

Term Structures of Defaultable Bonds

by

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Abstract

In this essay we introduce three types of credit pricing models: first and second generation structural-form models and reduced-form models with relative papers and their main results. This paper mainly focuses on the reduced-form credit pricing model by explaining Duffie and Singleton's paper *Modeling Term Structure of Defaultable Bonds* (1999). A detailed explanation of the paper's deductions and motivations is presented. We discuss the paper's drawbacks and summarize the relative improvements made by recent papers. In the end, we make applications to Duffie and Singleton's model and solve for explicit pricing formulas of defaultable bonds in different cases.

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Chapter 1

Introduction

In this chapter, we will first review the basic methods of pricing nondefaultable claims and defaultable claims. Then, we will introduce three types of credit pricing models: first and second generation structural-form models and reduced-form models with relative papers and their main results.

1.1 Nondefaultable Bonds

Definition: A zero coupon bond with maturity date T is a contract that guarantees to pay the bond holder Z dollars at time T . The face value Z is usually substituted by 1 for computational convenience. Denote the price of a bond with maturity date T at time t as $p(t, T)$ and $p(t, t) = 1$ for all t .

Note:

- (1) For fixed t , $p(t, T)$ is a function of T . We assume the function is differentiable w.r.t. T .
- (2) For fixed T , $p(t, T)$ turns out to be a stochastic process.

With observable prices of zero-coupon bonds in the market, we can define interest rates.

1.1 Nondefaultable Bonds

Definition: The instantaneous forward rate $f(t, T)$ with maturity T , contracted at time t , is defined by

$$f(t, T) = -\frac{\partial \log p(t, T)}{\partial T}. \quad (1.1)$$

The instantaneous short rate $r(t)$ at time t is defined as

$$r(t) = f(t, t).$$

$r(t)$ is the interest rate at which a person can borrow money for an infinitely small period of time starting from time t .

Some simple transformations of Equation (1.1) yield

$$p(t, T) = \exp\left[-\int_t^T f(t, s) ds\right].$$

With the definition of instantaneous forward rate $f(t, T)$, people want to model zero coupon bond's price process in arbitrage free world so that they can decide on a reasonably arbitrage free price when trading bonds.

It is natural to think of short rate of interest $r(t)$ as a stochastic process. We assume that, under real probability measure P , $r(t)$ is driven by

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))d\bar{W}(t), \quad (1.2)$$

where \bar{W} is a Brownian Motion under P measure. We also assume there exists an exogenously given asset whose price B is driven by

$$dB(t) = r(t)B(t)dt.$$

1.1 Nondefaultable Bonds

However, the price of a bond can not be uniquely determined by the r -dynamics in Equation (1.2) and the condition of arbitrage free because of a lack of enough underlying assets. In order to construct an arbitrage-free bond market, we need to set the price of a “benchmark” bond as given according to the **Meta-theorem** [refer to Bjork (2003) (7)] then the prices of all the other bonds will be fully determined.

Using a usual deduction method which include the application of Ito’s formula and a two-bond portfolio, Bjork (2003) (7) shows that, in an arbitrage-free bond market, there exists a process $\lambda(t)$ such that

$$\frac{\alpha_T(t) - r(t)}{\sigma_T(t)} = \lambda(t)$$

holds for all t and every choice of T . Here $\alpha_T(t)$ and $\sigma(t)$ satisfy the following equation for bond price P :

$$dP^T = P^T \alpha_T dt + P^T \sigma_T d\bar{W}.$$

As a result, P^T satisfies the term structure equation

$$\begin{cases} P_t(t, T) + (\mu - \lambda\sigma)P_r(t, T) + \frac{1}{2}\sigma^2 P_{rr}(t, T) - rP(t, T) = 0, \\ P(T, T, r) = 1. \end{cases}$$

We specify λ exogenously.

By **Feynman-Kac Theorem**, under a risk neutral probability measure Q , the price of a zero-coupon bond with maturity T at time t is given by

$$P(t, r; T) = E_{t,r}^Q[e^{-\int_t^T r(s)ds}]. \quad (1.3)$$

1.2 Defaultable Bonds

The price of a zero-coupon bond is the expected present value of 1 dollar under Q measure, discounted by stochastic process interest rate $r(t)$.

Note that unlike Black-Scholes, the risk-neutral measure Q is not unique. There is a one-on-one relationship between λ and Q , which is determined by market. We need to use market data to decide the value of λ .

In literature, we turn to specify r -dynamics directly under risk-neutral measure Q . Models like Vasicek model which we are going to use later in Chapter 3 are all under Q measure.

1.2 Defaultable Bonds

Zero-coupon bond issuers are supposed to pay back bond holders the face value at maturity date. In reality, however, for private companies and some countries such as Iceland who are under the threat of bankruptcy, there is no guarantee that the bondholders can get all their face value back at the end. Therefore, people would be unwilling to pay as much as undeflatable bonds. It is natural for people to lower the price of a defaultable bond. The amount by which the price of a defaultable bond is lower than an otherwise equivalent undeflatable bond is called a margin which depends on the trust of people towards the bond issuer. For companies who have an AAA credit rating, the margin is relatively small. Indeed, decreasing the price of defaultable bonds is equivalent to increasing the interest rate of borrowing money.

There are three main factors that affect the credit risk: default rate, recovery rate and the exposure at default. Many models for defaultable bonds take recovery rates and default rates into account. In Chapter 2, we will introduce a paper written by Duffie and Singleton (1999) (13) where they introduce a reduced-form credit pricing model. They increase the

1.3 Three types of credit pricing models

interest rate $r(t)$ in Equation (1.3) in a form of $r_t + h_t L_t$ to model the lower price at the left hand side of the Equation (1.3). h_t is the hazard rate for default under risk-neutral measure Q . L_t is the expected proportion of loss in market value. In another word, $(1 - L_t)$ is the recovery rate.

Duffie and Singleton (1999) (13) treat h_t and L_t as exogenously given and do not specify if they are independent from or correlated with each other. They set L_t as a constant 0.5 in numerical calculation.

Evidence shows that there is a correlation between default rates and recovery rates, which is often neglected by many credit risk models. This is because credit pricing models tend to pay attention to systematic credit risk and people assume that the recovery rate is determined endogenously (e.g. seniority, industry field) and hence is independent from default rate. In Chapter 2 of this article, we will introduce a paper that incorporates the negative relationship between default rates and recovery rates.

1.3 Three types of credit pricing models

Credit pricing models can be divided into three types: first and second generation structural-form models and reduced-form models [refer to Altman, Resti and Sironi (2004) (2)]. This paper will mainly focus on the third type, reduced-form model. But it would not hurt if we introduce the main idea of the first two types of models as well.

1.3 Three types of credit pricing models

1.3.1 First generation structural model

In “first generation structural-form” models, the process of default depends on the level of the company’s value. When the value of a firm goes below its liabilities, the firm defaults. The rate of recovery is endogenously determined as a function of the value of a defaulted company. The “first generation structural-form” model has the default rate and the recovery rate negatively related. See Merton (1974) (23).

The idea behind the “first generation structural-form” models is quite intuitive. However, it has some weak points. First, it assumes that firms only default at maturity. Second, seniority structures of multi-class debts need to be specified. Third, the model adopts an absolute-priority rule which in reality is often violated. Furthermore, the lognormal distribution in the model tends to overstate the recovery rates.

1.3.2 Second generation structural model

Second-generation structural-form models improve one of the first-generation’s drawbacks by assuming default can happen anytime before or at maturity. However, the recovery rate that these models use is exogenous and independent from default rate.

Both the first and second generation structural-form models suffer from the drawback that they require estimation of firm’s market value, which indeed is not easily observable. Also, credit-rating changes are not incorporated in the models. Furthermore, the assumption of continuous value of the firm makes the provability of default known for certain.

1.4 Define recovery rates and probability of default

1.3.3 Reduced-form model

Reduced-form models tend to overcome the drawbacks that are embedded in structural-form models. They use an exogenous rate of recovery that is independent from default probability. The recovery rate $(1 - L_t)$ and probability of default h_t as mentioned in Section 1.2 are stochastic process. A reduced-form model assumes that h_t is a Poisson process which is driven by an exogenous random variable. In Chapter 2, we will give specific definitions of h_t and L_t from Duffie and Singleton's paper.

1.4 Define recovery rates and probability of default

People have developed several different ways to define recovery rates and probability of default.

1.4.1 Relative papers and results

Jarrow and Turnbull (1995) (20) make an assumption that a defaulted bond has a market value that is an exogenously specified fraction of an otherwise equivalent default-free bond. Duffie and Singleton (1999) (13) allow for a random recovery rate that depends on the predefault market value of the bond. Their model assumes an exogenous process for the expected loss L_t at default. This means that the recovery rate does not depend on the value of the defaultable claim. Thus correlation between the recovery rate and the default process can be incorporated. They can be linked by firm-specific or macroeconomic variables.

Recovery rates in some other models are defined to be the same for those bonds that have the same issuer, seniority and face value, regardless of the time of default. Duffie (1998) (12) proposes a model that has an exogenously specified fraction recovery of face value, regardless of coupon level or maturity. And the fraction of face value would be the same for

1.4 Define recovery rates and probability of default

bonds of same seniority. Recovery parameters are drawn from data from rating agencies. Jarrow, Lando and Turnbull (1997) (19) set different recovery rates for different seniorities.

Empirical work does not show much support for reduced-form models. Duffee (1999) (11) finds Duffie and Singleton's (1999) (13) model hard to explain the real term structure of credit spreads across firms with different credit risk qualities.

Zhou (2001) (25) combines both the structural-form and reduced-form models by incorporating jump risk into default process and linking recovery rates and firm value at default.

Frye (2000) (14) and (15) makes recovery rate and probability of default negatively correlated by letting them depend on the state of the systematic factor. His empirical evidence also show that there is a strong negative correlation between default rates and recovery rates. However, Altman, Brady, Resti and Sironi (2005) (3) find that taking only the economy condition as systematic risk is less predictive than Frye's model would suggest.

Altman et al. (2005) (3) run Monte Carlo simulations and show that both the expected loss and the unexpected loss would be largely underestimated if probability of default and recovery rate are assumed to be uncorrelated, which will result in insufficient bank reserves. Acharya, Bharath and Srinivason (2003) (1) find that the industry condition at the time of default is robust and important determinants of recovery rates, which contradicts Altman et al.'s result. Hu and Perraudin (2002) (18) actually find that the correlation coefficients between quarterly recovery rates and default rates for bonds issued by US-domiciled obligors are 0.22 for post 1982 data (1982-2000) and 0.19 for 1971-2000 period. Bakshi et al. (2001) (5) allow for a flexible correlation between the risk-free rate, the default probability and the recovery rate.

1.4 Define recovery rates and probability of default

1.4.2 Most recent outcomes

Madan and Unal (2000) (22) introduce a two-factor model for the hazard rate in closed form so that it can better be computed. The likelihood of default is driven by the firm's non-interest sensitive assets and interest rates.

In recessions when defaults cluster, the average amount recovered on the bonds of defaulting firms tends to decrease. Bruche and Gonzalez-Aguado (2010) (8) assume that the negative relationship between the recovery rate and the default probability is driven by an unobservable Markov Chain, which they interpret as "credit cycle". They show that the model is to fit better than models in which the relationship is driven by observable macroeconomic variables.

Guo, Jarrow and Zeng (2009) (17) study the process of determining the recovery rates and derive a reduced-form model by using information reduction. Their model is able to provide analytic expressions for a firm's default intensity, bankruptcy intensity and zero-coupon bond prices both before and after default.

In the absence of forward-looking models for recovery rates, people tend to use exogenously assumed (constant) recovery rates in credit risk models. Das and Hanouna (2009) (9) introduce a flexible jump-to-default model that use observables such as stock prices and stock volatilities in conjunction with credit spreads to identify implied, endogenous dynamic functions of recovery rates and default probabilities.

Chapter 2

About Duffie and Singleton(1999)

In this chapter, we will explain Duffie and Singleton's paper *Modeling Term Structures of Defaultable Bonds* (1999). A detailed explanation of the paper's deductions and motivations is presented. We discuss the paper's drawbacks and summarize the relative improvements made by recent papers.

2.1 Basic ideas of the paper

Duffie and Singleton (1999) (13) introduce a reduced-form model of pricing defaultable bonds, focusing on applications to the term structure of interest rates for defaultable bonds. They take default as an unpredictable event governed by a hazard-rate process, and parameterize losses at default in terms of the fractional reduction in market value that occurs at default.

The model is under the assumption of an arbitrage-free setting where securities are priced in terms of some short rate process r and a risk-neutral measure Q . Denote h_t the hazard rate for default at time t and denote L_t the expected fractional loss in market value if default were to happen at time t conditional on the information available up to time t . For

2.1 Basic ideas of the paper

undefaultable bonds whose face value is X , the price as defined in Chapter 1 is

$$V(t, r; T) = E_{t,r}^Q[e^{-\int_t^T r(s)ds} X].$$

The idea of Duffie and Singleton's model is to decrease the price of a defaultable bond by increasing the interest rate to a default-adjusted short rate process R_t , where $R_t = r_t + h_t L_t$, hence to price the defaultable bonds as if they are in a default-free world.

The valuation formula turns into

$$V_t = E_t^Q[\exp(-\int_t^T R_s ds) X], \quad (2.1)$$

that is

$$V_t = E_t^Q[\exp(-\int_t^T (r_t + h_t L_t) ds) X].$$

The valuation thus takes into account the probability and timing of default, as well as the effect of losses on default.

The paper assumes $h_t L_t$ to be given exogenously, which means $h_t L_t$ does not depend on the value of the defaultable claim itself. This is a typical assumption of reduced-form models for defaultable bonds. However, Duffie and Singleton also warn that this independence of the value of contingent claims is counterfactual for some cases such as swap contracts with asymmetric counter-party credit quality.

Now let us turn to the details of Duffie and Singleton's framework. To make it easier to understand, the authors start from a discrete-time setting for Equation (2.1), and then formalize the intuition in continuous time.

2.2 Discrete-time motivation

Consider a defaultable zero-coupon bond that has a face value of $X_{t+\tau}$ and a maturity $t + \tau$.

For any time s , let

◆ h_t be, at time s under risk-neutral measure Q , the conditional probability of default between the time interval s and $s + 1$ given the information available at time s and no default happens by s .

◆ φ_s denote the recovery in units of account, say dollars, in the event of default at s .

◆ r_s be the default-free short interest rate.

Note: By the definition, h_s should depend on the condition of the bond issuer, say the company. Thus for two companies in different fields, $h1_s$ and $h2_s$ may be different. However, in bad times when large number of defaults occur, h_s 's may be correlated in some sense. Duffie and Singleton allow h_t and L_t to depend on firm specific or macroeconomic variables.

Note: φ_s should also depend on firm-specific variables while bears default correlation property.

Note: r_s is constant in time interval $[s, s + 1]$.

If by time t , the bond has not defaulted yet. Its market value V_t would be the expectation of the present values under two situations. The first situation is that the bond defaults between t and $t + 1$ and the issuer pays φ_{t+1} ; the second situation is the bond does not default during t and $t + 1$ and it worth V_{t+1} at time $t + 1$. This means

$$V_t = h_t e^{-r_t} E_t^Q(\varphi_{t+1}) + (1 - h_t) e^{-r_t} E_t^Q(V_{t+1}), \quad (2.2)$$

2.2 Discrete-time motivation

where E_t^Q is the expectation under Q measure, conditional on information available for investors at time t . Recursively solve Equation (2.2) over the life of the bond yields

$$\begin{aligned}
V_t &= h_t e^{-r_t} E_t^Q(\varphi_{t+1}) + (1 - h_t) e^{-r_t} E_t^Q(V_{t+1}) \\
&= E_t^Q[h_t e^{-r_t} \varphi_{t+1} + (1 - h_t) e^{-r_t} V_{t+1}] \\
&= E_t^Q[h_t e^{-r_t} \varphi_{t+1} + (1 - h_t) e^{-r_t} E_{t+1}^Q[h_{t+1} e^{-r_{t+1}} \varphi_{t+2} + (1 - h_{t+1}) e^{-r_{t+1}} V_{t+2}]] \\
&= E_t^Q[h_t e^{-r_t} \varphi_{t+1} + (1 - h_t) h_{t+1} e^{-(r_t+r_{t+1})} \varphi_{t+2} + (1 - h_t)(1 - h_{t+1}) e^{-(r_t+r_{t+1})} V_{t+2}] \\
&= E_t^Q[h_t e^{-r_t} \varphi_{t+1} + (1 - h_t) h_{t+1} e^{-(r_t+r_{t+1})} \varphi_{t+2} \\
&\quad + (1 - h_t)(1 - h_{t+1}) e^{-(r_t+r_{t+1})} E_{t+2}^Q[h_{t+2} e^{-r_{t+2}} \varphi_{t+3} + (1 - h_{t+2}) e^{-r_{t+2}} V_{t+3}]] \\
&= E_t^Q[h_t e^{-r_t} \varphi_{t+1} + (1 - h_t) h_{t+1} e^{-(r_t+r_{t+1})} \varphi_{t+2} + (1 - h_t)(1 - h_{t+1}) h_{t+2} e^{-(r_t+r_{t+1}+r_{t+2})} \varphi_{t+3}] \\
&\quad + E_t^Q[(1 - h_t)(1 - h_{t+1})(1 - h_{t+2}) e^{-(r_t+r_{t+1}+r_{t+2})} \varphi_{t+3}] \\
&= E_t^Q\left[\sum_{j=0}^2 h_{t+j} e^{-\sum_{k=0}^j r_{t+k}} \varphi_{t+j+1} \prod_{l=0}^j (1 - h_{t+l-1}) + E_t^Q[e^{-\sum_{k=0}^2 r_{t+k}} V_{t+3} \prod_{j=1}^3 (1 - h_{t+j-1})]\right] \\
&= \dots \\
&= E_t^Q\left[\sum_{j=0}^{\tau-1} h_{t+j} e^{-\sum_{k=0}^j r_{t+k}} \varphi_{t+j+1} \prod_{l=0}^j (1 - h_{t+l-1}) + E_t^Q[e^{-\sum_{k=0}^{\tau-1} r_{t+k}} V_{t+\tau} \prod_{j=1}^{\tau} (1 - h_{t+j-1})]\right].
\end{aligned} \tag{2.3}$$

Note $h_{t-1} = 0$ since the bond has not defaulted by time t .

It is very hard to calculate Equation (2.3) because of the requirement of dealing with the joint probability distribution of φ , r and h over various horizons. Duffie and Singleton simplify the equation by substituting $E_s^Q(\varphi_{s+1})$, the risk-neutral expected recovery at time s in the event of default at time $s + 1$, with a fraction of the risk-neutral expected market value at time $s + 1$, in the event of not defaulted by time $s + 1$.

This assumption can be written as

$$\mathbf{RMV:} \quad E_s^Q(\varphi_{s+1}) = (1 - L_s) E_s^Q(V_{s+1}),$$

2.2 Discrete-time motivation

where L_s is some adapted process L bounded by 1, i.e. L_s is the expected fractional loss in market value if default happens in time interval $[s, s + 1]$.

RMV actually sets a proportional relationship assuming higher market value yields higher recovery value.

Put **RMV** into Equation (2.2), we get

$$\begin{aligned} V_t &= (1 - h_t)e^{-r_t} E_t^Q(V_{t+1}) + h_t e^{-r_t} (1 - L_t) E_t^Q(V_{t+1}) \\ &= E_t^Q[e^{-\sum_{j=0}^{\tau-1} R_{t+j}} X_{t+\tau}], \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} e^{-R_t} &= (1 - h_t)e^{-r_t} + h_t e^{-r_t} (1 - L_t) \\ &= e^{-r_t} (1 - h_t L_t), \\ e^{-R_t + r_t} &= 1 - h_t L_t. \end{aligned} \quad (2.5)$$

Note that in Equation (2.4), $\sum_{j=0}^{\tau-1} R_{t+j}$ is the discrete-time form of integration $\int_t^\tau R_s ds$. Equation (2.4) is the discrete-time form of Equation (2.1).

For small x , $e^x \simeq 1 + x$. Then

$$1 - R_t + r_t = 1 - h_t L_t,$$

$$R_t = r_t + h_t L_t.$$

We get the reason why the authors propose $R_t = r_t + h_t L_t$.

Unrealistic aspects of **RMV** assumption:

RMV assumes people know the fraction of recovery beforehand, which is usually not the

2.2 Discrete-time motivation

case in real world. Setting the amount of recovery when default occurs is quite a complicated process which contains paying back company debts of bank, etc.

In **RMV**, L_s is stochastic though predetermined. Like h_s , L_s should also vary for different companies. Duffie and Singleton point out that h_s and L_s may be fitted to market data and are allowed to depend on firm-specific or macroeconomic variables.

When doing actual calculation, people use historical data to get past L_s , plug in valuation today and check if it is over priced or under priced. The thought that the historical behavior would continue in the future is not correct especially in bad times. It is a common defect of models which use historical data.

Equation (2.4) gives the discrete-time form of Equation (2.1), i.e. the valuation of defaultable bonds. It shows that the price of a defaultable claim can be expressed as the present value of the promised payoff $X_{t+\tau}$, treated as if it were default-free, discounted by the defaulted-adjusted short rate R_t . Duffie and Singleton prove it valid as well in continuous time later. As a result, default-free term structure models can be applied to pricing defaultable bonds directly.

Duffie and Singleton justify the rationality of the **RMV** assumption by plotting the distributions of recovery by seniority. Bonds with high seniorities tend to have high recovery rates. Duffie and Singleton also point out that, by strong evidence, hazard rates of default of corporate bonds tend to go higher in recessions, while interest rates and recovery rates are lower than average. They allow h_t and L_t to depend on or be correlated with riskless interest rate. Also remember in the previous argument about **RMV**, Duffie and Singleton allow h_t and L_t to depend on firm-specific or macroeconomic variables. However, they do not talk about how to handle the negative correlation between h_t and L_t . At the end of Chapter 2, we will introduce an article that incorporates the negative relationship between

2.3 Continuous-time valuation

default rates and recovery rates.

As a typical reduced-form model, the hazard rate h_t is not driven by the issuer's ability to meet its obligations but would, in a general formula, depend on the default boundary for assets, the volatility of the underlying asset process V at the default boundary, and the risk-neutral conditional distribution of the level of assets given the history of information available to investors.

The assumption of exogeneity of hazard rate h_t and fractional recovery rate L_t and the failure to accommodate endogeneity can sometimes lead to mispricing. Later in their article, Duffie and Singleton incorporate in the model with endogenous hazard and recovery rates.

2.3 Continuous-time valuation

In this section, Duffie and Singleton formalize the discrete-time model. Set probability space (Ω, \mathcal{F}, P) and filtration $\mathcal{F}_t : t \geq 0$. A short rate interest rate process r_t is also fixed, though the specific term structure model for r_t is not determined. Therefore, an investment of one unit of account in default-free deposits at any time t can roll over to $\exp(\int_t^s r_u du)$ at a later time s .

A contingent claim is a pair (Z, τ) , which means a random variable Z is paid at stopping time τ . Assume Z is \mathcal{F}_t measurable and there is an equivalent martingale measure Q relative to the short rate process r . Then the price of the contingent claim is defined by $U_t = 0$ for $t \geq \tau$, and

$$U_t = E_t^Q[\exp(-\int_t^\tau r_u du)Z], t < \tau, \quad (2.6)$$

2.3 Continuous-time valuation

where E_t^Q denotes the conditional expectation under probability measure Q given \mathcal{F}_t .

Define a defaultable claim to be a pair $((X, T), (X', T'))$ of contingent claims. The underlying claim (X, T) means, in the contract, the issuer has the obligation to pay X at time T . The secondary claim (X', T') means at stopping time T' the issuer defaults and the claim holder receives payment X' . Therefore an actual claim (Z, τ) generated by a defaultable claim $((X, T), (X', T'))$ can be expressed as

$$\tau = \min(T, T'); Z = X1_{\{T < T'\}} + X'1_{\{T \geq T'\}}. \quad (2.7)$$

Now try to characterize the price process U of the above defaultable bond. Suppose that the default time T' follows a risk-neutral default hazard rate process h . Also suppose a process Λ which is 0 before default and 1 after, i.e. $\Lambda_t = 1_{\{t \geq T'\}}$. Then Λ can be written as

$$d\Lambda_t = (1 - \Lambda_t)h_t dt + dM_t, \quad (2.8)$$

where M_t is a martingale under measure Q .

Note: By definition, $\Lambda_0 = 0$, and Equation (2.8) is actually

$$\begin{aligned} \Lambda_t &= \Lambda_0 + \int (1 - \Lambda_t)h_t dt + \int dM_t \\ &= \int (1 - \Lambda_t)h_t dt + \int dM_t. \end{aligned}$$

Λ_t is random, but we can not tell if it only takes 0 or 1 from the above equations.

Here is a heuristic explanation for the fact that Λ_t must jump somewhere. Take expec-

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tations under Q measure and get rid of the stochastic term. We have

$$\begin{aligned} E^Q(\Lambda_t) &= E^Q\left(\int (1 - \Lambda_t)h_t dt\right) + E^Q\left(\int dM_t\right) \\ &= E^Q\left(\int (1 - \Lambda_t)h_t dt\right). \end{aligned}$$

Suppose Λ_t never jump and remains 0, then the left hand side of the equation is zero, but on the right hand side, $\Lambda_t \equiv 0$. It turns into $E^Q(\int h_t dt)$ which is larger than 0 because $h_t \geq 0$. Therefore, with positive probability, Λ is 1.

Here is a more formal explanation for Equation (2.8). Consider the following process, for a random variable X_t taking only the values of 0 and 1,

$$dX = \mu dt + dM_t,$$

where M_t is a martingale. μ here is $(1 - \Lambda_t)h_t$ in Equation (2.8). Let us discretize the equation, so that the time step is $\Delta t = h$, and $t = 0, h, 2h, \dots, nh$,

$$\Delta X_n = \mu h + \Delta M_n = \mu h + M_{n+1} - M_n.$$

Again, X_n takes only the values of 0 and 1 and M_n is a martingale.

If $X_n = 0$, then $X_{n+1} = \mu h + \Delta M_n$, which must be equal to 0 or 1. So ΔM_n takes only two values, namely $-\mu h$ with probability p and $1 - \mu h$ with probability $(1 - p)$. Since M_n is a martingale, we can compute p ,

$$\begin{aligned} 0 &= E[\Delta M_n] = -\mu h p + (1 - p)(1 - \mu h) \\ &= 1 - \mu h - p. \end{aligned}$$

2.3 Continuous-time valuation

So $p = 1 - \mu h$. So the larger is μ , the higher is $(1 - p)$, i.e. the higher is the probability of jumping to $X_{n+1} = 1$.

To be more specific, M_{n+1} equals to 0 with probability p , and 1 with probability $(1 - p)$.

$$\mu h = \mu \Delta = (1 - \Lambda_t) h_t \Delta = h_t \Delta.$$

The risk neutral conditional probability, given information \mathcal{F}_t available at time t , of default before $t + \Delta$, in the event of no default by t , is approximately h_t for small Δ .

If $X_n = 1$, then $X_{n+1} = 1 + \mu h + M_{n+1} - M_n$ only takes 0 or 1. Because ΔM is $-\mu h$ with probability p , and is $1 - \mu h$ with probability $1 - p$. This means $X_{n+1} = 1$ with probability p (when $\Delta M = -\mu h$), and $X_{n+1} = 2$ with probability $1 - p$ (when $\Delta M = 1 - \mu h$) which can not happen. Therefore, $p = 1$, then $X_{n+1} = 1$ with probability 1.

This satisfies the character of Λ that is 0 before default and 1 afterwards.

To be consistent with the previous **RMV**, Duffie and Singleton suppose the claim pays

$$X' = (1 - L_t)U_{t-}, \quad (2.9)$$

where $U_{t-} = \lim_{s \rightarrow t} U_s$ is the price of the claim “just before” default and L_t is the random variable indicating the fractional loss of the market value of the claim at default. Assume L_t is bounded by 1 and predictable, which means roughly that the information determining L_t is available before time t . This assumption is unrealistic somehow. Duffie and Singleton extend their model to allow for conditionally uncertain jumps in market value at default.

As a preliminary step, it is useful to define a process V_t which is the market value of the defaultable claim if there has been no default by time t . In particular, $V_T = X$. $V_t = U_t$ for

2.4 Exogenous expected loss rate

$t < T'$.

2.4 Exogenous expected loss rate

In this section, Duffie and Singleton define L_t .

From previous reasoning, we know

$$V_t = E_t^Q[\exp(-\int_t^T R_s ds)X], \quad (2.10)$$

where

$$R_t = r_t + h_t L_t. \quad (2.11)$$

In order to confirm this conjecture, Duffie and Singleton use the fact that the gain process (price plus cumulative dividend), after discounting at the short-rate process r , must be martingale under Q . This discounted gain process G is defined as

$$G = \exp(-\int_0^t r_s ds)V_t(1 - \Lambda_t) + \int_0^t \exp(-\int_0^s r_u du)(1 - L_s)V_{s-}d\Lambda_s. \quad (2.12)$$

The first term is the discounted price of the claim. Note Λ_t is 0 before default and 1 afterwards. $(1 - L_s)V_{s-}$ is the recovery payment upon default as in Equation (2.9). $\exp(-\int_0^t r_s ds)$ and $\exp(-\int_0^s r_u du)$ are discount parts.

If the claim has not defaulted by time t , Λ_t is always 0 then G leaves only the first term. If the claim defaults during the time period, the first term turns to zero and we get the discounted payout of the claim which is the second term.

2.4 Exogenous expected loss rate

Duffie and Singleton suppose that V does not itself jump at the default time T' . From Equation (2.10), this is a primitive condition on (r, h, X) and the information filtration $\mathcal{F}_t : t \geq 0$. This means essentially that, although there may be “surprise” jumps in the conditional distribution of the market value of the default free claim (X, T) , h or L , these surprises occur precisely at the default time with probability zero.

Since V jumps at most a countable number of times, V_{s-} can be replaced in Equation (2.12) with V_s for the purpose of the following calculation. Apply Ito’s formula [see Protter (1990) (24)] to Equation (2.12), using Equation (2.9) and the assumption that V does not jump at the default time T' , Duffie and Singleton find that, for G to be a Q martingale, it is necessary and sufficient that

$$V_t = \int_0^t R_s V_s ds + m_t \quad (2.13)$$

for some Q martingale m . Given the boundary condition $V_T = X$, Equation (2.13) implies Equations (2.10)-(2.12). Therefore, the fact that G is a Q martingale and the boundary condition $V_T = X$ confirm the continuous-time valuation formula (2.10) and (2.11), i.e

$$V_t = E_t^Q \left[\exp\left(-\int_t^T R_s ds\right) X \right],$$
$$R_t = r_t + h_t L_t.$$

In other words, the property that G is a Q martingale and the fact that $V_T = X$ provide a complete characterization of arbitrage-free pricing of the defaultable claim.

The uniqueness of solutions of Equation (2.13) with $V_T = X$ is proved by Antonelli (1993) (4). Duffie and Singleton finally state the uniqueness of the prices of defaultable claims in **Theorem 1**.

2.5 Drawbacks and following improvements

Theorem 1: Given (X, T, T', L, r) , suppose the default time T' has a risk-neutral hazard rate process h . Let $R = r + hL$ and suppose that V is well defined by Equation (2.10) and satisfies $\Delta V(T') = 0$ almost surely. Then there is a unique defaultable claim $((X, T), (X', T'))$ and process U satisfying Equations (2.6), (2.7), and (2.9). Moreover, for $t < T'$, $U_t = V_t$.

In summary, Duffie and Singleton propose a pricing model for defaultable claims in which the usual short-term interest rate r is replaced by a default-adjusted short-rate process $R = r + hL$ and thus the defaultable claims are priced as if they are default-free.

The pricing formula of defaultable bonds is

$$V_t = E_t^Q[\exp(-\int_t^T R_s ds)X],$$
$$R_t = r_t + h_t L_t.$$

Duffie and Singleton first provide a heuristic proof of the pricing formula in discrete time setting. Then they show the validation of the pricing formula in continuous time and prove the uniqueness of the price. As a result, they provide a complete characterization of arbitrage-free pricing of the defaultable claim.

2.5 Drawbacks and following improvements

Duffie and Singleton provide people an easy and straight forward way to price defaultable claims. However, empirical work does not show much support. Duffee (1999) (11) finds Duffie and Singleton's (1999) (13) model hard to explain the real term structure of credit spreads across firms with different credit risk qualities.

2.5 Drawbacks and following improvements

Duffie and Singleton point out that h_t and L_t may be related to r and they may depend on firm specific or macroeconomic variables. They study special cases in one of which h_t and L_t each depends on interest rate r and the current market value of the defaultable claim. They also extend their model to allow for conditionally uncertain jumps in market value at default so that the fractional loss L_t is undetermined (L_t is predictable in their original model).

It needs to be pointed out that there are some drawbacks on the assumptions of hazard rate of default h_t and the expected fractional loss in market value L_t .

First, Duffie and Singleton treat h_t and L_t as exogenously given but do not mention the negative relationship between h_t and L_t . Bruche and Gonzalez-Aguado (2010) (8) improve this issue by assuming that the negative relationship between recovery rate and default probability is driven by an unobservable Markov Chain, which they interpret as “credit cycle”. They show that the model is to fit better than models in which the relationship is driven by observable macroeconomic variables.

Another aspect that Duffie and Singleton do not mention is default correlation. It is a fact that, in bad times, defaults can impact surviving firms, allowing for a greater clustering of default. The default rate for a group of credits tend to be higher in a recession and lower when the economy is booming. Moreover, historical data may not be applicable to recession times. Li (1999) (21) studies the problem of default correlation. Under the framework of Duffie and Singleton (1999) (13), he derives a credit curve and specify the distribution of the surviving time. He introduces a random variable called “time-until-default” to denote the survival time of each defaultable entity or financial instrument, and defines the default correlation between two credit risks as the correlation coefficient between their survival times. Li uses a copula function approach to specify the joint distribution of survival times.

2.5 Drawbacks and following improvements

Duffie and Singleton apply the default-adjusted short-rate process $R = r + hL$ to HJM model. With some necessary drift restrictions on forward rate, one can treat the dynamics of the term structure of interest rates on defaultable debt using HJM model as if they are default-free. Berndt et al.(2010) (6) find implementing the models delicate and computationally intensive due to the reason that the dynamics of all riskless forward rates and risky forward credit spreads are not Markov in a finite number of state variables. The problem is more complicated if the derivative security depends on the credit spreads of multiple names. Berndt et al.(2010) (6) overcome these problems by imposing modest restrictions on forward volatilities and jumps. Their article develops a Markovian framework in HJM paradigm for pricing credit derivatives on both single and multiple names. The models allow for arbitrary correlations between credit spreads of different firms and permit the default of some firms to cause jumps in the term structure of credit spreads of other surviving firms, a feature that allows defaults to cluster over time.

Duffie and Singleton use continuous-time affine models in their paper. However, the continuous-time affine models have not yet encountered the expected success among practitioners and regulators due to a lack of flexibility. Gouiroux et al.(2006) (16) explain that this lack of flexibility is mainly due to the continuous-time assumption. They conduct a discrete-time affine analysis of credit risk in which the different types of factors introduced in are less constrained hence are able to reproduce complicated cycle effects.

In Duffie and Singleton's paper, h_t and L_t are allowed to depend on macroeconomic variables. David (2008) (10) expands this idea and studies the credit spreads puzzle. Macroeconomic shocks carry risk premiums so that expected default losses are sensitive to changes in the price of risk. By incorporating state dependence and increasing the price of risk, the econometrician obtains high credit spreads while maintaining average default losses at historical levels.

Chapter 3

Applications to Duffie and Singleton's Model

In this Chapter, we are going to simplify the assumption of one of Duffie and Singleton's models and solve for explicit pricing formulas of defaultable bonds in different cases.

3.1 Deduce the formula

Duffie and Singleton deduce a continuous-time Markov formulation by assuming a state-variable process Y that is Markovian under Q measure and assuming the contingent claim is of the form $X = g(Y_T)$ for some function g , and $R_t = \rho(Y_t)$ for some function $\rho(\cdot)$.

We simplify the assumption to the case that Markovian Y_T is r_t and $R_t = \rho(Y_t) = \rho(r_t) = r_t + h_t L_t$ and try to get the same result as Duffie and Singleton have.

3.1 Deduce the formula

3.1.1 Feynman-Kac Formula

Feynman-Kac Formula:

Assume F is a solution to the boundary value problem:

$$\begin{cases} \frac{\partial F}{\partial t}(t, X) + \mu(t, X) \frac{\partial F}{\partial X} + \frac{1}{2} \sigma^2(t, X) \frac{\partial^2 F}{\partial X^2}(t, X) - rF(t, X) = 0, \\ F(T, X) = \Phi(X), \end{cases}$$

where X satisfies

$$\begin{cases} dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s, \\ X_t = x. \end{cases}$$

In addition, the process $\sigma(s, X_s) \frac{\partial F}{\partial X}(s, X)$ is in L^2 .

Then, F has the representation

$$F(t, X) = e^{-r(T-t)} E_{t,x}[\Phi(X_T)].$$

Note: The stochastic process X is defined in the same probability measure as in the expectation in the expression of F , no matter what specific probability measure it is.

3.1.2 Vasicek Model

Suppose interest rate r_t follows Vasicek Model:

$$dr = (b - ar)dt + \sigma dW.$$

Note: W is under Q measure.

3.1 Deduce the formula

Let $\mu = b - ar$. Based on Duffie and Singleton's paper, we simplify the model by taking

$$Y_t = r_t,$$

$$X = g(Y_t) = g(r_t),$$

$$R_t = \rho(Y_t) = \rho(r_t) = r_t + h_t L_t.$$

$J(r_t, t)$ is the price of a defaultable claim at time t . The claim promises to pay X at time T and it has not defaulted by time t .

From Chapter 2, the price of a defaultable claim can be expressed as the present value of the promised payoff X , treated as if it were default free, discounted by the default-adjusted short rate R_t .

$$J(r_t, t) = E^Q[\exp(-\int_t^T R_s ds)X|r_t].$$

It is wrong if we use $J(r_t, t) = E^Q[\exp(-\int_t^T r_s ds)X|r_t]$ because at time T , the company may have defaulted. This $J(r_t, t)$ may not exist. It is not a martingale.

3.1.3 Deduction of $J(r_t, t)$

Apply Ito's lemma to $J(r_t, t)$, we get

$$\begin{aligned} dJ(r_t, t) &= \frac{\partial J}{\partial t} dt + \frac{\partial J}{\partial r} \mu dt + \frac{\partial J}{\partial r} \sigma dW + \frac{1}{2} \frac{\partial^2 J}{\partial r^2} \sigma^2 dt \\ &= \left[\frac{\partial J}{\partial t} + \frac{\partial J}{\partial r} \mu + \frac{1}{2} \frac{\partial^2 J}{\partial r^2} \sigma^2 \right] dt + \frac{\partial J}{\partial r} \sigma dW, \end{aligned}$$

$$\frac{\partial \tilde{J}}{\tilde{J}} = dt \left[\frac{\partial J}{\partial t} + \frac{\partial J}{\partial r} \mu + \frac{1}{2} \frac{\partial^2 J}{\partial r^2} \sigma^2 - RJ \right] \frac{1}{J} + \frac{\partial J}{\partial r} \sigma \frac{1}{J} dW.$$

3.2 Solve the equation

Use R in the equation above instead of r . This is because in defaultable world, we treat defaultable claims as default-free by using short rate R_t instead of r_t .

\tilde{J} is a martingale if and only if

$$\frac{\partial J}{\partial t} + \frac{\partial J}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 J}{\partial r^2}\sigma^2 - RJ = 0. \quad (3.1)$$

J satisfies boundary condition

$$J_T = X = g(r_T). \quad (3.2)$$

Expand Equation (3.1),

$$\frac{\partial J}{\partial t} + \frac{\partial J}{\partial r}(b - ar_t) + \frac{1}{2}\frac{\partial^2 J}{\partial r^2}\sigma^2 - (r_t + h_t L_t)J = 0, \quad (3.3)$$

where a, b, σ are constant numbers.

By Feynman-Kac Formula, if J is a solution of Equations (3.1) and (3.2), then

$$\begin{aligned} J(r_t, t) &= E^Q[\exp(-\int_t^T R_s ds)X|r_t] \\ &= E^Q[\exp(-\int_t^T (r_s + h_s L_s) ds)g(r_T)|r_t]. \end{aligned}$$

3.2 Solve the equation

Next, we try to solve Equation (3.1) explicitly in order to get the price of the defaultable bond V_t at time t . Note that V_t is actually $J(Y_t, t)$, i.e. $J(r_t, t)$. By observing Equation

3.2 Solve the equation

(3.1), we find that it is similar to the one that is deduced in default-free world, i.e.

$$\frac{\partial J}{\partial t} + \frac{\partial J}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 J}{\partial r^2}\sigma^2 - rJ = 0,$$

and we want to make use of the current calculation results in default-free world.

We write J as a function of R_t and t , then $J(R_t, t)$ still follows

$$\begin{aligned} J(R_t, t) &= E^Q[\exp(-\int_t^T R_s ds)X|r_t] \\ &= E^Q[\exp(-\int_t^T R_s ds)X|R_t]. \end{aligned}$$

As concluded previously, through $R_t = r_t + h_t L_t$, the price of a defaultable bond can be expressed as the present value of the promised payoff X , treated as if it were default-free, discounted by the default-adjusted short rate R_t . Therefore, we have $J(R_t, t)$ which solves

$$\frac{\partial J}{\partial t} + \frac{\partial J}{\partial R}\phi + \frac{1}{2}\frac{\partial^2 J}{\partial R^2}\Sigma^2 - RJ = 0, \quad (3.4)$$

where

$$dR = \phi dt + \Sigma dW.$$

3.2.1 Constant hL in Vasicek model and Hull-White model

Suppose r_t follows Vasicek model, i.e.

$$dr = (b - ar)dt + \sigma dW,$$

3.2 Solve the equation

and

$$r_t = R_t - h_t L_t,$$

and suppose $h_t L_t$ is a constant number, say hL , then

$$\begin{aligned} dR = dr &= [b - a(R_t - hL)]dt + \sigma dW \\ &= [b + ahL - aR_t]dt + \sigma dW. \end{aligned}$$

We can see R also follows Vasicek model

$$dR = (\beta - \alpha R)dt + \Sigma dW,$$

where

$$\beta = b + ahL,$$

$$\alpha = a,$$

$$\Sigma = \sigma.$$

Use affine term structure as in Bjork (2003) (7), we solve for the explicit solution of term structure problem

$$\begin{cases} \frac{\partial J}{\partial t} + \frac{\partial J}{\partial R} \phi + \frac{1}{2} \frac{\partial^2 J}{\partial R^2} \Sigma^2 - RJ = 0, \\ dR = \phi dt + \Sigma dW, \\ J(R_T, T) = 1. \end{cases} \quad (3.5)$$

Here $J(R_T, T) = 1$ is for simplicity.

For r modeled by Vasicek model as in previous section,

$$dR = (\beta - \alpha R)dt + \Sigma dW,$$

3.2 Solve the equation

where $\beta = b + ahL$, $\alpha = a$, $\Sigma = \sigma$ are all constants.

Assume J has the form

$$J(t, R; T) = e^{A(t,T) - B(t,T)R}, \quad (3.6)$$

$J(T, R; T) = 1$ implies $A(T, T) = B(T, T) = 0$.

Plug Equation (3.6) into (3.5), we get

$$\begin{aligned} e^{A-BR}(A_t - B_tR) + e^{A-BR}(-B)\phi + \frac{1}{2}(-B)^2 e^{A-BR}\Sigma^2 - Re^{A-BR} &= 0 \\ \Rightarrow A_t - B_tR - \phi B + \frac{1}{2}B^2\Sigma^2 - R &= 0 \\ \Rightarrow A_t - 1 + B_tR - \phi B + \frac{1}{2}B^2\Sigma^2 &= 0. \end{aligned} \quad (3.7)$$

Suppose ϕ and Σ have the form

$$\begin{cases} \phi = x(t)R + y(t), \\ \Sigma = \sqrt{\gamma(t)R + \delta(t)}. \end{cases} \quad (3.8)$$

Equation (3.8) transforms Equation (3.7) into

$$\begin{aligned} A_t(t, T) - y(t)B(t, T) + \frac{1}{2}B^2(t, T)\delta(t) \\ - [1 + B_t(t, T) + x(t)B(t, T) - \frac{1}{2}\gamma(t)B^2(t, T)]R &= 0. \end{aligned} \quad (3.9)$$

Equation (3.9) holds for all t, T, R . The coefficient of R must be zero.

$$1 + B_t(t, T) + x(t)B(t, T) - \frac{1}{2}\gamma(t)B^2(t, T) = 0.$$

3.2 Solve the equation

And other terms in Equation (3.9) must also vanish, then

$$A_t(t, T) - y(t)B(t, T) + \frac{1}{2}B^2(t, T)\delta(t) = 0.$$

Conclusion: If ϕ and Σ in $dR = \phi dt + \Sigma dW$ have the form (3.8), then we have the solution of problem (3.5), which is

$$J = e^{A(t, T) - B(t, T)R},$$

where A and B satisfy the system

$$\begin{cases} B_t(t, T) + x(t)B(t, T) - \frac{1}{2}\gamma(t)B^2(t, T) = -1, \\ B(T, T) = 0, \end{cases} \quad (3.10)$$

$$\begin{cases} A_t(t, T) = y(t)B(t, T) - \frac{1}{2}B^2(t, T)\delta(t), \\ A(T, T) = 0. \end{cases} \quad (3.11)$$

Now we can use the results to compute the system in the situation of Vasicek model.

Recall $dR = (\beta - \alpha R)dt + \Sigma dW$,

where $\beta = b + ahL$, $\alpha = a$, $\Sigma = \sigma$.

3.2 Solve the equation

Then ϕ and Σ in Equation (3.8) are

$$\begin{aligned}\phi &= \beta - \alpha R = x(t)R + y(t), \\ x(t) &= -\alpha = -a, \\ y(t) &= \beta = b + ahL, \\ \Sigma &= \sigma = \sqrt{\gamma(t)R + \delta(t)}, \\ \gamma(t) &= 0, \\ \delta(t) &= \sigma^2.\end{aligned}$$

Now Equations (3.10) and (3.11) become

$$\begin{cases} B_t(t, T) - aB(t, T) = -1, \\ B(T, T) = 0, \end{cases} \quad (3.12)$$

$$\begin{cases} A_t(t, T) = (b + ahL)B(t, T) - \frac{1}{2}B^2(t, T)\sigma^2(t), \\ A(T, T) = 0. \end{cases} \quad (3.13)$$

These differential equations can be solved as

$$\begin{aligned}B(t, T) &= \frac{1}{a}\{1 - e^{-a(T-t)}\}, \\ A(t, T) &= \frac{\sigma^2}{2} \int_t^T B^2(s, T)ds - (b + ahL) \int_t^T B(s, T)ds \\ &= \frac{1}{a^2}[B(t, T) - T + t][a(b + ahL) - \frac{1}{2}\sigma^2] - \frac{\sigma^2 B^2(t, T)}{4a}.\end{aligned}$$

Similarly, we can find an explicit solution for a simplified version of the Hull-White extension of the Vasicek model, where the Q -dynamics of the short-rate r is defined by $dr = [\theta(t) - ar]dt + \sigma dW(t)$. Here a and σ are constants while θ is a deterministic

3.2 Solve the equation

function of time.

We can easily get the result for Hull-White model. The bond prices are given by

$$\begin{aligned} J(t, T) &= e^{A(t, T) - B(t, T)R(t)}, \\ B(t, T) &= \frac{1}{a} \{1 - e^{-a(T-t)}\}, \\ A(t, T) &= \int_t^T \left\{ \frac{1}{2} \sigma^2 B^2(s, T) - (\theta(s) + ahL)B(s, T) \right\} ds. \end{aligned}$$

3.2.2 Constant hL in CIR model

Suppose interest rate r follows CIR model

$$dr = a(b - r)dt + \sigma\sqrt{r}dW.$$

Use $dR_t = r_t + hL$, we derive dR

$$\begin{aligned} dR &= dr = a[b - (R - hL)]dt + \sigma\sqrt{R - hL}dW \\ &= a[b + hL - R]dt + \sigma\sqrt{R - hL}dW. \end{aligned}$$

Then the term structure problem becomes

$$\begin{cases} \frac{\partial J}{\partial t} + \frac{\partial J}{\partial R} a[b + hL - R] + \frac{1}{2} \frac{\partial^2 J}{\partial R^2} \sigma^2 (R - hL) - RJ = 0, \\ dR = a[b + hL - R]dt + \sigma\sqrt{R - hL}dW, \\ J(R_T, T) = 1. \end{cases} \quad (3.14)$$

3.2 Solve the equation

R no longer follows a CIR model which is much more difficult to handle. We can use numerical methods to solve this stochastic partial differential equation problem.

3.2.3 hL as a function of r and t

It is natural to think of the rate of default h_t and the expected fractional loss in market value L_t related to interest rate r . Because h and L enter the model as a whole product of hL , we do not treat them separately. Now we treat hL as a function of (r, t) , i.e. $hL(r, t)$.

$$\begin{aligned}R(t) &= r(t) + hL(r, t), \\ dr &= \mu dt + \sigma dW.\end{aligned}$$

Apply Ito's formula to $R(r, t)$, we have

$$\begin{aligned}dR &= \frac{\partial R}{\partial t} dt + \frac{\partial R}{\partial r} \mu dt + \frac{\partial R}{\partial r} \sigma dW + \frac{1}{2} \frac{\partial^2 R}{\partial r^2} \sigma^2 dt \\ &= \left[\frac{\partial R}{\partial t} + \frac{\partial R}{\partial r} \mu + \frac{1}{2} \frac{\partial^2 R}{\partial r^2} \sigma^2 \right] dt + \frac{\partial R}{\partial r} \sigma dW \\ &= \left[\frac{\partial hL}{\partial t} + \left(1 + \frac{\partial hL}{\partial r}\right) \mu + \frac{1}{2} \frac{\partial^2 hL}{\partial r^2} \sigma^2 \right] dt + \left(1 + \frac{\partial hL}{\partial r}\right) \sigma dW.\end{aligned}$$

Assume r follows Vasicek model, then $\mu = b - ar$, $\sigma = \sigma$,

$$dR = \left[\frac{\partial hL}{\partial t} + \left(1 + \frac{\partial hL}{\partial r}\right) (b - ar) + \frac{1}{2} \frac{\partial^2 hL}{\partial r^2} \sigma^2 \right] dt + \left(1 + \frac{\partial hL}{\partial r}\right) \sigma dW.$$

3.2 Solve the equation

Assume hL is a simple linear function of r and t :

$$hL(r, t) = \alpha r + \beta t + \eta,$$

where α , β and η are constant numbers.

$$\text{Then } \frac{\partial hL}{\partial r} = \alpha, \frac{\partial hL}{\partial t} = \beta, \frac{\partial^2 hL}{\partial r^2} = 0.$$

And

$$\begin{aligned} dR &= \left[\frac{\partial hL}{\partial t} + \left(1 + \frac{\partial hL}{\partial r}\right)b - a\left(1 + \frac{\partial hL}{\partial r}\right)r \right] dt + \left(1 + \frac{\partial hL}{\partial r}\right) \sigma dW \\ &= [\beta + (1 + \alpha)b - a(1 + \alpha)r] dt + (1 + \alpha) \sigma dW. \end{aligned}$$

R follows Vasicek model and thus we can get an explicit solution:

$$J(t, T) = e^{A(t, T) - B(t, T)R},$$

where $A(t, T)$ and $B(t, T)$ satisfy the system

$$\begin{aligned} B(t, T) &= \frac{1}{a(1 + \alpha)} \{1 - e^{-a(1 + \alpha)(T - t)}\}, \\ A(t, T) &= \frac{1}{a^2(1 + \alpha)^2} [B(t, T) - T + t] \{a(1 + \alpha)[\beta + (1 + \alpha)b] - \frac{1}{2}(1 + \alpha)^2 \sigma^2\} \\ &\quad - \frac{(1 + \alpha)^2 \sigma^2}{4a(1 + \alpha)} B^2(t, T). \end{aligned}$$

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