in Figure 4.39. Here $Q := \{F_1, F_4, F_5, F_6, F_9, F_{10}\}$. A subgraph with exactly one arc between neighboring faces satisfying hypothesis (H2) is given in figure 4.34. A control law which realizes that subgraph need only satisfy $u < 0$ on the 1-interior of $F_3 \cap F_4$ and $F_5 \cap (F_6 \cup F_7)$, $u > 0$ on $F_8 \cap F_9$ and $F_{10} \cap (F_1 \cup F_2)$, $u = -x_2$ on $x_1 + x_2 = 0$ and $u = -2x_2$ on $2x_1 + x_2 = 0$. For example, $W(x) := x_2$ is constant on $F_3 \cap F_4$, with $\dot{W}(x, u) = u$, so if we want the transition $F_3 \to F_4$ we should have the orbits go against grad$W$, i.e. $\dot{W} < 0$, and thus $u < 0$ is adequate. Likewise, $W(x) := x_1 + x_2$ is constant on $F_1 \cap F_4$, with $\dot{W}(x, u) = x_2 + u$, so if (F_1, F_4) is an undirected arc, i.e. $F_1 \cap F_4$ is a coincident edge, then we should have $\dot{W} = 0$ so $u = -x_2$ is adequate. As argued in § 4.4.1, hypotheses (H4) and (H5) are satisfied.

On the faces $F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_6 \cup F_7 \cup F_8 \cup F_9$ define the Liapunov function $V(x_1, x_2) := x_1$ so $\dot{V}(x_1, x_2) = x_2$. Then regardless of how we interpolate $u|\mathcal{P}$, the closed-loop dynamics will be face-monotone with these Liapunov functions. On the face F_5 define the Liapunov function

$$V(x_1, x_2) := 2x_1 + 3x_2, \tag{4.60}$$

noting that $V \to \infty$ as $x \to \infty$ in F_5. Then the Lie derivative is

$$\dot{V}(x_1, x_2) = 2x_2 + 3u, \tag{4.61}$$

and it may be shown on substitution that consistently $\dot{V} < 0$ on ∂F_5 according to the previous conditions, so we require that $u < -\frac{2}{3}x_2$ on intF_5 to satisfy (H3)&(H6). Likewise for F_{10}. One feedback law satisfying these conditions is $u := -2x_1 - 3x_2$.

Figure 4.40: An attracting invariant manifold which contains an attracting equilibrium point.

4.10 Example–A Liapunov Method.

As an alternative to Figure 4.28, we consider the next simplest relation between geometry and dynamics as in Figure 4.40. This section, though a mere special case of the rest of the thesis, could be interpreted as an independent method.

Consider a nonlinear control system

$$\dot{x} = f(x, u) \tag{4.62}$$

for $x \in M, u \in U$ as in §3.6.1. Suppose that there exists a function $V : M \to R$ with associated Lie derivative

$$\dot{V} : M \times U \to R : (x, u) \mapsto dV(x)f(x, u) \tag{4.63}$$

such that for every state $x \in M$, there are (not necessarily unique) controls $k_+, k_0, k_- \in U$ such that

$$\dot{V}(x, k_+) > 0 \tag{4.64}$$

$$\dot{V}(x, k_0) = 0 \tag{4.65}$$

$$\dot{V}(x, k_-) < 0 \tag{4.66}$$

as realized via §B.3. Then the level set $V^{-1}(0)$ may be made an invariant set by setting $u = k_0(x)$ satisfying (4.65) for $x \in V^{-1}(0)$. Suppose further that for some choice of such k_0, the induced flow on $V^{-1}(0)$,

$$\dot{x} = f(x, k_0(x)), \; x \in V^{-1}(0) \tag{4.67}$$

is globally asymptotically stable to some equilibrium point $\bar{x} \in V^{-1}(0)$. Then a candidate feedback control law is

$$u(x) := \begin{cases} k_-(x) & for \quad V(x) > 0 \\ k_0(x) & for \quad V(x) = 0 \\ k_+(x) & for \quad V(x) < 0, \end{cases} \tag{4.68}$$

where k_-, k_+ satisfy (4.66) and (4.64) respectively, i.e. $\dot{V} < 0$ and $\dot{V} > 0$. Such a control law is easy to implement as it only involves function evaluations, and not function inversions [Is].

4.10.1 Example: the n-integrator.

Consider the n-integrator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= u. \end{aligned} \tag{4.69}$$

Although a stabilizer may be derived almost immediately using classical linear strategies, we use this example to illustrate the method of §4.10. As a candidate Liapunov function we take

$$V(x_1, \cdots, x_n) = \beta \cdot x = \beta_1 x_1 + \cdots + \beta_n x_n. \tag{4.70}$$

The associated Lie-derivative

$$\dot{V}(x, u) = \beta_1 x_2 + \cdots + \beta_{n-1} x_n + \beta_n u \tag{4.71}$$

can be made positive, negative or zero at any $x \in R^n$ provided $\beta_n \neq 0$. Now

$$\dot{V}(x,u) = 0 \Leftrightarrow u = -\frac{\beta_1}{\beta_n}x_2 - \cdots - \frac{\beta_{n-1}}{\beta_n}x_n \tag{4.72}$$

and

$$V(x) = 0 \Leftrightarrow x_1 = -\frac{\beta_2}{\beta_1}x_2 - \cdots - \frac{\beta_n}{\beta_1}x_n \tag{4.73}$$

provided $\beta_1 \neq 0$, and the closed-loop dynamics on the invariant plane (4.73) under the control law (4.72) is given, in the coordinates $x_2, \cdots, x_n$, by

$$\begin{aligned} \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -\frac{\beta_1}{\beta_n}x_2 - \cdots - \frac{\beta_{n-1}}{\beta_n}x_n. \end{aligned} \tag{4.74}$$

By stability analysis for linear dynamical systems, (4.74) will be asymptotically stable if

$$\beta_i = \begin{pmatrix} n-1 \\ i-1 \end{pmatrix}, \tag{4.75}$$

for $1 \leq i \leq n$. Note that $(x_2, \cdots, x_n) \to 0$ as $t \to \infty$ implies $x_1 \to 0$ as $t \to \infty$ while on the plane (4.73). We could now construct **any** law as in (4.68), but rather here we will find a linear feedback law of that form. Specifically, a linear control law $u = k \cdot x = k_1x_1 + \cdots + k_nx_x$ satisfying (4.72) on the plane (4.73) may be shown to be, most generally, of the form

$$u(x) = k_1x_1 + (\frac{\beta_2}{\beta_1}k_1 - \frac{\beta_1}{\beta_n})x_2 + \cdots + (\frac{\beta_n}{\beta_1}k_1 - \frac{\beta_{n-1}}{\beta_n})x_n. \tag{4.76}$$

Substituting into (4.71), we conclude that

$$\dot{V}(x,u(x)) = \beta_1x_1 + \cdots + \beta_{n-1}x_n + \beta_nk_1x_1 + (\frac{\beta_n\beta_2}{\beta_1}k_1 - \beta_1)x_2 + \cdots \tag{4.77}$$

$$+(\frac{\beta_n^2}{\beta_1}k_1 - \beta_{n-1})x_n \tag{4.78}$$

$$= \beta_n\beta_1k_1V(x). \tag{4.79}$$

Consequently, if β_1, β_n are as in (4.75) and $k_1 = -1$, say, then $\dot{V} < 0$ when $V > 0$ and vice-versa.

In conclusion, we have constructed the stabilizing feedback control law

$$\begin{aligned} u(x) &= -x_1 - (\beta_1 + \beta_2)x_2 - \cdots - (\beta_{n-1} + \beta_n)x_n && (4.80) \\ &= -x_1 - \cdots - \begin{pmatrix} n \\ i \end{pmatrix} x_i - \cdots - x_n. && (4.81) \end{aligned}$$

Chapter 5

Construction of Smooth Partitions and Liapunov Functions

In this chapter we discuss an algebraic approach to the coupled problem of constructing smooth partitions and Liapunov functions. To construct and refine smooth partitions, we rather construct submanifolds of codimension 1, transverse to or coincident with the flow, and piece these together so that they become edges of smooth partitions.

The strategy proposed here is to make the edges either isolated level sets of scalar functions or the subsets of state space where there is a change in the structure of the class of all local Liapunov functions. Differential conditions for a general submanifold to be transverse to or coincident with some closed-loop response are used to classify both strategies.

Aside from Frobenius' Theorem, the author is not aware of the ideas presented in this chapter appearing anywhere in the established literature.

5.1 Tentative Construction

As motivated above, consider the problem of determining the class of all local Liapunov functions of a given flow.

The condition that the trajectories of a flow defined by a vector field $f \in \mathcal{X}(M)$ (locally $f(x) = f_1(x)\dfrac{\partial}{\partial x_1} + \cdots + f_n(x)\dfrac{\partial}{\partial x_n}$) are transverse to a hypothetical function may be given in terms of its gradient 1-form ω (locally $\omega(x) = \omega_1(x)dx_1 + \cdots + \omega_n(x)dx_n$), as,

$$\omega(x)f(x) = \omega_1(x)f_1(x) + \cdots + \omega_n(x)f_n(x) \neq 0. \tag{5.82}$$

Thus if we can solve (5.82) for $\omega_1, \cdots, \omega_n$, we have identified candidates for the Liapunov functions through their gradients. Furthermore, "changes" in the class of solutions of (5.82), as one varies x, will identify candidates for edges of smooth partitions.

We are thus led to the problem of recovering scalar functions from their gradients. Alternatively, we seek to recover the level sets of the scalar function, i.e. the Pfaffian problem.

5.2 Pfaffian Systems

5.2.1 A Classical Linear Pfaffian Problem.

In the classical problem [vW], one attempts to recover a (foliation of) codimension 1 submanifolds N, of a larger manifold M, from its normal field, as in figure 5.41.

At every point $x \in M$, M the larger manifold, suppose we have given a normal vector $\omega(x) \in (T_x M)^*$, i.e. a cotangent vector, so ω defines a 1-form. We seek a submanifold N such that

$$\langle \omega(x), T_x N \rangle = \omega(x)(T_x N) = 0, \tag{5.83}$$

that is, $T_x N \subseteq \ker\omega(x)$, for all $x \in N$, say passing through some given $(n-2)$-dimensional submanifold.

A necessary and sufficient condition for existence is afforded by Frobenius' Theorem. For details, see theorem B.6.1. Roughly, if $\omega \wedge d\omega = 0$ then there exist scalar functions $\lambda, V : M \to R$ such that $\omega = \lambda dV$ and $\lambda > 0$, so the submanifolds N are the level sets $\{x \in M : V(x) = V_0\}$.

The Pfaffian systems we will be considering, however, are nonlinear.

Figure 5.41: A normal field and a surface normal to it.

5.2.2 The Nonlinear 'OR' Pfaffian Problem

In this variant of the classical Pffafian problem, one seeks a codimension-1 submanifold N satisfying

$$\omega(T_xN) = 0 \tag{5.84}$$

for **some** $\omega \in \mathcal{N}(x)$ for each $x \in N$, where $\mathcal{N}(\mathrm{x})$ is just a given **set** of co-vectors, $\mathcal{N}(x) \subseteq T_x^*M$. The set-valued function $\mathcal{N}$ is an example of a **multifunction**.

For example, the flow $\dot{x_1} = 1, \dot{x_2} = 0$ is transverse to a 1-d submanifold N of R^2 iff $T_xN \neq span(1,0)$, for all $x \in N$, or equivalently, N satisfies the nonlinear Pfaffian problem above with

$$\mathcal{N}(x) = \{\omega_1 dx_1 + \omega_2 dx_2 : \omega_1 \neq 0\}. \tag{5.85}$$

A 1-form ω satisfying $\omega(x) \in \mathcal{N}(x)$ for all $x \in M$ is said to be a **selection** of the multifunction $\mathcal{N}$.

Existence of solutions to the nonlinear Pfaffian problem may likewise be asserted by Frobenius' Theorem. Specifically, if N is a codimension 1 submanifold of M solving the nonlinear Pfaffian problem, let the associated co-vector in $\mathcal{N}(x)$ satisfying (5.84) be $\omega(x)$, for each $x \in N$. Thus if we have a foliation $\{N_r\}_{r\in R}$ of these codimension-1 submanifolds, we have defined a 1-form ω on all of M. The foliation N_r satisfies the classical linear

Pfaffian problem with that 1-form $\omega(x)$. Examples of this reasoning will be given later.

5.3 Transversality Multifunctions

The following definition is used to classify the transversality of a control system to a submanifold as one varies an admissible control $u \in U$.

Definition 5.3.1 *Let $U \neq \emptyset$ be a set. A function $g : U \to R$ is said to be of* **sign class** $\mathcal{S}_\mathcal{C}$*, where $\emptyset \neq \mathcal{C} \subseteq \{-1, 0, +1\}$, if $\{\mathrm{sgn}\, g(u) : u \in U\} = \mathcal{C}$. We also define the* **transversality classes**

$$'0' := \mathcal{S}_{\{0\}},\ \ '1' := \mathcal{S}_{\{-1\}} \cup \mathcal{S}_{\{+1\}},$$

$$'01' := \mathcal{S}_{\{-1,0\}} \cup \mathcal{S}_{\{0,+1\}},\ \ '2' := \mathcal{S}_{\{-1,+1\}},$$

$$'02' := \mathcal{S}_{\{-1,0,+1\}},$$

as compound sign classes.□

Let $T_x'M := (T_x^*M) \setminus \{0\}$ denote the nonzero covectors.

Definition 5.3.2 *Consider a nonlinear control system*

$$\dot{x} = f(x, u) \tag{5.86}$$

as in § 3.6.1, with U denoting the constraint set of admissible controls. Let $\mathcal{C}$ denote one of the transversality classes above. Define the $\mathcal{C}$**-transversality multifunction**

$$\mathcal{N}_\mathcal{C}(x) := \{\omega \in T_x'M : u \mapsto \omega(f(x,u)) \text{ is of class } \mathcal{C}\}. \tag{5.87}$$

To calculate the transversality multifunctions in concrete examples, it may be useful to define the transversality function

$$\mathcal{T} : (x, u, \omega) \mapsto \omega(f(x,u)) \tag{5.88}$$

and apply a strategy similar to that in §B.3.

5.3.1 Example: the 2-integrator.

Consider the 2-integrator $\dot{x} = f(x,u)$ defined on $M := R^2, U := R$, using canonical coordinates, by

$$f(x_1, x_2, u) := \begin{pmatrix} x_2 \\ u \end{pmatrix}. \tag{5.89}$$

For any $\omega \in T_x^{'}M$, say $\omega = \omega_1 dx_1 + \omega_2 dx_2, (\omega_1, \omega_2) \neq 0$, we have that

$$\mathcal{T}(x, u, \omega) = \omega_1 x_2 + \omega_2 u. \tag{5.90}$$

The function $u \mapsto \mathcal{T}(\omega, x, u)$, for $\omega_2 \neq 0$, is always of class $'02'$, but for $\omega_2 = 0$ is of class $'0'$ or $'1'$ according to whether x_2 is zero or non-zero. Thus the transversality multifunctions are

$$\mathcal{N}_0 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{cases} \{\omega \in T_x^{'}M : \omega = \omega_1 dx_1, \omega_1 \neq 0\} & \text{if} \quad x_2 = 0 \\ \emptyset & \text{if} \quad x_2 \neq 0 \end{cases} \tag{5.91}$$

$$\mathcal{N}_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{cases} \emptyset & \text{if} \quad x_2 = 0 \\ \{\omega \in T_x^{'}M : \omega = \omega_1 dx_1, \omega_1 \neq 0\} & \text{if} \quad x_2 \neq 0 \end{cases} \tag{5.92}$$

$$\mathcal{N}_{02} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \{\omega \in T_x^{'}M : \omega = \omega_1 dx_1 + \omega_2 dx_2, \omega_2 \neq 0\}. \tag{5.93}$$

It is easily shown that the only maximal solutions of the nonlinear 'OR' Pfaffian problem with $\mathcal{N} = \mathcal{N}_1$ are the half-lines $\{L_{\pm,\lambda}\}_{\lambda \in R}$ given by

$$L_{+,\lambda} := \{(\lambda, x_2) : x_2 > 0\}, L_{-,\lambda} := \{(\lambda, x_2) : x_2 < 0\}. \tag{5.94}$$

Again, $L_{\pm,\lambda}$ were constructed to be the edges of smooth partitions. Across these edges the flow is consistently in one direction, as in Figure 5.42.

Figure 5.42: Foliation of a selection of $\mathcal{N}_1$.

5.4 Liapunov Functions

Consider the use of a Liapunov function $V : F \to R$ to assert that some closed-loop response of a control system is face-monotone across a face of a smooth partition. Conditions that a scalar function is indeed a Liapunov function are afforded by the transversality multifunctions.

Proposition 5.4.1 *Consider a nonlinear control system*

$$\dot{x} = f(x, u)$$

with transversality multifunctions $\mathcal{N}_{\mathcal{C}}$. Let F be a nonempty open subset of the state space.

- **(C1)** *If there is a Liapunov function $V : F \to R$ on F then the gradient 1-form dV is a selection of $\mathcal{N}_1 \cup \mathcal{N}_{01} \cup \mathcal{N}_2 \cup \mathcal{N}_{02}$ on F.*
- **(C2)** *Conversely, a selection ω of the multifunction $\mathcal{N}_1 \cup \mathcal{N}_{01} \cup \mathcal{N}_2 \cup \mathcal{N}_{02}$ on F satisfying Frobenius' condition $\omega \wedge d\omega = 0$ defines a Liapunov function on F. Specifically,*

by Frobenius' Theorem, there exist functions $\lambda, V : F \to R$ such that $\lambda > 0$ and $\omega = \lambda dV$ and in this case V is a Liapunov function.

Proof: Suppose that $V : F \to R$ is a Liapunov function. By definition 4.3.5, there exists a feedback law $k : F \to U$ so that

$$\dot{V}(x, k(x)) := dV(x)f(x, k(x)) \neq 0. \tag{5.95}$$

Thus, for every $x \in F$,

$$u \mapsto dV(x)f(x, u) \tag{5.96}$$

is not of class '0', so by definition 5.3.2,

$$dV(x) \in (\mathcal{N}_1 \cup \mathcal{N}_{01} \cup \mathcal{N}_2 \cup \mathcal{N}_{02})(x). \tag{5.97}$$

Conversely, suppose that ω is a selection of the multifunction $\mathcal{N}_1 \cup \mathcal{N}_{01} \cup \mathcal{N}_2 \cup \mathcal{N}_{02}$ on F satisfying Frobenius' condition $\omega \wedge d\omega = 0$. Then by Frobenius' Theorem B.6.1, there exists functions $\lambda, V : F \to R$ such that $\lambda > 0$ and $\omega = \lambda dV$. Now as ω is a selection of the said multifunction, for every $x \in F$ there exists a $u(x) \in U$ such that $\omega(f(x, u(x)) \neq 0$. Thus by the positivity of λ, $\dot{V}(x, u(x)) = dV(x)f(x, u(x)) \neq 0$, i.e. that V is a Liapunov function on F. □

In critical cases, just the existence of sections of $\mathcal{N}_1$ is sufficient.

Corollary 5.4.2 *Suppose that a nonlinear control system framed by a smooth partition $\mathcal{P}$ has a selection of $\mathcal{N}_1$ on a face $F \in \mathcal{P}$ satisfying Frobenius' condition. Then the control system has face-monotone closed-loop dynamics regardless of the control law defined on the interior of the face F.*

Proof: Suppose that such a selection exists. By the proof of the second part of proposition 5.4.1, there exists a function $V : F \to R$ such that $dV(x) \in \mathcal{N}_1(x)$ for all

$x \in F$, so by definition 5.3.2, $\dot{V}(x,u) = dV(x)f(x,u) \neq 0$ for all $x \in F$ and $u \in U$. Thus by definition 4.3.5, V is a Liapunov function for any closed-loop dynamics on F. $\square$.

For example see § 4.8, with face F_1 and the smooth selection $\omega := dx_1$ of $\mathcal{N}_1$.

The hypothesis of § 4.10 may be reformulated as saying that there is a global selection of $\mathcal{N}_{02}$.

5.5 Smooth Partitions

Very few useful comments can be made for deciding on an effective smooth partition of a control system. At least, one strategy for edge construction is, as stated above, to use isolated level sets of Liapunov functions.

Alternatively, in example 5.3.1, there is a change in the structure of the transversality multifunctions along the line $x_2 = 0$. This makes it a candidate for an edge. Off this edge, construction of Liapunov functions or edges of smooth partitions is not affected by the dynamics of the system.

It is instructive to calculate the transversality multifunctions of various control systems to observe the structure of local Liapunov functions.

5.6 Example

Stabilize, about $(0,0)$,

$$\begin{aligned} \dot{x_1} &= x_2, \quad (x_1,x_2) \in R^2 \\ \dot{x_2} &= x_1 u, \quad u \in R. \end{aligned} \tag{5.98}$$

Note that this system does not have relative degree [Is] defined at $(0,0)$, so by Theorem 2.6 of [Is], the state space exact linearization problem is not solvable, i.e., feedback linearization is inconclusive, and this problem is genuinely nonlinear. The author is not aware of a time-invariant static stabilizing feedback control law having been previously derived for this system. It seems virtually impossible, however, to show Liapunov's classical method must fail. Nevertheless, the partition utilized in this example could not be associated with discrete level sets of a classical Liapunov function.

To determine the transversality multifunctions, we again define the $\mathcal{T}$ function

$$\mathcal{T} : (n_1, n_2, x_1, x_2, u) \mapsto n_1 x_2 + n_2 x_1 u, \tag{5.99}$$

for $(n_1, n_2) \neq (0, 0), (x_1, x_2) \in R^2, u \in R$, as per (5.88). Then,

$$u \mapsto \mathcal{T}(n_1, n_2, 0, 0, u) = n_1(0) + n_2(0)u = 0$$

is of class '0', for all $(n_1, n_2) \in R^2$;

$$u \mapsto \mathcal{T}(n_1, 0, x_1, 0, u) = n_1(0) + 0(x_1)u = 0$$

is of class '0', for all $n_1 \in R, x_1 \in R$;

$$u \mapsto \mathcal{T}(0, n_2, 0, x_2, u) = 0(x_2) + n_2(0)u = 0$$

is of class '0', for all $n_2 \in R, x_2 \in R$;

$$u \mapsto \mathcal{T}(n_1, n_2, 0, x_2, u) = n_1 x_2 + n_2(0)u = n_1 x_2$$

is of class '1', for all $n_1 \neq 0, x_2 \neq 0$;

$$u \mapsto \mathcal{T}(n_1, 0, x_1, x_2, u) = n_1 x_2 + 0(x_1)u = n_1 x_2$$

is of class '1', for all $n_1 \neq 0, x_2 \neq 0$;

$$u \mapsto \mathcal{T}(n_1, n_2, x_1, x_2, u) = n_1 x_2 + n_2 x_1 u$$

is of class '02', for all $(n_1, n_2) \neq (0, 0), x_1 \neq 0, x_2 \in R$.

Thus it may be shown that

$$\mathcal{N}_0(x_1,x_2) = \begin{cases} T'_{(x_1,x_2)}R^2 & if \quad (x_1,x_2)=0 \\ \{n = n_1 dx_1 : n_1 \neq 0\} & if \quad x_1 \neq 0, x_2 = 0 \\ \{n = n_2 dx_2 : n_2 \neq 0\} & if \quad x_1 = 0, x_2 \neq 0 \\ \emptyset & if \quad x_1 \neq 0, x_2 \neq 0, \end{cases} \tag{5.100}$$

$$\mathcal{N}_1(x_1,x_2) = \begin{cases} \emptyset & if \quad x_2 = 0 \\ \{n = n_1 dx_1 + n_2 dx_2 : n_1 \neq 0\} & if \quad x_1 = 0, x_2 \neq 0 \\ \{n = n_1 dx_1 : n_1 \neq 0\} & if \quad x_1 \neq 0, x_2 \neq 0, \end{cases} \tag{5.101}$$

$$\mathcal{N}_{02}(x_1,x_2) = \begin{cases} \emptyset & if \quad x_1 = 0 \\ \{n_1 dx_1 + n_2 dx_2 : n_2 \neq 0\} & if \quad x_1 \neq 0. \end{cases} \tag{5.102}$$

The structure of the transversality multifunctions changes on the coordinate axis, so they are candidate edges. Clearly, the nonlinear Pfaffian problem with $\mathcal{N} = \mathcal{N}_0$ does not have any solutions, while for $\mathcal{N} = \mathcal{N}_1$, we have again the two families $\{l_{\alpha,\pm}\}_{\alpha \in R}$ given by (5.94). Likewise, the nonlinear Pfaffian problem with $\mathcal{N} = \mathcal{N}_{02}$ has, amongst others, the foliations $\{h_\beta\}_{\beta \in R}$, $\{c_{r,\pm}\}_{r>0}$,$\{p_{\beta,\pm}\}_{\beta>0}$, where

$$h_\beta := \{(x_1,\beta) : x_1 \in R\}, \tag{5.103}$$

$$c_{r,+} := \{(x_1, \sqrt{r^2 - x_1^2}) : x_1 \in (0,r)\}, \tag{5.104}$$

$$c_{r,-} := \{(x_1, -\sqrt{r^2 - x_1^2}) : x_1 \in (-r,0)\}, \tag{5.105}$$

$$p_{\beta,+} := \{(x_1, x_1^2 + \beta) : x_1 < 0\}, \tag{5.106}$$

$$p_{\beta,-} := \{(x_1, -x_1^2 - \beta) : x_1 > 0\}, \tag{5.107}$$

as in figure 5.43.

Consider the smooth partition in Figure 5.44 constructed from the edges h_0, $l_{0,+}$,$l_{0,-}$ and $p_{2n,+}$, $p_{2n,-}$, $c_{2n+1,+}$, $c_{2n+1,-}$, where $n \geq 0$ in Z, and the associated graph in figure 5.45.

Figure 5.43: Foliations of selections of $\mathcal{N}_{02}$.

Figure 5.44: A smooth partition for (5.98).

Figure 5.45: The graph associated with (5.98) and (5.44).

A subgraph satisfying hypothesis (H2) of Theorem 4.4.1 is given in Figure 5.46. We wish to find a control law so that the closed-loop dynamics is as framed in that graph. By section § 4.6, it need only satisfy the conditions,

$$u(x_1, x_2) = 2x_2 \tag{5.108}$$

on the edges $p_{2n,\pm}$ for $n \geq 0$,

$$u(x_1, x_2) = -1 \tag{5.109}$$

on the edges $c_{2n+1,\pm}$ for $n \geq 0$, and $u(x_1, 0) < 0$ for $x_1 \neq 0$. The first two conditions were obtained by differentiating the defining equations for the p and c submanifolds, and solving for u.

Consider any feedback control law satisfying these conditions. The closed-loop dynamics is face-monotone (H3) by proposition 5.4.2 for the Liapunov function $V(x_1, x_2) := x_1$, which also satisfies hypothesis (H6). Hypothesis (H5) is clear as the flow on $p_{0,+} \cup p_{0,-}$ is topologically equivalent to the flow $\dot{x} = -x$ on R. Hypotheses (H4) is also clear. Therefore, by Theorem 4.4.1, the closed-loop dynamics will be globally asymptotically stable.

Figure 5.46: A stable subgraph of the graph in Figure 5.45.

Chapter 6

Conclusions

All methods for nonlinear systems compromise scope and viability. For example, Liapunov's method [AV] has a very general scope but is difficult to apply in practice, while feedback linearization [Is] limits itself to affine systems yet is almost entirely computable. The method proposed in §4.9 and chapter five for the construction of static stabilizing feedback laws for nonlinear but smooth control systems modelled by a system of deterministic differential equations is also no exception.

The algorithm strictly contains Liapunov's method, and appears to also accommodate

1. I/O systems [So], that is, systems with outputs rather than complete detection of state,

2. systems with non-Euclidean geometry, such as those in chapter three, and

3. discrete-time systems.

A cursory inspection of the proof of theorem 4.4.1, however, will reveal dependency in its hypotheses. Thus there is inefficiency in the intermediate hypotheses checking steps 2-6 of algorithm §4.9. Furthermore, weakened hypotheses and conclusion would permit the design of closed-loop dynamics (3.25)+(3.27) in figure 3.23, where the equilibrium point $(0, 0)$ attracts only a dense set. In conclusion, the author believes that a revision of theorem 4.4.1, reflecting a more efficient algorithm of greater scope is feasible.

In the algorithm's current form §4.9, however, the viability of step 1, the choice of effective partition, is not clear. The author believes that this problem may be partially circumvented by embedding the existing algorithm in a recursive loop, where the partition is refined at each iteration. Such an algorithm would, arguably, constitute a significant improvement over Liapunov's classical method.

Theorem 4.4.1 asserts the global asymptotic stability of flows whose associated vector fields are of class globally C^1. Strictly speaking then, theorem 4.4.1 cannot be used to prove that a **piecewise** C^1 feedback law designed using algorithm 4.9 actually is a stabilizer. I strongly believe, although I have not checked all the details, that this smoothness hypothesis (H7) of theorem 4.4.1 may be weakened to accommodate piecewise C^1 vector fields, that is, algorithm 4.9 may be extended to the design of piecewise C^1 feedback laws.

Finally, the ideas of this thesis could be used to study other problems. For example, the graphs in §4.7 could be used to study tracking directly and not through stabilization.

Appendix A

Prerequisite Mathematical References

It is pedagogically impossible to develop the mathematical prerequisites of this thesis in an appendix, as is standard in geometric control theory texts [Is], [So]. Rather we outline the logistics involved in learning the fundamental ideas of the prerequisites through the following lists, comments and citations. A rather complete encyclopedic reference is [CD].

A.1 Basics

A.1.1 Basic Topology

Refer to [Mu][Si] for definitions of the following concepts which appear repeatedly in various contexts throughout the thesis.

Sets, elements; topological structure on a set: open and closed sets; closure, interior and boundary; basis and neighborhoods; relative topologies; metric spaces; continuity and convergence.

A.1.2 Algebra

Refer to [HK][HS] for definitions of the following concepts.

Vector spaces; operators; tensor products; modules; exterior products; inverses; eigenvalues and eigenvectors; dual and tensor spaces; ideals; groups; group actions.

A.2 Geometric Analysis

A.2.1 Advanced Calculus = Local Differential Geometry

Refer to [CS] for notation, definition and theorems concerning

Vector-valued functions $f : R^n \to R^m$; differentiability; Jacobians; implicit function theorem.

A.2.2 Manifolds

Refer to [Bo][vW] for definitions concerning

Topological manifolds; coordinate charts; differentiable manifolds, atlases and compatibility; submanifolds, immersions, embeddings and codimension; Lie groups; mappings between manifolds.

Note that the basic geometric structures of this thesis are differentiable manifolds with corners. The distinction is that each point has a neighborhood which is homeomorphic to a relatively open subset of $[0, +\infty)^n$. See [KS][Do].

A.2.3 Bundles

Refer to [vW][CD] for definitions, examples and theorems concerning

Product manifolds; vector bundles; vector bundle morphisms; group actions; principal fibre bundles; fibre bundle morphisms; bundle of frames; connections; sections of bundles; pullback of bundles.

A.2.4 Algebraic Structures over Manifolds

Refer to [vW][Fl].

Tangent spaces, cotangent spaces tensor spaces; tangent cotangent and tensor bundles; vector fields; differential forms; differentiation as a bundle map.

A.2.5 Topological Dynamics

Refer to [Ar1][HS][Wi].

Vector fields defining differential equations on manifolds; Lie derivatives; flows; equilibrium points; linearization; stability, asymptotic stability; invariant manifolds.

A.2.6 Exterior Differential Systems

Refer to [vW].

Forms; Exterior differential systems; codistributions; integral-manifolds; exactness and closedness; integrability; ideals.

A.2.7 Geometric Mechanics

Refer to [Go][Ar2][Mar].

State; Newtonian, Lagrangian, and Hamiltonian mechanics; modelling state spaces by vector bundles and governing equations by differential equations on manifolds.

A.2.8 Functional Analysis

Refer to [NS]

Hilbert and Banach spaces; L^p spaces; linear operators; nonlinear operators; inverses; realization by differential equations.

A.3 Control and System Theory

Refer to [So][Ka][Og].

Linear systems; signals, systems and states; feedback; input and output; closed-loop; stabilization and compensation; tracking; design specifications; controllers; Pole-placement.

A.4 Miscellaneous

A.4.1 Graph Theory

Refer to [Lu][Ha] for definitions, examples, notation and algorithms concerning Directed and undirected graphs; arcs, vertices; directed paths; tree-growing algorithms.

In this thesis we propose a graph theory version of "attractivity":

Definition A.4.1 *A graph $\mathcal{G} := (V, T)$ with directed and undirected arcs and a finite set of distinguished vertices $Q \subseteq V$ is said to be* Q-**stable** *if*

(H1) there is no infinite directed path in V. That is, there does not exist a sequence $\{t_i\}_{i\geq 1}$ of directed arcs in T and vertices $\{\alpha_i\}_{i\geq 1}$ in V, such that $\alpha_{i-1} \xrightarrow{t_i} \alpha_i$, for all $i \geq 1$, and

(H2) for every vertex α in $V \setminus Q$, there is a vertex β in V such that $\alpha \to \beta$ is a directed arc in T.

Intuitively, Q is the "global attractor" for such a graph as each directed path is finite and (every continuation) terminates at a vertex in Q. Note that hypothesis (H1) excludes cycles and paths diverging away from Q.

A.4.2 Automata Theory

Refer to [So].

Finite-state machines; state, input and transitions; determinism.

A.4.3 Multifunctions

A function $\mathcal{N} : M \to \mathcal{P}$ such that the value $\mathcal{N}(x)$ of the function at a point $x \in M$ is actually a **set** is called a **multifunction**. Thus the range $\mathcal{P}$ of $\mathcal{N}$ is a set of sets. If

$\mathcal{N}(x) \subseteq W$, for every $x \in M$, that is $\mathcal{P}$ is a subset of the power set of W, then we write $\mathcal{N} : M \rightrightarrows W$.

Appendix B

Cited Theorems

B.1 Long Tubular Flows

Following [PdM], a **tubular flow** for a vector field $X \in \mathcal{X}^r(M)$ on a smooth manifold M of dimension n is a pair (Ω, Φ) where Ω is an open set in M and Φ is a C^r diffeomorphism of Ω onto a cube $I^n := I \times I^{n-1} = \{(x, y) \in R \times R^{n-1} : |x| < 1, \|y\|_\infty < 1\}$ which take trajectories of X in Ω to the straight lines $I \times \{y\} \subset I \times I^{n-1}$.

Theorem B.1.1 (Long Tubular Flow [PdM]) *Let $\gamma \subset M$ be an arc of a trajectory of X that is compact and not closed. Then there exists a tubular flow (Ω, Φ) of X such that $\gamma \subset \Omega$, as in figure B.47.*

Since the transversality of a vector field X to a submanifold E is preserved under diffeomorphism, it follows from the Long Tubular Flow theorem that qualitatively all such configurations look as in figure B.48.

Figure B.47: A flow ϕ rectified by a diffeomorphism Φ.

Figure B.48: Rectified flow past a transverse submanifold.

Figure B.49: E attracts M and $\bar{x}$ attracts E.

B.2 Flow-Connectedness

Recall from the proof of theorem 4.4.1 that we had a local invariant submanifold E of M such that for every orbit Γ_M in M, $\omega(\Gamma_M) \subseteq E$, while for every orbit Γ_E in E, $\omega(\Gamma_E) = \{\bar{x}\}$ [figure B.49]. Then, using additional hypotheses on the flow on E we concluded that for every orbit Γ_M in M, $\omega(\Gamma_M) = \{\bar{x}\}$.

Additional hypotheses are indeed necessary as demonstrated in the counter-example

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= 2x_1 - 3x_1^2 - \mu x_2(x_1^3 - x_1^2 + \tfrac{1}{2}x_2^2) \end{aligned} \tag{B.110}$$

$$M := \{(x_1, x_2) \in R^2 : x_1 > 0,\ H(\frac{2}{3}, 0) < H(x_1, x_2) \leq 0\} \tag{B.111}$$

Figure B.50: A flow with an attracting homoclinic orbit.

$$E \;:=\; \{(x_1, x_2) \in R^2 : x_1 > 0,\ H(x_1, x_2) = 0\}, \tag{B.112}$$

where

$$H(x_1, x_2) := \frac{1}{2}x_2^2 + x_1^3 - x_1^2 \tag{B.113}$$

as illustrated in figure B.50 where $\omega(\Gamma) = E$ for every orbit Γ in $M \setminus E$, when $\mu > 0$.

In the case that E is 2-dimensional, we could use the hypothesis that the only cycle-graph in E is $\{\bar{x}\}$ itself and apply the fundamental theorem on ω-limit sets. A higher-dimensional analogue is afforded in "flow-connectedness". Specifically,

Definition B.2.1 ([DF]) *Suppose that X is a compact metric space and $T : X \to X$ is a homeomorphism. We say that X is T-connected if for every closed proper subset A of X we have that*

$$TA \cap \overline{CA} \neq \emptyset, \tag{B.114}$$

where $\overline{CA}$ denotes the closure (in X) of the compliment of A in X.

Theorem B.2.2 ([DF]) *A necessary and sufficient condition that X be an ω-limit set under a flow $\{\phi_t\}_{t \in R}$ is that X be ϕ_1-connected.*

B.3 Location of Zeros

Given a set $M \subseteq R^n$, a subset $E \subseteq M$ defined as the level set of a differentiable function $V : M \to R$, $U \subseteq R^m$, consider the problem, as in §4.6 and §4.10, of finding, if possible, for each $x \in E$, values $u_-, u_0, u_+ \in U$ such that

$$f(x, u_-) < 0, \tag{B.115}$$

$$f(x, u_0) = 0, \tag{B.116}$$

$$f(x, u_+) > 0. \tag{B.117}$$

Aside from ad-hoc reasoning, this problem may be effectively solved on a computer. Specifically, discretize E and U, say via some $\{x_\lambda\}_{\lambda\in\Lambda} \subset E$ and $\{u_\sigma\}_{\sigma\in\Sigma} \subset U$ respectively. Then, for each $\lambda \in \Lambda$, if

$$\{\text{sgn} f(x_\lambda, u_\sigma) : \sigma \in \Sigma\} \tag{B.118}$$

is $\{-1\}, \{0\}, \{+1\}, \{-1, 0\}, \{0, +1\}$ we speculate that we can solve (B.115), (B.116), (B.117), (B.116)&(B.116), and (B.116)&(B.117) only, respectively, for $x = x_\lambda$ with adequate values u_-, u_0, u_+ constructed automatically. If (B.118) is only $\{-1, +1\}$ then by the implicit function theorem, (B.115) (B.116) and (B.117) can be solved, and an adequate value of u_0 may be realized by Newton's method. The interpolation of the values u_0 from the discretization follows from the implicit function theorem.

B.4 Non-triviality of the Tangent bundle to the 2-Sphere

Proposition B.4.1 *The tangent bundle* $\pi_T : TS^2 \to S^2 : X_n \mapsto n$ *(*$X_n \in T_nS^2, n \in S^2$*) is not trivial. That is, there is no continuous bijection* $\Phi : S^2 \times R^2 \to TS^2$ *for which* $\pi_1 = \pi_T \circ \Phi$*, where* $\pi_1 : S^2 \times R^2 \to S^2 : (n, u) \mapsto n$*.*

For a proof see any modern differential geometry text such as [vW].

B.5 Contractability of Domains of Attraction

We paraphrase Theorem 4.8.14 from [So].

Theorem B.5.1 *A domain of attraction of a continuous flow $\phi : R \times M \to M$ on a topological manifold must be a contractable set. By continuous we mean that ϕ is a continuous function.*

B.6 Frobenius' Theorem

We paraphrase [vW].

Theorem B.6.1 *Suppose that M is a differentiable manifold on which we are given a one-form $\omega : U \to T^*U$ defined some non-empty open subset of M. Then there exists a scalar function $f(\cdot) : U \to R$ such that*

$$N_k := \{x \in U : f(x) = k,\ df(x) \neq 0\}$$

are integral manifolds for ω for each $k \in \mathrm{range} f$, (that is

$$T_x N_k \subseteq \ker \omega(x)$$

for every $x \in N_k$,) if, and only if,

$$\omega \wedge d\omega = 0$$

on U.

B.7 Input-Output Stability via Asymptotic Stability

Following [Vid], consider an Input-Output system

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)) \\ y(t) &= g(t, x(t), u(t)) \end{aligned} \tag{B.119}$$

with 0 as an equilibrium point, $f(t,0,0) = 0$ and $g(t,0,0) = 0$ for all $t \in R$, $f|_{u=0}$ is of class C^1 and f and g are locally Lipschitz at $(x,u) = (0,0)$.

Definition B.7.1 *The system (B.119) is said to be* **small signal L^p-stable with finite gain and zero bias** *if there exists constants $r_p > 0$ and $\gamma_p < \infty$ such that*

$$x(0) = 0, \|u(t)\|_p \leq r_p, u \in L^p \tag{B.120}$$

$$\Rightarrow y \in L^p, \|y\|_p \leq \gamma_p \|u\|_p. \tag{B.121}$$

Theorem B.7.2 *Suppose that $x = 0$ is an exponentially stable equilibrium point of the unforced system*

$$\dot{x}(t) = f(t, x(t), 0). \tag{B.122}$$

Then (B.119) is small signal L^p-stable with finite gain and zero bias for each $p \in [0, +\infty]$.

B.8 LaSalle's Invariance Principle

Definition B.8.1 *A* **Local Liapunov function** *for a nonlinear control system (with state space modelled on a differentiable manifold M) relative to an equilibrium point x^0 is a continuous function $V : M \to R$ for which there exists a neighborhood $\mathcal{O}$ of x^0 such that the following properties hold:*

1. *V is* **proper** *at x^0, that is, the sub-level set $\{x \in M : V(x) \leq \epsilon\}$ is a compact subset of $\mathcal{O}$ for each $\epsilon > 0$ small enough.*
2. *V is* **positive definite** *on $\mathcal{O}$, that is, $V(x^0) = 0$ and $V(x) > 0$ for each $x \in \mathcal{O}, x \neq x^0$.*
3. *For each initial state $\xi \in \mathcal{O}$ there exists a response $x(\cdot) : R_+ \to M, x(0) = \xi$ of the control system to some admissible control such that $t \mapsto V(x(t))$ is non-increasing and non-constant on the interval where $x(t) \in \mathcal{O}$.*

The function V is furthermore, a **global Liapunov function**, *if it satisfies (2) and (3) above with $\mathcal{O} = M$ and if V is* **globally proper**, *that is, the sub-level set $\{x \in M : V(x) \leq L\}$ is compact for each $L > 0$.*

Theorem B.8.2 *If there exists a local (respectively global) Liapunov function of a nonlinear control system relative to an equilibrium point x^0, then the control system is locally (respectively globally)* **asymptotically controllable**. *That is, for each neighborhood $\mathcal{V}$ of x^0 there exists a neighborhood $\mathcal{W}$ of x^0 such that each $x \in \mathcal{W}$ can be asymptotically controlled to x^0 without leaving $\mathcal{V}$ (respectively and also every $x \in M$ can be asymptotically controlled to x^0.) We say z can be* **asymptotically controlled to** y **without leaving** $\mathcal{V}$ *if there exists an admissible response $x(\cdot) : R_+ \to M$ such that $x(0) = z$,*

$$\lim_{t\to\infty} x(t) = y$$

and $x(t) \in \mathcal{V}$ for all $t \geq 0$.

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