PRICING AND HEDGING OPTIONS UNDER STOCHASTIC VOLATILITY

by

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Abstract

In this essay, I mainly discuss how to price and hedge options in stochastic volatility (SV) models. The market is incomplete in the SV model, whereas it is complete in the Black-Scholes model. Thus the option pricing and hedging methods are a little different for the SV model and for the Black-Scholes model. The no-arbitrage argument and the risk-neutral valuation method are two general methods for pricing options. Both methods can be applied to the SV model. Heston's SV model is discussed in more details, since it is the most popular SV model in practice. I also implement Heston's model and investigate the effects of the model parameters by looking at the pricing results.

Contents

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Chapter 1

Introduction

In their seminal paper on option pricing, Black and Scholes (1973) presented a famous option pricing formula. They assume the stock price follows a geometric Brownian motion with a constant volatility. A riskless portfolio is formed by a call option and some shares of stock. Based on no-arbitrage arguments, a partial differential equation can be derived for the price of a call option. This partial differential equation can be easily solved and gives a closed-form solution. Because of its simplicity and its independence of investors' expectations about future asset returns, the Black-Scholes (B-S) formula is widely used among practitioners for pricing and hedging options. However, a number of studies have shown that the B-S model has systematic biases across moneyness and maturity. The most well-known result of the B-S model is the so-called volatility smile or skew. The implied volatilities from the market prices of options tend to vary by strike prices and maturities. It is clear that some assumptions of the B-S model are too strong. Empirical studies find stock returns usually have a higher kurtosis compared to the normal distribution which is assumed by the B-S model. Some people also model stock volatilities as a stochastic process, either discrete or continuous.

In order to reduce the option pricing error, researchers have presented several approaches. Many traders prefer to remain in the B-S framework and simply use the volatility matrix or a fitted volatility function and then apply the B-S formula in a modified way. Other one-factor models include the constant elasticity of variance (CEV) model (Cox and Ross (1976)), and the local volatility models (Derman and Kani (1994), etc.). The market is complete in one-factor models. Some models are extended to have multiple factors. This group includes the jump-diffusion model (Merton (1976)), the continuous-time stochastic volatility model (Hull and White (1987), Stein and Stein (1991), Heston (1993)), and the GARCH option pricing model (Duan (1995), Heston and Nandi (2000)). Other models are hybrids, combining some aspects of standard models. Bates (1996)'s stochastic volatility/jump-diffusion model is a good example.

In this essay, I will discuss how to price and hedge options in the SV model. In finance literature, several methods are employed to price options. We can form a riskless portfolio and apply no-arbitrage arguments to give an option price based on the underlying stock price. We can also find an equivalent martingale measure and apply risk-neutral valuation methods. Both methods can be used in the SV model as well as in the B-S model. But the application procedures are different as the market is complete in the B-S model but incomplete in the SV model. Some people also apply utility functions to pricing options. But in practice, pricing methods based on utility functions are very difficult to implement.

In the B-S model, a perfect hedging is possible due to the market completeness. However, in SV models, the market is incomplete and a perfect hedging is impossible. Practitioners can either use other options to hedge the target option, or only use the underlying stock to partially hedge the option. Popular hedging strategies for SV models in finance literature include superhedging, mean-variance hedging and shortfall hedging.

The rest of the essay is organized as follows. Chapter 2 introduces the option pricing methods and the results in the B-S model. This serves to give a comparison with the results in the SV model. The pricing biases in the B-S model and the econometric evidence of stochastic volatility will be discussed. I will also give a very brief introduction to some popular models in option pricing literature. Chapter 3 discusses pricing and hedging options in SV models. Chapter 4 takes Heston's SV model as an example to discuss some implementation issues. The implied volatility surface in Heston's model is plotted. Effects of some model parameters are investigated. I will also present some parameter estimation methods for SV models. Chapter 5 discusses some limitations of SV models. Chapter 6 concludes this essay.

Chapter 2

Evidence of Stochastic Volatility

2.1 Black-Scholes Model and Volatility Smile

In the B-S world, the stock price, S, follows a geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dz, \tag{2.1}$$

where μ and σ are known constants, z is a standard Brownian motion. To derive the option pricing formula, Black and Scholes (1973) make some other assumptions, such as the market is *frictionless*, the short-term interest rate is known and a constant, r, short selling is allowed without restrictions, the stock pays no dividends. The essential step in the B-S methodology is the construction of a riskless portfolio and the no-arbitrage argument. The main derivation goes as follows.

Suppose that f is the price of a call option or other derivative contingent on S. By Ito's lemma,

$$df = (\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial f}{\partial S}\sigma S dz. \tag{2.2}$$

Next, we set up a portfolio consisting of a short position in a call option and a long position of Δ units of stock. Define Π as the value of the portfolio,

$$\Pi = -f + \Delta S. \tag{2.3}$$

The change in the value of this portfolio in a small time interval is given by

$$d\Pi = -df + \Delta dS. \tag{2.4}$$

Substituting (2.1), (2.2) into (2.4) yields

$$\begin{aligned} d\Pi &= -(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt - \frac{\partial f}{\partial S}\sigma S dz + \Delta\mu S dt + \Delta\sigma S dz \\ &= (-\frac{\partial f}{\partial S}\mu S - \frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + \Delta\mu S)dt + (-\frac{\partial f}{\partial S}\sigma S + \Delta\sigma S)dz. \end{aligned} \tag{2.5}$$

To make the portfolio riskless, we choose $\Delta = \frac{\partial f}{\partial S}$. Then,

$$d\Pi = (-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt. \tag{2.6}$$

On the other hand, in the absence of arbitrage opportunities, this riskless portfolio must earn a risk-free rate, r,

$$d\Pi = r\Pi dt. \tag{2.7}$$

Substituting from (2.3) and (2.6), this becomes

$$(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt = r(-f + \frac{\partial f}{\partial S}S)dt,$$

or

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf. \tag{2.8}$$

Equation (2.8) is the famous B-S partial differential equation. The solution depends on the boundary conditions. In the case of a European vanilla call option, the final condition is that the option price is simply its payoff at maturity.

$$f = max(S - K, 0), \quad t = T, \tag{2.9}$$

where K is the strike price. So the equation can be solved by backward in time with the final condition. The B-S pricing formula for the European call option is

$$c = S_0 N(d_1) - Ke^{-rT}N(d_2), \tag{2.10}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}, \tag{2.11}$$

and S_0 is the current stock price, T is the time to maturity, σ is the stock price volatility, $N(x)$ is the cumulative probability distribution function for the standard normal distribution. The price of the European put option can be computed by the *put-call parity.*

The expected return μ does not appear in the B-S equation. This means the pricing formula is independent of the individual's preference. This amazing property together with its simplicity makes the B-S pricing formula very popular among practitioners as well as academic researchers.

Another approach to deriving the B-S formula is the *risk-neutral valuation* method. The price of the option, c, is the expected value of the option at maturity in a risk-neutral world discounted at the risk-free interest rate, that is

$$c = e^{-rT} E^Q[max(S_T - K, 0)], \tag{2.12}$$

where E^Q denotes the expected value in a risk-neutral world. Q is also called the *equivalent martingale measure.* In the risk-neutral world, $S_T = S_0 e^{(r-\sigma^2/2)T+\sigma z}$. So the expectation in (2.12) can be calculated by integrating over the normal distribution. We can get the same pricing formulas as (2.10) and (2.11). The above two pricing methods, no-arbitrage valuation and risk-neutral valuation, are two general approaches to pricing options in modern finance literature.

Although the B-S formula is powerful to price stock options and simple to use, many empirical results show that it may systematically misprice a number of options. The well-known phenomenon related to the biases of the B-S model is the volatility smile or skew.

The implied volatility is the volatility used in the B-S model such that the observed market price of the option equals the model price.

$$c^{BS}(\sigma) = c^{Market}. \tag{2.13}$$

Consider call or put options on a given stock or an index. These options have the same maturity but different strike prices. We apply the B-S model to back out the implied volatilities and plot them against the strike prices. We expect the implied volatilities to be identical because the constant volatility is the assumption of the B-S model. However, it is likely not the case in practice. Most option markets exhibit persistent patterns of non-constant volatilities. In some markets, the implied volatilities form a 'U-shape', which is called the volatility smile. In-the-money options and out-of-the-money options have higher implied volatilities than at-the-money options. Generally, the shape of the volatility smile is not symmetric. It is more of a skewed curve. People also call it volatility skew or volatility smirk. Figure 2.1 is the implied volatilities of S&P 500 European call options at various strike prices on Feb. 24, 2005. All these options mature on Jun. 17. Obviously, the phenomenon of volatility smile is not consistent with the B-S model.

In addition to calculating a volatility smile, we can also calculate a volatility term structure, a function of maturity for a fixed strike price. The implied volatilities also vary with maturity. Combining the volatility smile and the volatility term structure, we can generate a volatility surface, one dimension for strike price and the other for maturity. Usually, the smile is significant for short maturity options and tends to be flat for long maturity options.

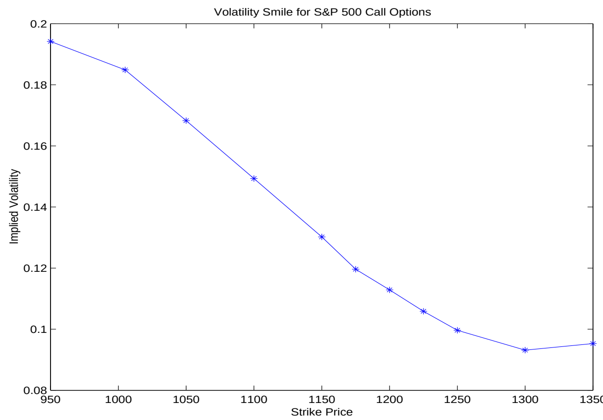

Figure 2.1: Implied volatilities for S&P 500 call options. Maturity is Jun. 17, 2005. Valuation date is Feb. 24, 2005. The S&P 500 index is 1200.20 on the valuation day. Use $r = 0.011$.

There are various explanations for the phenomenon of volatility smile. Some of explanations are related to the idealized assumption of the B-S model which says the asset price follows a geometric Brownian motion with a constant volatility. I provide some econometric evidence against this assumption of the B-S model.

2.2 Econometric Evidence

If the stock price follows a geometric Brownian motion as in equation (2.1), the stock price is lognormally distributed, or, the logarithmic return is normally distributed. From Figure 2.2, we see that price changes are small in sequential days in some periods and large in other periods. This is called volatility clustering. This feature implies that the volatility is autocorrelated.

People also find that the return distribution has leptokurtosis, which has higher central peak and fatter tails compared with the normal distribution. Leptokurtosis is consistent with the volatility smile. It is usually a result of a mixture of several distributions with different variances.

Based on the above facts, people have tried to model the volatilities. In econometrics, ARCH-type models are very popular to model volatilities. An ARCH (Autoregressive conditional heteroskedasticity) process can be defined in the following way. Assume the dependent variable y_t is generated by

$$y_t = x_t'\zeta + \varepsilon_t, \quad t = 1, 2...T, \tag{2.14}$$

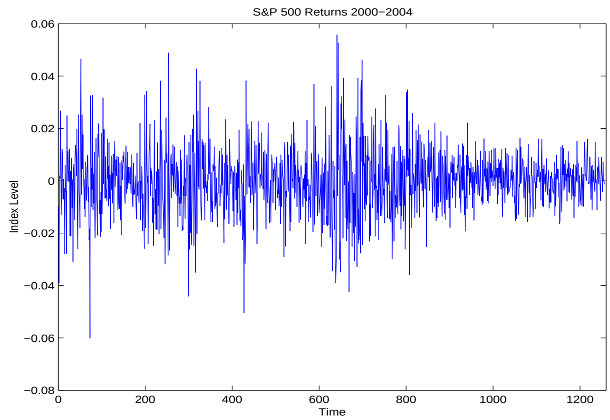

Figure 2.2: S&P 500 daily returns from Jan. 3, 2000 to Dec. 31, 2004

where x_t is a vector of exogenous variables, which may include lagged dependent variable, and ζ is a vector of regression parameters. Engle (1982) characterizes the stochastic error ε as an ARCH process,

$$\varepsilon_t|\phi_{t-1} \sim N(0, h_t), \tag{2.15}$$

$$h_t = \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + ... + \alpha_q\varepsilon_{t-q}^2, \tag{2.16}$$

with $\alpha_0 > 0$, and $\alpha_i \geq 0, i = 1, ..., q$, to ensure that the conditional variance is positive. Bollerslev (1986) proposes a GARCH (Generalized ARCH) model. The conditional variance in GARCH models is specified as

$$h_t = \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + ... + \alpha_q\varepsilon_{t-q}^2 + \beta_1 h_{t-1} + ... + \beta_p h_{t-p}, \tag{2.17}$$

with $\alpha_0 > 0$, $\alpha_i \geq 0, i = 1, ..., q$, and $\beta_j \geq 0, j = 1, ..., p$. A GARCH process with orders p and q is denoted as GARCH(p,q). There are several other versions of GARCH models, such as exponential GARCH, GJR GARCH, non-linear GARCH, etc. GARCH models are the most successful models to model the volatilities of financial time series. They have been applied in several finance fields.

2.3 Some Approaches to Dealing with Volatility Smile

Practitioners and researchers have proposed many methods to deal with the problem of volatility smile. The simplest way to incorporate the volatility smile is to use the volatility matrix. Market prices of options are used to generate implied volatilities. The volatility matrix replicates the volatility surface. When we want to price a new option, we can pick up a corresponding volatility and apply the B-S model to get the price. An alternative way to account for the volatility smile is to smooth the implied volatility relation across strike prices and maturities. For example, Dumas, Fleming and Whaley (1998) use the following function to fit the volatility surface,

$$\sigma = a_0 + a_1 K + a_2 K^2 + a_3 T + a_4 T^2 + a_5 KT, \tag{2.18}$$

where K is the strike price and T is the time to maturity. Applying the B-S model in this way is internally inconsistent because the B-S model assumes a constant volatility. But this procedure is quite useful in practice as a means of predicting option prices. Many empirical researchers employ this *ad hoc* Black-Scholes model when they test other complicated models. In many cases, this simple model is found to be able to beat those complicated models.

One way to modify the constant volatility assumption of the B-S model is to assume that volatility is a deterministic function of time and stock price: $\sigma = \sigma(t, S)$. In the special case of $\sigma = \sigma(t)$, a deterministic function of time, the option price will just be the B-S formula with volatility $\sqrt{\overline{\sigma^2}}$, where $\overline{\sigma^2} = \frac{1}{T}\int_0^T \sigma^2(s)ds$.

Another famous model with deterministic volatility is the *constant elasticity of variance* (CEV) model proposed by Cox and Ross (1976). The stock price in this model is

$$dS = \mu S dt + \sigma S^\alpha dz, \tag{2.19}$$

where α is a positive constant. So the stock price has volatility $\sigma S^{\alpha-1}$. When $\alpha = 1$, we have the Black-Scholes case. When $\alpha < 1$, the volatility increases as the stock price decreases. This can generate a distribution with a fatter left tail and a thinner right tail. When $\alpha > 1$, the situation is reversed. So the volatility smile can be incorporated in this model. Cox and Ross (1976) also provide valuation formulas for European call and put options in the CEV model. Several studies have reported that the CEV model outperforms the B-S model in most cases. The problem of the model is that the option price in the CEV model will approach either zero or infinite in the long run.

Derman and Kani (1994) proposed a so-called *implied tree* model, which also assumes that the volatility is a deterministic function of stock price and time. The implied tree model is also called the *local volatility* model. The model has several versions developed by different researchers. These models incorporate the volatility smile in the construction of the tree. And this tree is particularly used for pricing exotic options. By the implied tree, the prices of exotics are consistent with all traded vanilla options. Implied trees are quite heavily used among traders. But it is difficult to get them to calibrate vanilla market data, because the local volatility surface can be very irregular.

Deterministic volatility models allow volatility to change in a deterministic way. But empirical evidence shows the variance of the stock returns is not stationary. The relation between the volatility and the stock or time changes with time. We model the volatility as a function of the stock price and time this week. But next week, this function will be quite different. Hence, it is not enough to allow volatility to change deterministically. Subsequent researches model the volatility as a stochastic variable. In SV models, volatility is modelled as a separate stochastic process. Details about SV models will be discussed in Chapter 3.

Another important strand in option pricing models is modelling stock prices with jumps. These models are motivated by the fact that the price exhibits jumps rather than continuous changes. Merton (1976) added random jumps to the geometric Brownian motion. The stochastic process for the stock price is

$$\frac{dS}{S} = (\mu - \lambda k)dt + \sigma dz + dp, \tag{2.20}$$

where λ is an average number of jumps per year, k is the average jump size, dp is the Poisson process generating the jumps, other parameters are similar to those in the B-S model. This jump diffusion model is useful when the underlying asset price has large changes, because continuous-time models cannot capture this property.

Some other researchers even model the stock price as a pure jump process. They also combine stochastic volatility and jump diffusion. Stock prices in these complicated models can have many good properties. But these models are hard to implement.

In this essay, I focus on SV models and take Heston's SV model as an example to discuss the implementation. The SV model has become more and more popular among practitioners and academic researchers. It is said to be the next generation of the option pricing model. Heston's SV model is easy to implement. This makes Heston's SV model the most popular SV model.

Chapter 3

Stochastic Volatility Models

3.1 A General Model

A general representation of the continuous-time SV model we are considering may be written as

$$dS_t = \mu S_t dt + \sigma_t S_t dW_{1t}, \tag{3.1}$$

$$\sigma_t = f(Y_t), \tag{3.2}$$

$$dY_t = a(t, Y_t)dt + b(t, Y_t)dZ_t, \tag{3.3}$$

$$dW_{1t}dZ_t = \rho dt. \tag{3.4}$$

Here the drift μ is still a constant. σ_t is the volatility of the stock price. f is some positive function. Y_t is some underlying process which determines the volatility. W_{1t} and Z_t are two correlated standard Brownian motions. The constant parameter ρ is the correlation coefficient between these two Brownian motions. We can also rewrite Z_t as

$$Z_t = \rho W_{1t} + \sqrt{1-\rho^2}W_{2t}, \tag{3.5}$$

where W_{2t} is a standard Brownian motion independent of W_{1t}. There are some economic arguments for a negative correlation between stock price and volatility shocks. Empirical studies also show $\rho < 0$ from stock data.

This general SV model contains many famous SV models in financial literature. The following are three examples.

(1) Hull-White Hull and White (1987) assume a geometric Brownian motion for the variance,

$$dY_t = \alpha Y_t dt + \beta Y_t dZ_t, \tag{3.6}$$

where α and β are constants. $f(y) = \sqrt{y}$. This is the first SV model for pricing options in financial literature. Hull and White derived a closed-form formula for the European option in a particular case in which $\rho = 0$. Generally, numerical techniques are required when implementing this model.

(2) Stein-Stein Stein and Stein (1991) assume the driving process Y_t is an Ornstein-Uhlenbeck (OU) process,

$$dY_t = \alpha(\omega - Y_t)dt + \beta dZ_t. \tag{3.7}$$

It is a mean-reverting process. From econometric studies, people believe that volatility is mean-reverting. So OU process is employed by many researchers to model the volatility. But it is not appropriate to simply assume that the volatility is an OU process, because Y_t can be negative in OU process. Stein and Stein (1991) assume $f(y) = |y|$. They also give a closed-form formula for the European option when $\rho = 0$. Some other people use $f(y) = e^y$ instead, e.g. Scott (1987).

(3) Heston Heston (1993) assumes Y_t follows a Cox-Ingersoll-Ross (CIR) pross,

$$dY_t = \kappa(\theta - Y_t)dt + \xi\sqrt{Y_t}dZ_t, \tag{3.8}$$

and $f(y) = \sqrt{y}$. Y_t is strictly positive when $2\kappa\theta \geq \xi^2$ and non-negative when $0 \leq 2\kappa\theta < \xi^2$. This model is very important because it provides a closed-form formula for the European option and ρ can be non-zero. It will be discussed in details in the next section.

3.2 Pricing Options

In order to price options in the SV model, we can apply no-arbitrage arguments, or use the risk-neutral valuation method. First we discuss the no-arbitrage method. The riskless portfolio is constructed as in the Black-Scholes model. But the construction method is different. In the SV option pricing model, there is only one traded risky asset S but two random sources W_{1t} and Z_t. So the market is incomplete. We cannot perfectly replicate the option solely with the underlying stock. No-arbitrage arguments are not enough to give the option price. We need additional assumptions. In the following derivation, equilibrium arguments are also employed. We know that the market can be completed by adding any option written on stock S. Simply, the market is complete when we have two traded assets, the underlying asset S and a benchmark option G. Then all other options can be replicated by these two traded assets. A riskless portfolio Π consists of an option F which we want to price, $-\Delta_1$ shares of the underlying asset S, and $-\Delta_2$ shares of the benchmark option G.

$$\Pi = F - \Delta_1 S - \Delta_2 G. \tag{3.9}$$

The portfolio is self-financing, so that

$$d\Pi = dF - \Delta_1 dS - \Delta_2 dG. \tag{3.10}$$

F and G are functions of variables t, S_t and Y_t. By applying the two-dimensional vertion of Ito's formula, (3.10) becomes

$$\begin{aligned} d\Pi &= \left[\frac{\partial F}{\partial t} + \frac{1}{2}f(Y)^2S^2\frac{\partial^2 F}{\partial S^2} + \frac{1}{2}b^2\frac{\partial^2 F}{\partial Y^2} + \rho f(Y)Sb\frac{\partial^2 F}{\partial S\partial Y}\right]dt + \frac{\partial F}{\partial S}dS + \frac{\partial F}{\partial Y}dY \\ &\quad -\Delta_1 dS - \Delta_2\left\{\left[\frac{\partial G}{\partial t} + \frac{1}{2}f(Y)^2S^2\frac{\partial^2 G}{\partial S^2} + \frac{1}{2}b^2\frac{\partial^2 G}{\partial Y^2} + \rho f(Y)Sb\frac{\partial^2 G}{\partial S\partial Y}\right]dt \right. \\ &\qquad \left. + \frac{\partial G}{\partial S}dS + \frac{\partial G}{\partial Y}dY\right\}. \end{aligned} \tag{3.11}$$

We can rewrite it by collecting the terms of dt, dS and dY,

$$\begin{aligned} d\Pi &= \left[\frac{\partial F}{\partial t} + \frac{1}{2}f(Y)^2S^2\frac{\partial^2 F}{\partial S^2} + \frac{1}{2}b^2\frac{\partial^2 F}{\partial Y^2} + \rho f(Y)Sb\frac{\partial^2 F}{\partial S\partial Y}\right]dt \\ &\quad -\Delta_2\left[\frac{\partial G}{\partial t} + \frac{1}{2}f(Y)^2S^2\frac{\partial^2 G}{\partial S^2} + \frac{1}{2}b^2\frac{\partial^2 G}{\partial Y^2} + \rho f(Y)Sb\frac{\partial^2 G}{\partial S\partial Y}\right]dt \\ &\quad + \left[\frac{\partial F}{\partial S} - \Delta_2\frac{\partial G}{\partial S} - \Delta_1\right]dS + \left[\frac{\partial F}{\partial Y} - \Delta_2\frac{\partial G}{\partial Y}\right]dY. \end{aligned} \tag{3.12}$$

To make the portfolio riskless, we choose

$$\frac{\partial F}{\partial S} - \Delta_2\frac{\partial G}{\partial S} - \Delta_1 = 0, \tag{3.13}$$

$$\frac{\partial F}{\partial Y} - \Delta_2\frac{\partial G}{\partial Y} = 0 \tag{3.14}$$

to eliminate dS term and dY term. Solving (3.13) and (3.14) gives

$$\Delta_2 = \frac{\partial F}{\partial Y}\bigg/\frac{\partial G}{\partial Y}, \tag{3.15}$$

$$\Delta_1 = \frac{\partial F}{\partial S} - \frac{\partial G}{\partial S}\frac{\partial F}{\partial Y}\bigg/\frac{\partial G}{\partial Y} \tag{3.16}$$

The portfolio is riskless if we rebalance it according to (3.15) and (3.16). On the other hand, the riskless portfolio must earn a risk-free rate, otherwise there would be an arbitrage opportunity. So

$$\begin{aligned} d\Pi &= \left[\frac{\partial F}{\partial t} + \frac{1}{2} f(Y)^2 S^2 \frac{\partial^2 F}{\partial S^2} + \frac{1}{2} b^2 \frac{\partial^2 F}{\partial Y^2} + \rho f(Y) S b \frac{\partial^2 F}{\partial S \partial Y}\right] dt \\ &\quad -\Delta_2 \left[\frac{\partial G}{\partial t} + \frac{1}{2} f(Y)^2 S^2 \frac{\partial^2 G}{\partial S^2} + \frac{1}{2} b^2 \frac{\partial^2 G}{\partial Y^2} + \rho f(Y) S b \frac{\partial^2 G}{\partial S \partial Y}\right] dt \\ &= r \Pi dt, \end{aligned} \tag{3.17}$$

where Π is given by (3.9). Substituting (3.15) and (3.16) into (3.17) gives

$$\begin{aligned} &\left[\frac{\partial F}{\partial t} + \frac{1}{2} f(Y)^2 S^2 \frac{\partial^2 F}{\partial S^2} + \frac{1}{2} b^2 \frac{\partial^2 F}{\partial Y^2} + \rho f(Y) S b \frac{\partial^2 F}{\partial S \partial Y} - rF + rS \frac{\partial F}{\partial S}\right] \Big/ \frac{\partial F}{\partial Y} \\ = &\left[\frac{\partial G}{\partial t} + \frac{1}{2} f(Y)^2 S^2 \frac{\partial^2 G}{\partial S^2} + \frac{1}{2} b^2 \frac{\partial^2 F}{\partial Y^2} + \rho f(Y) S b \frac{\partial^2 G}{\partial S \partial Y} - rG + rS \frac{\partial G}{\partial S}\right] \Big/ \frac{\partial G}{\partial Y} \end{aligned} \tag{3.18}$$

Notice that the left-hand side is a function of F only and the right-hand side is a function of G only. The only way that this equation holds is that both sides are equal to some function, $-k(t, S, Y)$, which is independent of any particular option. The equation for the option F can be written as

$$\frac{\partial F}{\partial t} + \frac{1}{2} f(Y)^2 S^2 \frac{\partial^2 F}{\partial S^2} + \frac{1}{2} b^2 \frac{\partial^2 F}{\partial Y^2} + \rho f(Y) S b \frac{\partial^2 F}{\partial S \partial Y} - rF + rS \frac{\partial F}{\partial S} = -k(t, S, Y) \frac{\partial F}{\partial Y}. \tag{3.19}$$

We denote $k(t, S, Y)$ by

$$k(t, S, Y) = a(t, Y) - b(t, Y) \Lambda(t, S, Y), \tag{3.20}$$

where $a(t, Y)$ is the drift term of the driving process Y_t, $\Lambda(t, S, Y)$ is called the *market price of volatility risk*. $\Lambda(t, S, Y)$ can not be determined by the arbitrage theory alone. In theory, it is completely determined by the benchmark option G. So the market determines the price of volatility risk. Finally, the PDE for the option becomes

$$\frac{\partial F}{\partial t} + \frac{1}{2} f(Y)^2 S^2 \frac{\partial^2 F}{\partial S^2} + \frac{1}{2} b^2 \frac{\partial^2 F}{\partial Y^2} + \rho f(Y) S b \frac{\partial^2 F}{\partial S \partial Y} - rF + rS \frac{\partial F}{\partial S} + (a - b\Lambda) \frac{\partial F}{\partial Y} = 0. \tag{3.21}$$

Given the terminal condition for F, the PDE (3.21) is solvable under some special driving processes.

We can also apply the risk-neutral valuation method to the SV model. The market is incomplete. But it is still free of arbitrage. The equivalent martingale measure is not unique. We have to choose one of all these measures to price the options. So the price of the option is also not unique. It will depend on which equivalent martingale measure we use.

The problem here is how to construct equivalent martingale measures and what the asset price process and the volatility process will be under these measures. To change measure from the objective measure P to an equivalent martingale measure Q, we need to use the Girsanov Theorem. Recall that we have two Brownian motions in the SV model, W_{1t} and Z_t. And Z_t can be expressed as $Z_t = \rho W_{1t} + \sqrt{1-\rho^2} W_{2t}$, where W_{1t} and W_{2t} are independent. Under any equivalent martingale measure Q, the underlying stock S_t will have a drift rS_t. Then the discounted stock price will be a martingale under Q. But it is not the case for volatility σ_t or the underlying driving process Y_t, because volatility is not a traded asset.

To make the the drift term of the stock price process be rS_t under Q, we set

$$d\widetilde{W}_{1t} = dW_{1t} + \frac{\mu - r}{f(Y_t)} dt. \tag{3.22}$$

Since W_{2t} is independent of W_{1t}, shift of W_{2t} will not change the drift of the stock price process. We choose the shift of W_{2t} as

$$d\widetilde{W}_{2t} = dW_{2t} + \gamma_t dt. \tag{3.23}$$

By the Girsanov Theorem, $\widetilde{W}_{1t}$ and $\widetilde{W}_{2t}$ are two independent standard Browmian motions under the measure Q, which is defined by the following Radon-Nikodym derivative

$$\frac{dQ}{dP} = exp\left(-\frac{1}{2}\int_0^T (\theta_{1t}^2 + \theta_{2t}^2) dt - \int_0^T \theta_{1t} dW_{1t} - \int_0^T \theta_{2t} dW_{2t}\right), \tag{3.24}$$

where

$$\begin{aligned} \theta_{1t} &= \frac{\mu - r}{f(Y_t)}, \\ \theta_{2t} &= \gamma_t. \end{aligned}$$

Here γ_t is an adapted process. To make Q be a probability measure, we need to make some assumptions on f and γ_t. Under the measure Q

$$dS_t = rS_t dt + \sigma_t S_t d\widetilde{W}_{1t}, \tag{3.25}$$

$$\sigma_t = f(Y_t), \tag{3.26}$$

$$dY_t = [a(t, Y_t) - b(t, Y_t)\Lambda(t, S_t, Y_t)]dt + b(t, Y_t) d\widetilde{Z}_t, \tag{3.27}$$

where

$$\widetilde{Z}_t = \rho\widetilde{W}_{1t} + \sqrt{1-\rho^2}\widetilde{W}_{2t}, \tag{3.28}$$

$$\Lambda(t, S_t, Y_t) = \rho\theta_{1t} + \sqrt{1-\rho^2}\theta_{2t} = \rho\frac{\mu - r}{f(Y_t)} + \sqrt{1-\rho^2}\gamma_t. \tag{3.29}$$

For every choice of regular enough process γ_t, we get an equivalent martingale measure Q which we can use it to price options. Let V_t be the price of an option with payoff H at maturity T, then the price of the option is

$$V_t = E^Q[e^{-r(T-t)}H|\mathcal{F}_t]. \tag{3.30}$$

When $\gamma = \gamma(t, S_t, Y_t)$, we have the Markovian setting. Applying Feynman-Kac theorem, we can get the same partial differential equation as (3.21). In practice, we can also use the Monte Carlo simulation to evaluate (3.30).

$\Lambda(t, S_t, Y_t)$ in equation (3.21) is the same as that in (3.27), although the derivation is independent. Recall that $\Lambda(t, S_t, Y_t)$ is called the market price of volatility risk. This name is very natural when we look at the drift term of the process dY_t in (3.27). Notice that in equation (3.21), the coefficient of $\frac{\partial F}{\partial Y}$ is just the risk-neutralized drift term of the process dY_t, which is $a(t, Y_t) - b(t, Y_t)\Lambda(t, S_t, Y_t)$. While the coefficient of $\frac{\partial F}{\partial S}$ is also the risk-neutralized drift term rS. So now we know why we choose $k(t, S, Y)$ to have the form (3.20). Further more, we know from (3.29) that $\Lambda(t, S_t, Y_t)$ can be decomposed into two parts which correspond to the two independent Brownian motion.

3.3 Hedging Options

In this section, I will give a brief introduction to some hedging strategies under stochastic volatility. In the B-S world, the market is complete. An option can be perfectly hedged by dynamically trading the underlying stock. The *delta* of an option, Δ, was introduced in Section 2.1 when deriving the B-S equation. A portfolio consisting of a short position in a call option and Δ units of stocks is locally riskless. Generally, the *delta* can be defined as the rate of change of option price F with respect to the stock price S,

$$\Delta = \frac{\partial F}{\partial S}. \tag{3.31}$$

However, the market is incomplete in the SV model. A perfect hedging is impossible by trading only the underlying stock. In the following several paragraphs, I introduce some option hedging strategies for the SV model.

We know from Section 3.2 that the market can be completed by adding a benchmark option G. So a perfect hedging can be achieved by trading the option G as well as the underlying stock S. Suppose that we want to hedge a short position in a call option F with strike price K. Option G can be a call option with the same maturity but a different strike price K' . From Section 3.2, the hedger will need a position in (i) Δ_1 units of the underlying stock, and (ii) Δ_2 units of option G, where Δ_1 and Δ_2 are given by (3.15) and (3.16). Traders define the *vega* of a portfolio (or an option) as the rate of change of the value of the portfolio with respect to the volatility of the underlying stock,

$$\nu = \frac{\partial F}{\partial \sigma}. \tag{3.32}$$

So the *vega* represents the volatility risk of the option. The hedging strategy we discussed above is similar to the construction of a *delta*-neutral and also *vega*-neutral portfolio.

We know now by dynamically trading the underlying asset and another option, both the price risk and the volatility risk can be hedged. In practice, continuous trading is impossible. Traders can rebalance the portfolio daily or at other frequencies. If the frequency is high, the transaction cost can be a problem. The transaction cost is much higher for trading options than for trading stocks. So traders prefer to use stocks to hedge options. Researchers have proposed some hedging strategies using only stocks in incomplete markets. *Superhedging*, *mean-variance hedging* and *shortfall hedging* are three frequently discussed strategies in financial literature.

Consider a path-independent claim with a payoff $H(S_T)$ at maturity T. If the payoff $H(S_T)$ is affine in the stock, then this claim can be perfectly replicated even if in the incomplete market. However, most claims are not affine in the stock, e.g. European call options. A *superhedging* is a trading strategy π, which starts with an initial wealth x, and has a terminal payoff $X^{x,\pi}(T) \geq H(S_T)$ almost surely. A simple example of superhedging is to buy one unit of stock to hedge a short position in a call option. The task here is to find the cheapest superhedging, which is an upper bound of the option price,

$$P^{up} = inf\{x \geq 0, \exists \pi, X^{x,\pi}(T) \geq H(S_T)\}. \tag{3.33}$$

Cvitanic, Pham, and Touzi (1999) analyze superheding for the SV model. From the viewpoint of the option writer, the superhedging is very attractive, since his hedging portfolio always has a value no less than the written option. But in practice, the superhedging solution for the SV model is not very satisfactory. An investor cannot charge P^{up} when he enters a short position in an option contract. Usually the initial cost P^{up} for a superhedging strategy is unfeasibly high in the SV model. In realistic situations, the option premium is lower than P^{up}. So the investor with a short position cannot implement a superhedging strategy. So he is unable to eliminate all the risk. Therefore he has to find a more realistic hedging strategy under a cost constraint which is the option premium.

Follmer and Sondermann (1986) use a quadratic loss function to assess the quality of trading strategies. Since a perfect replication is impossible in incomplete markets, they instead try to partially hedge the risk as much as possible, i.e. to find a trading strategy that most closely approximate the payoff under some criteria. One natural criteria is minimizing the mean-squared error. Following the previous denotation, the optimal replication problem becomes,

$$\min_{\pi} E[X^{x,\pi}(T) - H(S_T)]^2. \tag{3.34}$$

This leads to the notion of *mean-variance hedging*.

Another choice of the criteria is

$$\min_{\pi} E[l((X^{x,\pi}(T) - H(S_T))^+)] \tag{3.35}$$

where l is a convex function corresponding to risk aversion. The simple quadratic loss function is replaced by a loss function representing a shortfall risk. The problem of using a quadratic loss function is both the shortfalls and overshoots are penalized. But in practice the overshoot should not receive a penalty. In (3.35) only the shortfall is penalized. The shortfall risk is defined as the expectation of the shortfall weighted by the loss function l. Interested readers can find the results of the *shortfall hedging* strategy in Follmer and Leukert (2000).

Chapter 4

A Special Case: Heston's SV Model

4.1 An Introduction to the Model

SV models have been used by many traders to price and hedge options. Among the several existed SV models, Heston's SV model is the most popular one. This chapter will focus on Heston's SV model. I will first introduce Heston's setting for the volatility process. The partial differential equation for the option price under Heston's model will be provided. Then we will discuss how to solve this PDE for a European option by a method based on characteristic functions. Greeks will also be derived. And then we will discuss the implementation of Heston's model and some properties of the model results.

Heston (1993) assumes Y_t in (3.3) follows a Cox-Ingersoll-Ross (CIR) pross,

$$dY_t = \kappa(\theta - Y_t)dt + \xi\sqrt{Y_t}dZ_t, \tag{4.1}$$

and $f(y) = \sqrt{y}$. We rewrite the model in the following way for convenience,

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dZ_{1t}, \tag{4.2}$$

$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}dZ_{2t}, \tag{4.3}$$

$$dZ_{1t}dZ_{2t} = \rho dt, \tag{4.4}$$

where Z_{1t} and Z_{2t} are two standard Brownian motions with correlation ρ. θ is the long-run variance, κ is the rate of mean reversion, ξ is called *volatility of volatility* or *volatility of variance*. Here we assume there is no dividend payment and the interest rate is constant. By the same argument of Section 3.2, the value of any option $U(S,v,t)$ must satisfy the same PDE (3.21),

$$\frac{\partial U}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2}+\frac{1}{2}\xi^2 v\frac{\partial^2 U}{\partial v^2}+\rho\xi vS\frac{\partial^2 U}{\partial S\partial v}-rU+rS\frac{\partial U}{\partial S}+\{\kappa[\theta-v]-\Lambda(S,v,t)\xi\sqrt{v}\}\frac{\partial U}{\partial v}=0. \tag{4.5}$$

$\Lambda(S,v,t)$ is the *market price of volatility risk*[1]. Heston (1993) chooses the market price of volatility risk to be proportional to the volatility, i.e. $\Lambda(S,v,t) = k\sqrt{v}$. Or $\Lambda(S,v,t)\xi\sqrt{v} = k\xi v$. Let $\lambda = k\xi$, so the coefficient of $\frac{\partial U}{\partial v}$ in (4.5) becomes $\{\kappa[\theta - v] - \lambda v\}$. This choice of market price of volatility risk gives us analytical advantages. The drift term of the specified process (4.3) is an affine function of the state variable itself. The affinity makes the model easier to solve. The market price of volatility risk is assigned to be proportional to the square root of the variance. Since the diffusion of the variance process is also proportional to the square root of the variance, the product of the market price of risk and the diffusion is proportional to variance itself. As a result, the drift term will remain affine under the equivalent martingale measure. This particular market price of volatility risk helps the model to have a closed-form solution.

4.2 A Closed-Form Solution

Heston (1993) solves this equation for a European option by using a technique based on characteristic functions. This method can also be used to solve some other important models, so we give a brief introduction here. The solution of the equation has the form which is similar to the Black-Scholes formula,

$$C(S,v,t) = SP_1 - Ke^{-r(T-t)}P_2. \tag{4.6}$$

Let

$$x = ln(S), \tag{4.7}$$

and substitute the proposed solution (4.6) into the (4.5). Then we have two PDEs which P_1 and P_2 should satisfy,

$$\frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2}+\rho\xi v\frac{\partial^2 P_j}{\partial x\partial v}+\frac{1}{2}\xi^2 v\frac{\partial^2 P_j}{\partial v^2}+(r+u_j v)\frac{\partial P_j}{\partial x}+(a-b_j v)\frac{\partial P_j}{\partial v}+\frac{\partial P_j}{\partial t}=0, \tag{4.8}$$

[1] In Heston (1993), the coefficient of $\frac{\partial U}{\partial v}$ in (4.5) is $\{\kappa[\theta - v] - \Lambda(S,v,t)\}$. The difference comes from the different definition of $\Lambda(S,v,t)$. Our definition is consistent with the notation in Section 3.2. The results will become the same after we specify $\Lambda(S,v,t)$.

for $j = 1, 2$, where

$$u_1 = \frac{1}{2}, \ \ u_2 = -\frac{1}{2}, \ \ a = \kappa\theta, \ \ b_1 = \kappa + \lambda - \rho\xi, \ \ b_2 = k + \lambda. \tag{4.9}$$

These PDEs must be solved subject to the terminal condition

$$P_j(x, v, T; ln(K)) = 1_{\{x \geq ln(K)\}}. \tag{4.10}$$

They can be interpreted as "risk-neutralized" probabilities. The corresponding characteristic functions $f_1(x, v, t; \phi)$ and $f_2(x, v, t; \phi)$ will satisfy similar PDEs,

$$\frac{1}{2}v\frac{\partial^2 f_j}{\partial x^2} + \rho\xi v\frac{\partial^2 f_j}{\partial x \partial v} + \frac{1}{2}\xi^2 v\frac{\partial^2 f_j}{\partial v^2} + (r + u_j v)\frac{\partial f_j}{\partial x} + (a - b_j v)\frac{\partial f_j}{\partial v} + \frac{\partial f_j}{\partial t} = 0 \tag{4.11}$$

with the terminal condition

$$f_j(x, v, T; \phi) = e^{i\phi x}. \tag{4.12}$$

Due to the linearity of the coefficients we guess a functional form

$$f_j(x, v, t; \phi) = e^{C(T-t;\phi) + D(T-t,\phi)v + i\phi x}. \tag{4.13}$$

Substituting this functional form into the PDE (4.11), we have two ordinary differential equations,

$$-\frac{1}{2}\phi^2 + \rho\xi\phi i D + \frac{1}{2}\xi^2 D^2 + u_j\phi i - b_j D + \frac{\partial D}{\partial t} = 0, \tag{4.14}$$

$$r\phi i + aD + \frac{\partial C}{\partial t} = 0, \tag{4.15}$$

subject to the terminal condition $C(0; \phi) = 0$, $D(0; \phi) = 0$. [2] The solutions to these two ODEs are

$$C(\tau; \phi) = r\phi i\tau + \frac{a}{\xi^2}\left\{(b_j - \rho\xi\phi i + d)\tau - 2ln\left[\frac{1 - ge^{d\tau}}{1 - g}\right]\right\}, \tag{4.16}$$

$$D(\tau; \phi) = \frac{b_j - \rho\xi\phi i + d}{\xi^2}\left[\frac{1 - e^{d\tau}}{1 - ge^{d\tau}},\right] \tag{4.17}$$

and

$$g = \frac{b_j - \rho\xi\phi i + d}{b_j - \rho\xi\phi i - d}, \tag{4.18}$$

[2] The ODE(4.14) is a little different from the one in Heston 1993). I think there are some typos in that paper.

$$d = \sqrt{(\rho\xi\phi i - b_j)^2 - \xi^2(2u_j\phi i - \phi^2)}. \tag{4.19}$$

Then the risk-neutral probabilities, P_1 and P_2 can be recovered by inverting the corresponding characteristic functions,

$$P_j(x, v, t; ln(K)) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi ln(K)} f_j(x, v, t; \phi)}{i\phi}\right] d\phi. \tag{4.20}$$

So the price of European call options is given by equations (4.6), (4.13), and (4.20). The price of European put options can be derived by the put-call parity. The essence of the characteristic function methodology is using Fourier analysis. The general inverse Fourier transform for pricing options is discussed in Lewis (2000). Fast Fourier transform algorithm is introduced by Carr and Madan (1999) to price options.

Heston (1993) also incorporates the stochastic interest rates into the model and apply the SV model to bond options and foreign currency options.

Since we have known the closed-form solution of the European call option, we can calculate the Greeks easily. Here I give the expressions of the three most frequently used Greeks: *delta*, *gamma* and *vega*.

The *delta* of the call is simply given by

$$\Delta = \frac{\partial C(S, v, t)}{\partial S} = P_1. \tag{4.21}$$

The *gamma* of an option is defined as the second derivative of the option price with respect to the stock price. So in Heston's model, *gamma* is given by

$$\Gamma = \frac{\partial^2 C(S, v, t)}{\partial S^2} = \frac{\partial \Delta}{\partial S} = \frac{\partial \Delta}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{S}\frac{\partial P_1}{\partial x}, \tag{4.22}$$

and

$$\frac{\partial P_1}{\partial x} = \frac{1}{\pi}\int_0^\infty Re[e^{-i\phi ln(K)} f_j(x, v, t; \phi)] d\phi. \tag{4.23}$$

Usually the *vega* of an option refers to the first derivative of the option with respect to volatility. For convenience, in Heston's model we define *vega* as the first derivative with respect to the spot variance,

$$\nu = \frac{\partial C(S, v, t)}{\partial v} = S\frac{\partial P_1}{\partial v} - Ke^{-r(T-t)}\frac{\partial P_2}{\partial v}, \tag{4.24}$$

where

$$\frac{\partial P_j}{\partial v} = \frac{1}{\pi}\int_0^\infty Re\left[\frac{D(T-t;\phi)e^{-i\phi ln(K)} f_j(x, v, t; \phi)}{i\phi}\right] d\phi. \tag{4.25}$$

Other Greeks can be derived similarly.

4.3 Implementations and Results

In Figure 4.1 a sample path of stock price under Heston's model is simulated using parameters in the Table 4.1. The movement is under the objective measure P.

Table 4.1: **Parameters for stock price simulation under Heston's model**

Spot stock price	$S_0 = 1$
Drift	$\mu = 0.2$
Time horizon	1 year
Rate of mean reversion	$\kappa = 2$
Long-run variance	$\theta = 0.04$
Spot variance	$v_0 = 0.04$
Correlation	$\rho = -0.5$
Volatility of variance	$\xi = 0.1$

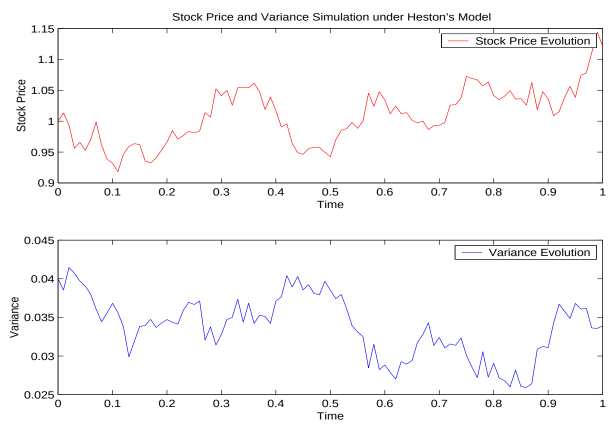

Figure 4.1: Simulation of stock prices under Heston's model

To exam the effects of some parameters on the option prices, we first consider the stock price process and the variance process under the equivalent martingale measure Q. From section 4.1 , the risk-neutralized process for the variance is

$$dv_t = \kappa^*(\theta^* - v_t)dt + \xi\sqrt{v_t}dZ_{2t}, \tag{4.26}$$

where $\kappa^* = \kappa + \lambda$ and $\theta^* = \kappa\theta/(\kappa + \lambda)$. The variance moves towards a long-run average variance θ^*, with a speed determined by κ^*. An increase in θ^* can increase option prices. And κ^* determines the relative weights of the spot variance and the long-run average variance. The process (4.26) is employed if we want to price options by using Monte Carlo simulation.

The closed-form solution (4.6), (4.13), and (4.20)for Heston's model can be implemented in some softwares. The solution contains complex numbers. A tool pack for the complex arithmetics is needed for some softwares, e.g. *VBA* and *C++*. But it is slow. A better way is to code the arithmetics by ourself. *Matlab* can calculate the complex numbers directly. I implement the solution in both *VBA* and *Matlab* to check the coding correctness. Another problem in the closed-form solution is the infinite integral. I use Gauss-Laguerre approximation for infinite integrals with 18 points. The integrands are oscillating. But when the strike price is not too far from the stock price, the approximation is reasonable and the error is quite small.

After we can calculate the option prices in Heston's model, we can investigate other model parameters. Here I focus on the two most important parameters: correlation ρ and volatility of variance ξ. We want to see whether Heston's model can explain the pricing biases of the B-S model. There are many empirical literature to test SV models, e.g. Bakshi, Cao, and Chen (1997). Usually, researchers use market data for options and stocks to estimate the model parameters and then exam the in-the-sample and out-of-the-sample pricing errors. Here I use another approach. First I calculate option prices for a set of different strike prices and time-to-maturities under Heston's model. Then these option prices are used as inputs to the B-S model and the implied volatilities are backed out. Recall that we use market prices to back out the market implied volatilities. We want to see whether Heston's model can generate volatility smiles and volatility surfaces as implied by the market prices.

In Figure 4.2, we investigate the effect of ρ on the implied volatility surface generated under Heston's model. In Panel A, $\rho = 0$. Away-from-the-money options have higher implied volatilities than near-the-money options. This is consistent with the 'smile' shape of implied volatilities in some financial market, e.g. currency options markets. This observation can be explained by the fat-tailed distribution of returns. We can also find that the 'smile' flattens when the time to maturity increases. This is also consistent with the real financial markets. The B-S model tends to work well for options with long maturities as a result of the corresponding flattened smile.

In Panel B, ρ is negative. We can find that in-the-money calls have higher implied volatilities, whereas out-of-the-money calls have lower implied volatilities. This is consistent with the phenomenon of 'volatility skew' in some financial markets, especially the equity options markets. Panel C has a positive ρ, which can generate

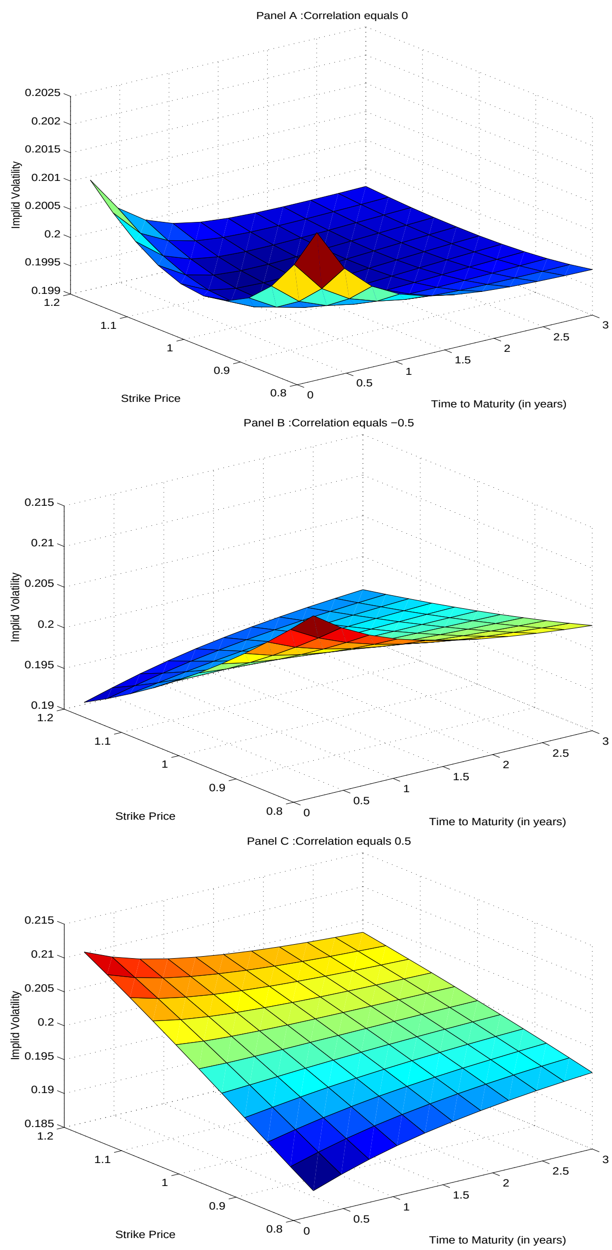

Figure 4.2: A plot of implied volatilities for option prices under Heston's model. Strike prices vary between 0.8 to 1.2. Time to maturities are between 0.2 year to 3 years. $S = 1$, $r = 0.01$, $\kappa = 2$, $\theta = 0.04$, $v_0 = 0.04$, $\xi = 0.1$, $\lambda = 0$. Three different correlations ρ are shown.

an opposite skew shape compared with Panel B. Such kind of volatility skew may appear in energy options markets.

The effect of ξ is investigated in Figure 4.3 for different correlations. We can find that ξ controls the significance of the 'smile' or the 'skew'. A small ξ results in a flat smile, while a large ξ gives us a significant smile.

From the viewpoint of the return distribution, the parameter ρ is related to the skewness. A negative ρ corresponds to a distribution with a negative skewness. Empirical findings show that stock prices usually have a negative skewness. If the model only allows $\rho = 0$, we cannot generate a skewed volatility surface. So Heston's model is better than the Hull-White model as Hull and White give a closed-form solution only for the case $\rho = 0$. The parameter ξ is related to the kurtosis of the distribution. Stock price distribution is found to be leptokurtic, which means the return distribution has fatter tails compared with the normal distribution. So a positive ξ is also important to capture the return distribution implied by the market prices.

4.4 Estimation of Parameters

Heston's model can capture the smile effect. However, to apply Heston's model, or other SV models, we need to know the model parameters. Strike price K and time to maturity T are specified in the contract. And We know spot stock price S_t, interest rate r from the market. However, the spot variance and its related structural parameters $(\kappa, \theta, \xi, \rho)$ are unobservable and need to be estimated. Market price of volatility risk λ is also unknown. Estimation of these parameters is not an easy task. The likelihood functions are not known in closed form for continuous-time SV models, since the observations are discrete. So the maximum likelihood method is very difficult to implement.

There are several econometric literature on estimation of SV models. One can use the underlying stock prices only to estimate the structural parameters. Indirect inference method is first proposed by Gourieroux, Monfort, and Renault (1993). This method is a simulation based moment matching method. Some studies have applied indirect inference method to the SV models. Ait-Sahalia and Kimmel (2004) use closed-form approximations to the true but unknown likelihood function and then employ the maximum likelihood method. Alternatively, researchers extract the variance from option prices. For example, the implied volatility of at-the-money options are used to be a proxy for the instantaneous volatility of the stock. When a time series of implied volatilities or variances is available, subsequent estimation techniques can be applied. Frequently used methods for SV models include the gen-

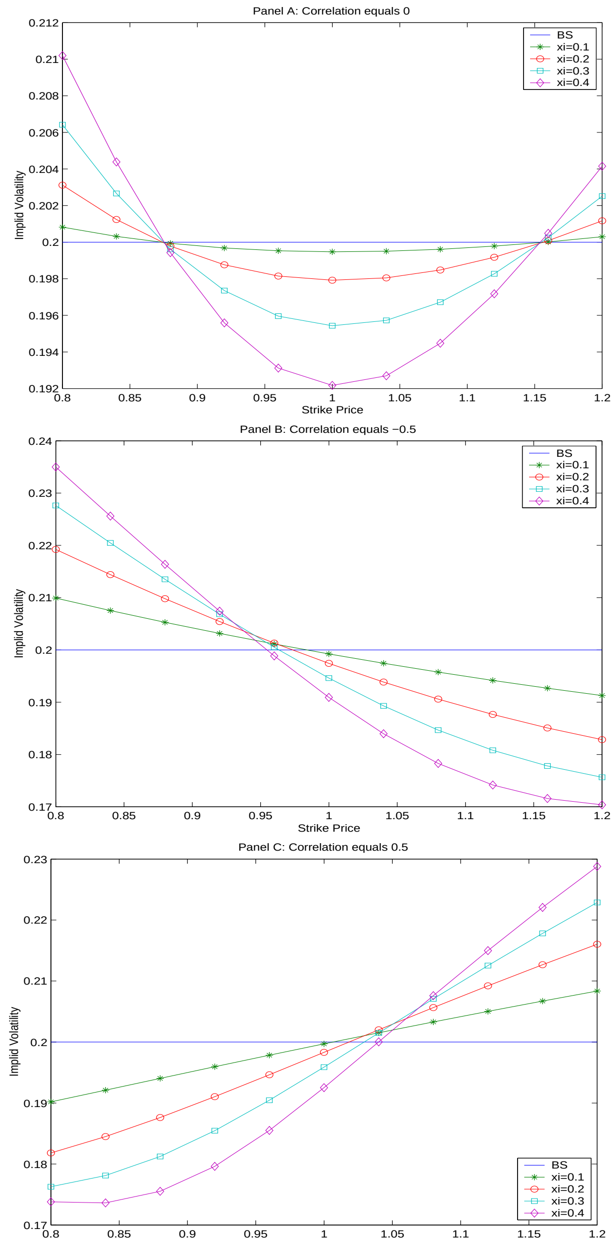

Figure 4.3: A plot of implied volatilities for option prices under Heston's model. $S = 1$, $r = 0.01$, $\kappa = 2$, $\theta = 0.04$, $v_0 = 0.04$, $\lambda = 0$. In each panel, ξ changes from 0 to 0.4. When $\xi = 0$, volatility is constant. We are back to the Black-Scholes world. Three different correlations ρ are shown.

eralized method-of-moments (GMM) developed by Hansen and Scheinkman (1995), and the efficient method-of-moments (EMM) proposed by Gallant and Tauchen (1996).

But in practice, it is not convenient to employ these econometric tools. An alternative and popular method is to use option prices directly. The procedure is simple. First, we collect N_t options on the same stock in the same day. These options have different time to maturities and strike prices. Let $c_{i,t}^{Market}$ be the price of the i-th option, and $c_{i,t}^{Model}$ be its price determined by the model. The parameters we want to estimate include structural parameters, κ, θ, ξ, ρ, λ, and the spot variance v_t. By the expression of the option price under Heston's model, we know we can always choose $\lambda = 0$ in practice. The parameter set $\Phi = \{\kappa, \theta, \xi, \rho, v_t\}$ is then determined by

$$\hat{\Phi} = arg \min_{\Phi} \frac{1}{N_t} \sum_{i=1}^{N_t} (c_{i,t}^{Market} - c_{i,t}^{Model})^2. \tag{4.27}$$

This procedure is used in Bakshi, Cao, and Chen (1997). But in some cases, we don't have enough options on the same stock traded in one day. Bates (1996) holds the model parameters constant through time and uses many days of option prices. This approach also have some problems. The spot variance is a parameter in the model. If we use many days of option prices, we will have a number of spot variances to estimate, one for each day. The computing time will be demanding since (4.27) is a non-linear optimization problem. Anyway, these approaches using loss functions are useful in practice and are not very difficult to implement.

Chapter 5

Limitations of Stochastic Volatility Models

The SV model is superior to the Black-Scholes model. Theoretically, SV models make more realistic assumptions; and empirically, researchers also find the SV model outperforms the B-S model in pricing options. But SV models are not great enough. The B-S model is easy to understand and implement. In contrast, SV models are not easy to use in practice. Further more, SV models cannot eliminate all the pricing biases. There are some evidence showing that SV models are still misspecified. Some assumptions of SV models may be further relaxed.

In Section 4.4, we introduced some calibration methods. In practice, calibration of SV models is a big problem. Volatility is unobservable. It is very hard to estimate model parameters by the underlying stock prices. One can extract implied volatility to be a proxy. But in some cases, options are thinly traded. Bid and ask spreads are large. The quoted prices are not reliable. If one use the method of minimizing the difference between the market prices and the model prices, the computation time will be demanding. For many SV models, closed-form solutions are not available. Some numerical methods are used. But usually it is time-consuming to get the price using these numerical methods. The optimization problem (4.27) is solvable in practice only if the option price under SV models can be computed quickly, because the optimization algorithm involves a number of trial-and-error loops. Although Heston's SV option pricing model has a closed-form solution, the infinite integral is still solved by a numerical method. It is much faster than other SV models. But the optimization problem is still slow. And there are some problems in this non-linear optimization problem. For example, the solution might be a local minimum point rather than the global minimum point.

An alternative to the continuous-time SV model is the GARCH option pricing model. GARCH models have an advantage that the current volatility is observable from the history of stock prices. The model parameters can be readily estimated by the discrete observations of stock prices. So the estimation procedure is considerably simplified. The stock price process and variances are discrete under GARCH option pricing models. Duan (1995) assumes

$$\ln\frac{S_t}{S_{t-1}} = r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \sqrt{h_t}\varepsilon_t, \quad \varepsilon_t|\phi_{t-1} \sim N(0,1), \tag{5.1}$$

$$h_t = \beta_0 + \beta_1 h_{t-1} + \beta_2 h_{t-1}(\varepsilon_{t-1} - \theta), \tag{5.2}$$

where r is the risk-free interest rate. λ can be interpreted as the unit risk premium. ϕ_{t-1} is the information set till time t–1.

In econometrics, GARCH models are the most successful models to model the volatilities of financial time series. But the GARCH option pricing model has not been commonly used by traders and finance researchers. They still prefer the continuous-time models. Duan (1996) shows that most of the existing bivariate diffusion models that have been used to model stock returns and volatilities in SV models, can be represented as limits of a family of GARCH models. So continuous-time SV option pricing models and GARCH option pricing models are closely related. Based on this fact, Lewis (2000) models the prices by a bivariate diffusion process, but estimate the parameters of the model using GARCH techniques.

Another weakness of SV models is they model stock prices in a continuous context. In the real financial markets, prices exhibit jumps rather than continuous changes. Large price changes cannot be generated by pure diffusion processes in SV models. Bates (1996) finds that some parameters of Heston's model need to be implausibly high when fitting the market data. One explanation for this is the absence of price jumps. We know the correlation paramter, ρ, controls the level of skewness and the volatility of variance, ξ, controls the level of kurtosis. But the ability of Heston's model to generate enough short term kurtosis is limited. And a high level of ξ means the short-term kurtosis is very high. Bates (1996) adds Poisson jumps in the stock price process and proposes a so-called stochastic volatility/jump-diffusion (SVJ) model. Discontinuous price jumps and crashes can generate additional skewness and kurtosis. So the SVJ model can generate more desirable return distributions. Bakshi, Cao, and Chen (1997) also find that the SVJ model outperforms the SV model, especially in pricing short-maturity options. Price jumps mainly capture short-term excess kurtosis and skewness, whereas stochastic volatility captures such moment properties in the long run. However, incorporating jumps increases the implementation cost.

Chapter 6

Conclusion

In this essay, I first introduced the B-S model for pricing options. The phenomenon of volatility smile or skew was described. The implied volatilities from market option prices vary by strike price and time to maturity. It contradicts to the constant volatility assumption of the B-S model. A number of empirical studies have revealed that the true distribution of stock returns is a skewed and leptokurtic distribution rather than a normal distribution. To deal with this problem, researchers have presented several models that incorporating more realistic features of stock returns.

Particularly, SV models have received remarkable attentions. They are said to be the next generation of option pricing models. Modelling volatilities in a stochastic way corrects the simple constant volatility assumption of the B-S model. By changing the model parameters, almost all kinds of asset distributions can be generated. Allowing volatility to be stochastic, the model can generate a more leptokurtic return distribution. A negative skewness can also be generated by a negative correlation between the stock price process and the volatility process. The SV model can also have an implied volatility surface which is similar to the one generated by market data. Therefore, the SV model gives a good modification to describe the real financial markets.

The market incompleteness is one of the most distinctive features of the SV model when we want to price and hedge options. The market price of volatility risk was introduced when pricing options in SV models. Since a perfect hedging is impossible in incomplete markets, superhedging, mean-variance hedging and shortfall hedging were introduced.

Among several SV models, Heston's SV model is the most popular one. A closed-form solution was derived by a method based on characteristic functions. We investigated the effects of some important parameters of the model by looking at the implied volatility surface. Estimation of parameters was also discussed, as this step is necessary when we want to implement the model in practice.

Although SV models have many good features and are found to correct some pricing biases of the B-S model, SV models have some limitations. A discrete version of the SV model, the GARCH option model, is said to be easily implemented, since the parameters can be estimated from underlying stock prices directly. Price jumps are found to be very common in financial markets. Adding jumps in the standard model can improve the performance. A so-called stochastic volatility/jump-diffusion model was introduced.

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