INTRODUCTION TO COLLATERALIZED DEBT OBLIGATIONS

by

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Abstract

Collateralized Debt Obligations(CDOs) are one of the most interesting innovations of the securitization market in the 90s. They create new, customized asset classes, by allowing various investors to share the risk and return of an underlying pool of debt obligations. I review first different types of CDOs used in capital markets and their economic rational. In the second part I present the market valuation, diversity score, and risk of CDOs, in a simple jump-diffusion setting for correlated default intensities.

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Chapter 1
Introduction

1. The structure of Collateralized Debt Obligations

Collateralized Debt Obligations(CDOs) are one of the most interesting innovations of the securitization market in the 90s. They create new, customized asset classes, by allowing various investors to share the risk and return of an underlying pool of debt obligations. The attractiveness to investors is determined exactly by the underlying debt and the rules for sharing the risk and return.

A CDO is a securitization in which a portfolio of securities is transferred to a special purpose vehicle (SPV). The SPV in turn issues tranches of debt securities (notes) of different seniority and equity to fund the purchase of the portfolio. Securitization is considered a reallocation of the risk. It is a process of converting assets into securities backed by those assets, in order to lower funding costs, access the capital markets, generate management fees and reach some accounting purposes.

A Collateralized Bond Obligation (CBO) involves mostly bonds, while a Collateralized Loan Obligation (CLO) involves mostly loans.

The CDOs borrow their structural template from Collateralized Mortgage Obligations. A CDOs can hold mortgage-backed securities, asset-backed securities, real-estate investment trusts (REITS) and even other CDOs.

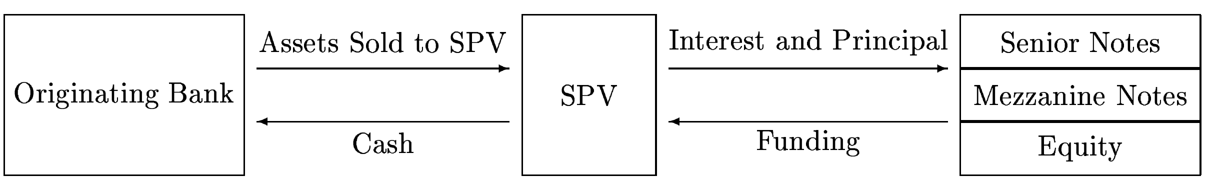

Figure 1.1: CDO Diagram

I have set up a typical CDO structure in Figure 1.1. The assets are transferred to the SPV that funds these assets, from cash proceeds of the notes it has issued.

The CDO structure allocates interest income and principal repayment from a pool of different debt instruments to a prioritized collection of securities notes called tranches. They are typically rated based on portfolio quality, diversification, and structural subordination. There are always at least two tranches.

The losses in interest or principal to the collateral are absorbed first by the lowest level tranche and then in order to the next tranche and so on. The mechanism for distributing the losses to the various tranches is called waterfall. Senior notes are paid before mezzanine and lower rated notes. Any residual cash flow is paid to the equity piece. This makes the senior CDO liabilities significantly less risky than the collateral.

The market classifies CDOs into two broad categories: market value structures and cash flow structures. The two categories are not completely disjoint. There are examples of CDOs which have characteristics of both classes. In any case, all CDOs have an investment manager who manages the collateral. He must follow the rules set out in the offering circular.

On every payment date, equity receives cash distributions after the scheduled debt payments and other costs have been paid off. The equity is also called the first-loss position in the collateral portfolio, because it is exposed to the risk of the first dollar loss in the portfolio.

Losses occur when there is some kind of credit event. A credit event is usually either a default of the collateral, or a credit downgrade of the collateral. In either case, the market value of the collateral drops. The lowest tranche is the riskiest and is called the equity tranche. All the tranches except the equity tranche have credit ratings. The highest tranche is usually rated AAA.

The CDO rating is based on its ability to service debt with the cash flows generated by the underlying assets. The debt service depends on the collateral diversification, subordination and structural protection (credit enhancement and liquidity protection).

As we move down the CDOs capital structure, the level of risk increases. The equity holders bear the highest risk.

The typical CDO consists of a ramp-up period, during which the collateral portfolio is formed, a reinvestment period, during which the collateral portfolio is actively managed, and an unwind period, during which the liabilities are repaid in order of seniority, using collateral principal proceeds.

In the repayment period, excess interest payments gradually decrease as the collateral portfolio principal proceeds are used to repay the debt in order of seniority. After all the debt classes have been redeemed, the remaining principal payments pass to the equity.

Classes	**Rating**	**Coupon**	**% Capital Structure**
A	Aaa/AAA	Libor + 45bp	70%
B	A2/A	Libor + 145bp	15%
C	Baa2/BBB-	Libor + 245bp	7%
D	Ba3/NR	Libor + 645bp	4%
Equity	Not Rated	Expected Return 25% - 30%	4%

Figure 1.2: CDO Capital Structure

Figure 1.2 displays an example of capital structure, where the high yield bonds collateralize CDO liabilities.

1.1. Arbitrage and Balance Sheet CDOs

Most CDOs can be placed into either of two main groups: arbitrage and balance sheet transactions.

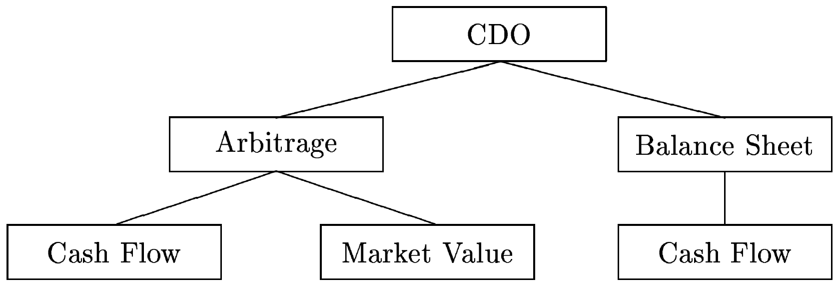

Figure 1.3: CDO Structure

Figure 1.3 shows the conceptual breakdown between the two structures.

- Cash flow CDOs
  A cash flow CDO is one where the collateral portfolio is not subjected to active trading by the CDO manager. The uncertainty concerning the interest and principal repayments is determined by the number and timing of the collateral assets that default. Losses due to defaults are the main source of risk.

- Market value CDOs
  A market value CDO is one in which the CDO tranches receive payments based essentially on the mark-to-market returns of the collateral pool, as determined in large part by the trading performance of the CDO manager.

- Balance sheet cash flows CDOs
  Balance sheet deals are structures for the purpose of capital relief, where the assets securitized are low yielding debt instruments. The capital relief reduces funding costs or increases return on equity, by removing from the balance sheet the assets that take too much regulatory capital.

  These transactions rely on the quality of the collateral that is represented by guaranteed bank loans with a very high recovery rate. In the majority of the cases, the sold assets are loan-secured portfolios.

- Arbitrage CDOs
  An arbitrage CDO, often underwritten by an investment bank, is designed to capture an arbitrage between yield on collateral acquired in the capital markets (largely sub-investment grade) and investment-grade notes issued to investors.

- Arbitrage market value CDOs
  Arbitrage market value CDOs go through a very extensive trading by the collateral manager, necessary to exploit perceived price appreciations.

  This type of CDO relies on the market value of the pool securitized, which is monitored on a daily basis. Every security traded in capital markets, with an estimated price volatility, can be included in this type of CDO. During the revolving period, the collateral manager can increase or decrease the funding costs that changes the leverage of the structure.

- Arbitrage cash flow CDOs
  Most collateral assets are bonds. As arbitrage deals, the collateral assets can be refinanced by re-tranching the credit risk and funding cost in a more diversified portfolio. Unlike arbitrage market value CDOs, the collateral assets are not traded very frequently.

1.2. Credit enhancement in cash flow transactions

Senior notes in cash flow transactions are protected by subordination, over-collateralization and excess spread. The senior notes have a priority claim on all cash flows generated by the collateral, therefore, non-senior notes performance is subordinated to the good performance of senior notes.

Over-collateralization(OC) provides a further protection to senior notes by imposing a minimum collateral value with two coverage tests: par value and interest coverage tests.

The par value test requires that the senior notes (and subsequently the other notes) are at least a certain percentage of the underlying collateral (for example 115%).

The par value test is applicable to lower rated notes (mezzanines). In this case, the trigger percentage below that fails the test is selected at a lower rate (for example 105%).

An interest coverage test is applied to ensure that collateral interest income is sufficient to cover losses and still make interest payment to the senior notes. This credit support is also known as excess spread.

1.3. Credit enhancement in market value transactions

Advance rates are the primary form of credit enhancement in market value transactions. It is defined as the maximum percentage of the market rate that can be used to issue debt. Rating agencies assign different advance rates to different types of collateral. They depend on the volatility of the asset return, and on the liquidity of the asset in the market. Assets with a higher return volatility and lower liquidity are given lower advance rates.

A collateral manager must ensure that the advance rate test is not violated due to fluctuations in the underlying prices. When a breach of the test happens, the collateral manager must remedy it within a cure period by either selling securities with a lower advance rate and buy ones with a higher advance rate, or by selling securities with a lower advance rate and repay the debt starting with the most senior notes.

The Minimum Net Worth Test is also designed to offer credit protection to the senior notes holders in market value transactions, by creating an equity cushion. This is achieved by imposing that the excess market asset value, minus the debt notes is equal or greater than the equity face value, times a percentage.

In cases where the test is breached, the manager has a cure period to bring the CDO into compliance, by either redeeming part or all of the senior notes, or by generating enough capital gains through selling of some assets.

1.4. The Manager

The manager of the CDO is responsible for the credit performance of the collateral portfolio and for ensuring that the transaction meets the diversification, quality and structural guidelines specified by the rating agencies. In return for managing the collateral portfolio, the manager receives a fee. During the reinvestment period, the CDO manager continuously evaluates the state of the collateral portfolio and of the overall market. He trades out positions at risk for credit deterioration, and takes advantage of appreciation opportunities.

The key to a successful market value CDO is the manager's ability to generate high risk-adjusted returns through research, market knowledge and trading ability. The return performance of CDO equity depends hugely on the long-horizon returns of the underlying portfolio realized by the manager.

2. The Economic rational for CDOs

CDOs make most economic sense for collateral securities in markets where there is limited (inefficient) information, with the possibility of high risk-adjusted returns through active management.

Risky assets, such as the debt of leveraged corporations, are often difficult to analyze and value, thus limiting their potential investor base and creating a gap in the economy between the demand and supply of risky finance. As result, corporate debts are relatively illiquid in the secondary market.

The CDO structure addresses this market inefficiency by bringing a specialized manager to the transaction and allocating much of the risk, in the form of a liquidity premium in the equity class.

The CDO cash flow structure acts as a cushion and hedges the debt from defaults and the direct impact of mark-to- market changes in the value of the collateral.

In trying to reach its economic target, an issuer would have two main constraints: to minimize the total cost of notes (i.e. the floating or fixed rate attached to each note) and to minimize the size of the subordinated notes (among them the equity piece).

The originator's return is given by the excess spread of the notes (average rate of the loan portfolio minus the average rate of the notes) over the funding cost of his collateral.

The example from Figure 1.4 (taken from [6]) shows the return on equity(ROE) before and after a CDO. The bank has a portfolio of loans of 100 million Euros on its balance sheet (left box), for which the average spread over the Libor is 100 bps. The loans receive a risk weight of 100% where the Regulatory Capital (the equity) is 8%, and the loan portfolio ROE is of 12.5%.

With the CDO structure in Figure 1.4 (right box) the bank retains only 2% of the original loan portfolio (the loss piece) and securitizes the remaining 98%. In this example, the bank would receive a huge relief of regulatory capital. This would drop from 8 million euros to 2 million euros. The new ROE is 33.1% (calculated by dividing the gross return from the loan portfolio of 100 bps minus the average cost of the three notes (100 bps 34.4 bps) by 2, which is the amount of regulatory capital.

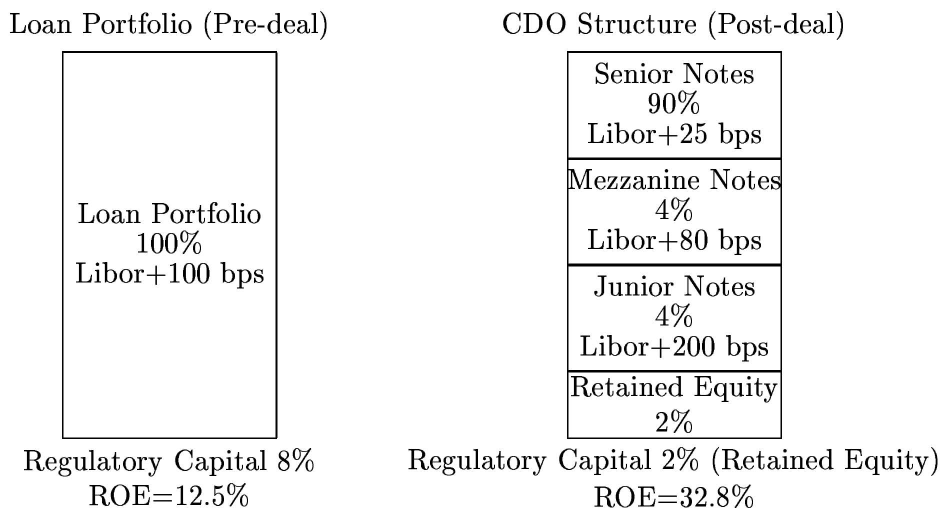

Figure 1.4: Regulatory Capital Relief and ROE

The valuation model presented in Chapter 2 does not deal directly with the effects of market imperfections. It takes as given the default risk of the underlying participations, and assumes that investors are symmetrically informed. It is generally difficult to infer separate risk premia for default timing and default-recovery from the prices of the underlying debt and market risk-free interest rates (see [4]). These risk premia play separate roles in the valuation of CDO tranches. I will take these risk-premia as given, in the form of parametric models for default timing and recovery distributions.

Chapter 2
The Valuation Model for CDOs

1. Default Risk Model

This section presents the basic default modelling for the underlying collateral. First, I introduce a simple model for the default risk of one obligor as described in [2]. Then I turn to the multi-issuer setting.

1.1. Obligor Default Intensity

Each underlying obligor defaults at some conditional expected arrival rate. At each time t before the default time τ of the given obligor, the default arrives at some intensity $\lambda(t)$, given all currently available information $\mathcal{F}_t$. For each small interval Δt we have the approximation

$$\mathrm{P}(\tau < t + \Delta t \mid \mathcal{F}_t) \approx \lambda(t)\Delta t.$$

We call the stochastic process λ a pre-intensity (also called 'compensator' in Stochastic Calculus literature) for a stopping time τ if for any $t < \tau$, the current intensity is $\lambda(t)$ and

$$\mathrm{P}(\tau > t + s \mid \mathcal{F}_t) = \mathrm{E}[\exp(\int_t^{t+s} -\lambda_u du) \mid \mathcal{F}_t].$$

This pre-intensity model is a special case of an affine family of processes that used for modelling short-term interest rates. Therefore, each obligor's default time has some pre-intensity process λ solving a stochastic differential equation of the form

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t + dJ_t. \tag{2.1}$$

Here W is a standard Brownian motion. $\Delta J(t)$ denotes any jump that occurs at time t of a pure-jump process J, independent of W. The jump sizes of J are independent and exponentially distributed with mean μ and jump times of J are those of an independent Poisson process with mean jump arrival rate l. (Jump times and jump sizes are also independent.)

A process of this form (2.1) is called a basic affine process with parameters $(\kappa, \theta, \sigma, \mu, l)$. These parameters can be adjusted in several ways to control the manner in which default risk changes over time. For example, one can vary the mean-reversion rate κ, the long-run mean

$$\overline{m} = \theta + \frac{l\mu}{\kappa}, \tag{2.2}$$

or the relative contributions to the total variance of λ_t that are attributed to jump risk and to diffusive volatility. The long-run variance of λ_t is given by

$$\mathrm{var}_\infty = \lim_{t\to\infty} \mathrm{var}(\lambda_t) = \frac{\sigma^2 \overline{m}}{2\kappa} + \frac{l\mu^2}{\kappa}. \tag{2.3}$$

For details, see Appendix 2.

1.2. Multi-Issuer Default Model

A basic affine model can be written as the sum of independent basic affine models, provided the parameters κ, σ, and μ governing, respectively, the mean-reversion rate, diffusive volatility, and mean jump size are common to the underlying pair of independent basic affine processes. This allows studying the implications of changing the correlation in the default times of the various participations (collateralizing bonds or loans) of a CDO, while holding constant the default-risk model of each underlying obligor.

Proposition 1 *Suppose X and Y are independent basic affine processes with respective parameters $(\kappa, \theta_X, \sigma, \mu, l_X)$ and $(\kappa, \theta_Y, \sigma, \mu, l_Y)$. Then $Z = X + Y$ is a basic affine process with parameters $(\kappa, \theta, \sigma, \mu, l)$, where $\theta = \theta_X + \theta_Y$ and $l = l_X + l_Y$.*

See Appendix A for a proof of this result.

In this way is possible to introduce the correlation among different obligors' default times. Suppose that there are N participations in the collateral pool, whose default times $\tau_1, .., \tau_N$ have pre-intensity processes $\lambda_1, .., \lambda_N$, respectively, that are basic affine processes. To introduce correlation, suppose that $\lambda_i = X_c + X_i$, where X_i and X_c are basic affine processes with respective parameters $(\kappa, \theta_i, \sigma, \mu, l_i)$ and $(\kappa, \theta_c, \sigma, \mu, l_c)$. Moreover, $X_1, .., X_N, X_c$ are assumed independent. By Proposition 1, λ_i is also a basic affine process with parameters $(\kappa, \theta, \sigma, \mu, l)$, where $\theta = \theta_i + \theta_c$ and $l = l_i + l_c$. The process X_c can be viewed as a state process governing common aspects of economic performance in an industry, sector, or currency region, and X_i as a state variable governing the default risk specific to obligor i.

1.3. Sectoral, Regional, and Global Risk

Various kinds of risks contribute to credit risk. Industry, country, or economic risk are systemic risks in that they simultaneously affect more than one entity. So two entities that share an industry, country, or economy may have nonindependent credit risk. Specific risk is that risk which is specific to one entity.

One can incorporate, by a repeated use of Proposition 1, multi-factor risk elements (regional, sectoral, and other sources). For example, suppose that the default time τ_i for the obligor i has a pre-intensity $\lambda_i = X_i + Y_{c(i)} + Z$. Here the sector factor $Y_{c(i)}$ is common to all issuers in the sector $c(i) \subset 1, .., N$, for S different sectors and Z is common to all issuers. As before, $X_1, .., X_N, Y_1, .., Y_S, Z$ are independent basic affine processes. Using the independence of the underlying state variables one can see that

$$\mathrm{E}[\exp(\int_t^T -\lambda_i(u)du) \mid \mathcal{F}_t] = \exp[\alpha(T-t) + \beta_i(T-t)X_i(t) + \beta_{c(i)}(T-t)Y_{c(i)}(t) + \beta_Z(T-t)Z(t)]$$

where $\alpha(\cdot) = \alpha_i(\cdot)+\alpha_{c(i)}(\cdot)+\alpha_Z(\cdot)$, and where all of the α and β coefficients are obtained explicitly from Appendix B, from the respective parameters of the underlying basic affine processes X_i, $Y_{c(i)}$, and Z.

The setup presented in Appendix A allows for interest rates that are jointly determined by an underlying multi-factor affine jump-diffusion model.

Further research can discuss the more general multi-factor affine model in which the underlying state variables are not independent.

1.4. Recovery Risk

We suppose that, at default, any given piece of debt in the collateral pool may be sold for a fraction of its face value. The risk-neutral conditional expectation of this fraction, given all information $\mathcal{F}_t$ available at any time t before default, is a constant $\overline{f} \in (0, 1)$ that does not depend on t. The recovery fractions of the underlying participations are assumed to be independently distributed, and independent of default times and interest rates. Assume as well that the recovered fraction of face value is uniformly distributed on $[0, 1]$.

1.5. Collateral Credit Spreads

Suppose that for an issuer whose default time τ has a basic affine pre-intensity process λ, a zero-coupon bond maturing at time t has an initial market value of

$$p(t, \lambda(0)) = \delta(t)e^{\alpha(t)+\beta(t)\lambda(0)} + \overline{f}\int_0^t \delta(u)\pi(u)du \tag{2.4}$$

where $\delta(t) = \mathrm{E}[e^{-\int_0^t r_u du}]$ is default-free zero-coupon discount and

$$\pi(t) = -\frac{\partial}{\partial t}\mathrm{P}(\tau > t) = -e^{\alpha(t)+\beta(t)\lambda(0)}(\alpha'(t) + \beta'(t)\lambda(0)). \tag{2.5}$$

The first term in (2.4) is the market value of a claim that pays 1 at maturity in the event of survival. The second term of (2.4) is the market value of a claim to any default recovery between times 0 and t. It is clear that (2.5) is the probability density function of the default event.

Using this defaultable discount function $p(\cdot)$, we can value any straight coupon bond, or determine par coupon rates. For example, for n coupon periods, the (annualized) par coupon rate $c(s)$ for maturity in s years is determined at any time t by the identity

$$1 = p(s, \lambda_t) + \frac{c(s)}{n}\sum_{j=1}^{ns} p(\frac{j}{n}, \lambda_t).$$

1.6. Diversity Scores

To accurately measure risk and return one must quantify the diversification. An investor in a particular tranche (or investment-rating agency) wants to know the likelihood of sustaining a loss and the likely severity of that loss. In statistical language he wants to know the exact probability distribution of losses to the underlying pool of debt. The probability distribution depends on both the probability of a credit event and relationship between two or more credit events. In statistical language this is called non-independence of events (see [7] for details). Correlation is a measure of non-independence, but it is only one number and does not capture the complicated nature of credit events.

Credit events are uncertain and the number is variable. The fewer the number of credit events the greater the benefits to all investors. In addition, the less variability in the number of credit events, the greater the benefits to all investors. This is another way of saying the better the diversification, the better the investment.

A key measure for CDO risk analysis developed by Moody's is the diversity score. The diversity score of a given pool of participations is the number n of bonds in a idealized comparison portfolio that meets the following criteria

1. The total face value of the comparison portfolio is the same as the total face value of the collateral pool.
2. The bonds of the comparison portfolio have equal face values.
3. The comparison bonds are equally likely to default, and their default is independent.
4. The comparison bonds are, in some sense, of the same average default probability as the participations of the collateral pool.
5. The comparison portfolio has, according to some measure of risk, the same total loss risk as does the collateral pool.

Example 1 *Consider a collateral portfolio of $n = 60$ bonds all having the same default probability and face value with the following structure*

No of Firms in Same Industry	1	2	3	4	5	
Diversity Score - from Tables	1	1.5	2	2.3	2.6	
No of Credit Events	2	7	6	4	2	Total
Diversity Score	$2 \cdot 1$	$7 \cdot 1.5$	$6 \cdot 2$	$4 \cdot 2.3$	$2 \cdot 2.6$	$38.9 \approx 39$

Figure 2.1: Diversity score for a comparison portfolio

Hence, the comparison portfolio has a diversity score of $d = 39$. The maximum diversity is here $n = 60$, corresponding to independent collateral bonds.

A diversity score of n and a comparison-bond default probability of p imply, using the independence assumption for the comparison portfolio, that the probability of k defaults out of the n bonds of the comparison portfolio is

$$q(k, n) = \binom{n}{k} p^k (1 - p)^{n-k}.$$

A more mathematical treatment of the diversity score is provided in Appendix A. An alternative to Moody's binomial diversity score is presented in Chapter 2, Section 3.2.

2. Pricing Model

The basic affine framework presented offers the ability to quickly calibrate the model to the underlying participations, in terms of given correlations, default probabilities, yield spreads, and so on. I discuss different CDO cash-structures and default-risk parameters. The basic CDO structure consists of a special purpose vehicle that acquires a collateral portfolio of participations (debt instruments of various obligors), and allocates interest, principal, and default-recovery cash from the collateral pool to the CDO tranches.

The value of the CDOs is the value of the proceeds of the various tranches, which is the same as the value of the underlying pool of debt (plus or minus management fees). If one miscalculates the risk of one tranche, and, therefore that tranche, then one automatically miscalculates the other tranches.

2.1. The Collateral Pool

Consider N participations in the collateral pool. Each asset $i \in \{1, .., N\}$ has a face value M_i and an annualized coupon rate C_i, paid n times a year to the SPV, until maturity or default. At default, a participation is sold for its recovery value, and the proceeds from sale are also made available to the SPV. Let $A(k)$ be the set of non-defaulted assets at coupon period k, and $B(k) = A(k-1) \setminus A(k)$ the set of assets defaulting between coupon periods $k-1$ and k. If L_i is the loss in face value at default of collateral asset i, then the total cash flow $Z(k)$ available in coupon period k is given by

$$Z(k) = \sum_{i \in A(k)} \frac{C_i}{n} M_i + \sum_{i \in B(k)} (M_i - L_i). \tag{2.6}$$

This cash flow collected by SPV from the collateral pool (coupons,principal, and recoveries on defaulted assets) are allocated to the prioritized CDO-tranches. There are several tranches of sinking fund bonds and an equity piece (the most subordinated).

2.2. Sinking Fund Bond

Following [2], I will turn now to a CDO structure that pays SPV cash to a prioritized sequence of sinking-fund bonds, as well as a junior subordinate residual.

A sinking-fund bond with n coupon periods per year has some remaining principal $F(k)$ at coupon period k, an annualized coupon rate c, and a scheduled interest payment at coupon period k of $F(k)c/n$. If the actual interest paid $Y(k)$ is less than the scheduled interest payment, any difference $F(k)c/n - Y(k)$ is accrued at the bond's own coupon rate c. It generates an accrued unpaid interest at period k of $U(k)$, where $U(0) = 0$ and

$$U(k) = (1 + \frac{c}{n})U(k-1) + \frac{c}{n}F(k) - Y(k).$$

Any pre-payment of principal $D(k)$ in period k, and contractual unpaid reduction in principal $J(k)$ in period k, are linked by

$$D(k) + J(k) \leq F(k-1) \text{ and } F(k) = F(k-1) - D(k) - J(k).$$

At maturity, coupon period number K, any unpaid accrued interest and unpaid principal, $U(K)$ and $F(K)$ respectively, are paid to the extent provided in the CDO contract. The total actual payment in any coupon period k is $Y(k) + D(k)$. The par coupon rate on a given sinking-fund bond is the scheduled coupon rate c with the property that the initial market value of the bond is equal to its initial face value, $F(0)$.

Any sinking-fund bond that pays all remaining principal and all accrued unpaid interest by or at its maturity date has a par-coupon rate equal to the default-free coupon rate, no matter the timing of the interest and principal payments.

To summarize, suppose that there are a senior sinking fund bond, with principal $F_1(0) = P_1$, coupon rate c_1, a mezzanine sinking fund bond,with principal $F_2(0) = P_2$, coupon c_2 and an equity tranche with initial market value of $P_3 = P - P_1 - P_2$. Here P is the principal value of the collateral. The excess cash flow from the collateral is deposited in a reserve account which earns interest at the risk-free rate $r(k)$, during coupon period k.

These funds may be used to purchase additional collateral. Future research can investigate these effects of investing the reserve account in participations added to the collateral pool.

At maturity, coupon period K, any remaining funds in the reserve account, after payments towards the two tranches, are paid to the subordinated residual tranche. The management fees are neglected.

2.3. Prioritization schemes

I describe the valuation for two prioritization schemes: uniform-prioritization scheme and fast-prioritization scheme.

Based on the definition of the sinking funds given in 2.2, I have to define the actual interest payments $Y_1(k)$ and $Y_2(k)$ for the senior and mezzanine sinking funds, respectively, any payments of principal, $D_1(k)$ and $D_2(k)$, and any contractual reductions in principal, $J_1(k)$ and $J_2(k)$.

Under the uniform prioritization scheme, the interest $W_(k)$ collected from the surviving participations is allocated in priority order, with the senior tranche getting $Y_1(k) = \min\{U_1(k), W(k)\}$ and the mezzanine getting $Y_2(k) = \min\{U_2(k), W(k) - Y_1(k)\}$. The available reserve $R(k)$, before payments at period k, is thus defined by

$$R(k) = \left(1 + \frac{r(k)}{n}\right)[R(k-1) - Y_1(k-1) - Y_2(k-1)] + Z(k).$$

Unpaid reductions in principal from default losses occur in reverse priority order, so that the junior residual tranche suffers the reduction

$$H(k) = \max\{\sum_{i \in B(k)} L_i - (W(k) - Y_1(k) - Y_2(k)), 0\} \quad (2.7)$$

$$J_3(k) = \min\{F_3(k-1), H(k)\}. \quad (2.8)$$

$H(k)$ is the total of default losses since the previous coupon date, less collected and undistributed interest income. The mezzanine and senior tranches are successively reduced in principal by

$$J_2(k) = \min\{F_2(k-1), H(k) - J_3(k)\} \tag{2.9}$$
$$J_1(k) = \min\{F_1(k-1), H(k) - J_3(k) - J_2(k)\}. \tag{2.10}$$

Under uniform prioritization, there are no early payments of principal, so $D_1(k) = D_2(k) = 0$ for $k < K$.

At maturity, the remaining reserve is paid in priority order, and principal and accrued interest are treated identically, so that, without loss of generality, for purposes of valuation, we take $Y_1(K) = Y_2(K) = 0$.

$$D_1(K) = \min\{F_1(K) + U_1(K), R(K)\} \tag{2.11}$$
$$D_2(K) = \min\{F_2(K) + U_2(K), R(K) - D_1(K)\}. \tag{2.12}$$

The residual tranche collects

$$D_3(K) = R(K) - D_1(K) - D_2(K).$$

In the fast-prioritization scheme setup, the senior tranche is allocated interest and principal payments as quickly as possible, until maturity or until its principal remaining is reduced to zero, whichever is first. Until the senior tranche is paid in full, the mezzanine tranche accrues unpaid interest at its coupon rate. Then the mezzanine tranche is paid interest and principal as quickly as possible until maturity or until retired. Finally, any remaining cash is allocated to the residual tranche.

In coupon period k, the senior tranche is allocated the interest and principal payment, respectively

$$Y_1(k) = \min\{U_1(k), Z(k)\} \tag{2.13}$$
$$D_1(k) = \min\{F_1(k-1), Z(k) - Y_1(k)\}. \tag{2.14}$$

The mezzanine receives the interest and principal payments as follows

$$Y_2(k) = \min\{U_2(k), Z(k) - Y_1(k) - D_1(k)\} \tag{2.15}$$
$$D_2(k) = \min\{F_2(k-1), Z(k) - Y_1(k) - D_1(k) - Y_2(k)\}. \tag{2.16}$$

Finally, any residual cash are paid to the junior subordinated tranche.

$$D_3(k) = Z(k) - Y_1(k) - D_1(k) - Y_2(k) - D_2(k).$$

For this scheme, there are no contractual reductions in principal ($J_i(k) = 0$).

3. Analytical Results

Analytical results are provided for the probability distribution for the number of defaulting participations, and the total of default losses of principal, including the effects of random recovery. An important space will be devoted as well to Moodys' approach of computing the Diversity Score (see 1.6).

The main point is the ability to compute explicitly the probability of survival of all participations in any chosen sub-group of obligors.

For a given time horizon T , let d_j denote the event that obligor j defaults by T. That is, $d_j = \{\tau_j < T\}$. We let M denote the number of defaults. Assuming symmetry (invariance under permutation) in the unconditional joint distribution of default times,

$$\mathrm{P}(M = k) = \binom{N}{k}\mathrm{P}(d_1 \cap .. \cap d_k \cap d^c_{k+1} \cap .. \cap d^c_N).$$

Let $q(k, N) = \mathrm{P}(d_1 \cap .. \cap d_k \cap d^c_{k+1} \cap .. \cap d^c_N)$. The probability $p_j = \mathrm{P}(d_1 \cup .. \cup d_k)$ that at least one of the first j names defaults by T is computed later; for now, we take this calculation as given. One can prove that

Proposition 2 $q(k, N) = \sum_{j=1}^{N} \binom{k}{N-j} (-1)^{j+k+N+1} p_j$.

Proof of Proposition 2:
The argument proceeds by induction in k and N. Assume, therefore, that the claim holds for all positive pairs (k, n) such that $n < N$ and $k \le n$. Now let $n = N$. First of all, as an immediate consequence of the inclusion-exclusion principle

$$q(n, n) = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} p_j$$

which is clearly consistent with the claim. Assume now that the claim holds for all pairs $(k + 1, n), .., (n, n)$. We have

$$q(k + 1, n) = \mathrm{P}(d_1 \cap .. \cap d_{k+1} \cap d^c_{k+2} \cap .. \cap d^c_n) \tag{2.17}$$
$$q(k, n) = \mathrm{P}(d_1 \cap .. \cap d_k \cap d^c_{k+1} \cap .. \cap d^c_n). \tag{2.18}$$

Adding these two equations, $q(k + 1, n) + q(k, n) = \mathrm{P}(d_1 \cap .. \cap d_k \cap d^c_{k+2} \cap .. \cap d^c_n) = q(k, n - 1)$.

whence $q(k, n) = q(k, n - 1) - q(k + 1, n)$. It is now a simple matter to verify the proposed formula for $q(k, n)$, using the fact that $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$. This completes the proof.

Using the fact that the pre-intensity of the first-to-arrive $\tau^{(j)} = \min\{\tau_1, .., \tau_j\}$ of the stopping times $\tau_1, .., \tau_j$ is $\lambda_1 + .. + \lambda_j$, and using the independence of $X_1, .., X_N, X_c$, we have

$$\begin{aligned} p_j &= 1 - \mathrm{P}(\tau^{(j)} > T) = 1 - \mathrm{E}[-\exp\left(\int_0^T \sum_{i=1}^{j} \lambda_i dt\right)] \\ &= 1 - \exp(\alpha_c(T) + \beta_c(T)X_c(0) + j\alpha_i(T) + j\beta_i(T)X_i(0)). \end{aligned} \tag{2.19}$$

Here $\alpha_c(T), \beta_c(T)$ are given explicitly as the solutions of the ODEs (B.4) and (B.5), for the case $n = \kappa$, $\rho = -\sigma^2$, $q = j$, $l = l_c$, $m = -\kappa\theta_c$ and $p = 0$, while $\alpha_i(T), \beta_i(T)$ are the explicitly-given solutions of (B.1)-(B.2) for the case $n = \kappa$, $\rho = -\sigma^2$, $q = 1$, $l = l_i$, $m = -\kappa\theta_i$ and $p = 0$.

3.1. Computation of Diversity Scores

We define the diversity score S associated with a portfolio of bonds of total principal F to be the number of identically and independently defaulting bonds, each with principal F/S, whose total default losses have the same variance as the target-portfolio default losses. The computation of S entails computation of the variance of losses on the target portfolio of N bonds, which we address in this appendix.

Letting d_i denote the indicator of the event that participation i defaults by a given time T, assumed to be after its maturity, and letting L_i denote the random loss of principal when this event occurs, we have

$$\begin{aligned}\mathrm{var}(\sum_{i=1}^{N} L_i d_i) &= \mathrm{E}[(\sum_{i=1}^{N} L_i d_i)^2] - (\mathrm{E}[\sum_{i=1}^{N} L_i d_i])^2 \\ &= \sum_{i=1}^{N} \mathrm{E}(L_i^2)\mathrm{E}(d_i^2) + \sum_{i \neq j} \mathrm{E}(L_i L_j)\mathrm{E}(d_i d_j) - \sum_{i=1}^{N} (\mathrm{E}(L_i))^2 (\mathrm{E}(d_i))^2 - \\ &\quad - \sum_{i \neq j} \mathrm{E}(L_i)\mathrm{E}(L_j)\mathrm{E}(d_i)\mathrm{E}(d_j). \end{aligned} \tag{2.20}$$

Given an affine intensity model, one can compute all terms involved. In the symmetric case, letting $p_{(1)}$ denote the marginal probability of default of a bond, and $p_{(2)}$ the joint probability of default of any two bonds, the above reduces to

$$\mathrm{var}(\sum_{i=1}^{N} L_i d_i) = N p_{(1)} \mathrm{E}(L_i^2) + N(N-1) p_{(2)} (\mathrm{E}(L_i))^2 - N^2 p_{(1)}^2 (\mathrm{E}(L_i))^2.$$

Equating the variance of the original pool to that of the comparison pool I get

$$\frac{N}{S}\left(p_{(1)}\mathrm{E}(L_i^2) - p_{(1)}^2(\mathrm{E}(L_i))^2\right) = p_{(1)}\mathrm{E}(L_i^2) + (N-1)p_{(2)}(\mathrm{E}(L_i))^2 - N p_{(1)}^2 (\mathrm{E}(L_i))^2.$$

This equation can be easily solved for the diversity score S. To end the computation, one uses the identities: $p_{(1)} = p_1$, $p_{(2)} = 2p_1 - p_2$. Here p_1 and p_2 are computed according to equation 2.19. Assuming that losses are uniformly distributed on $[0, 1]$, I have as well that $\mathrm{E}(L_i^2) = \frac{1}{3}$ and $(\mathrm{E}(L_i))^2 = \frac{1}{4}$.

3.2. Alternative to the computation of the Diversity Score

I introduce an alternative to the Moody's Diversity score proposed in [5].

An m-to-default contract is a type of OTC credit derivative whose payoff depends on the occurrence and timing of the first m defaults in an underlying basket of defaultable bonds. It is a generalization of a first-to-default swap or a basket default swap. The valuation of m-to-default contracts is useful for the contracts themselves as well as for pricing each of the tranches in collateralized debt obligations.

This problem is often attached by approximating the underlying pool with a homogeneous pool in which firms are independent, so that, the ordering of defaults is no longer important. Furthermore, in this case the number of defaults at any time has a binomial distribution and the valuation problem simplifies. Moody's has developed a way to construct an approximating pool based on so called diversity scores, as in [1]. However, even if we can create an approximating pool where the number of defaults have the same expectation and variance as in the original pool the distribution of defaults in the original pool might be very different from the binomial distribution. Hence, we can not be sure how well the m-to-default contract is approximated.

In [5] is proposed a simple way to approximate the value of an m-to-default contract. Instead of making the entire pool homogeneous one can divide the pool into a number of subsets and make firms in each of the subsets homogeneous. Imagine a pool of $b \cdot a$ heterogeneous firms. This pool can be divided into b buckets each containing a firms. The new approximating pool is still a pool with $b \cdot a$ firms but now we only have b types of firms. This speeds up the pricing considerably. One advantage over an approximating pool where the number of defaults is binomially distributed is that here firms in the approximating pool do not need to be independent.

The idea is to decompose the general m-to-default contracts into a collection of first-to-defaults. As in Proposition 1, the first default from a pool of firms can be described as a new default process with a higher intensity.

We will define an m-to-default contract as a contract defined on an underlying pool with n firms. Let $U^{m,n}$ be a contract that pays out an amount for each of the first m defaults. The payment might depend on the defaulted firm. Furthermore, let $W^{m,n}$ be a contract which only has a payoff, at the final time T, conditional on less than m defaults. The value of the pool is $U^{m,n} + W^{m,n}$.

First, we will assume that there is no additional payment at maturity, T. Let firm i be one of the first m to default, and let the time of this default be $\tau_i < T$. Then the contract pays $Y_i(\tau_i)$ at time τ_i.

An m-to-default contract can be computed recursively in the following way. After the time of the first default there are only $n-1$ firms left so the remaining contract is an $(m-1)$-to-default of $n-1$ firms. One can create a portfolio of $(m-1)$-to-default contracts that exactly matches the m-to-default contract.

The case $m = 2$ presented in Figure 2.2 assumes that the first time to default is i at time t_1 and the second is j at time t_2. First, buy n first-to-default contracts of firms $n-1$. In each of the contracts one party should be excluded from the underlying pool. When one party defaults $n-1$ of the n contracts are paid out. The only one not paid out is the one where the defaulting party has been excluded. Since we only want one payment at the time of the first default we sell $n-2$ first-to-default contracts including all n firms. This leaves one payment at the time of the first default and one contract that is a first to default out of the $n-1$ firms that are still alive.

Position	**Firm excluded**	**Number of contracts**	**Payment at first default**	**Payment at second default**
Long	1	1	$Y_i(t_1)$	0
	. .	. .	. .	. .
	$j-1$	1	$Y_i(t_1)$	0
	j	1	$Y_i(t_1)$	0
	$j+1$	1	$Y_i(t_1)$	0
	. .	. .	. .	. .
	$i-1$	1	$Y_i(t_1)$	0
	i	1	0	$Y_j(t_2)$
	$i+1$	1	$Y_i(t_1)$	0
	. .	. .	. .	. .
	n	1	$Y_i(t_1)$	0
Short	None	$n-2$	$-(n-2)Y_i(t_1)$	0
Total			$Y_i(t_1)$	$Y_j(t_2)$

Figure 2.2: The arbitrage argument for the case of a 2-to-default contract

Define $U_l^{m,j}(t)$ as the value of an m-to-default of j contract. The sub-vector, l, signifies which firms have been excluded of the original pool of firms. If the k-th entry is 1 then the k-th firm has been excluded, otherwise the entry is 0 and the firm is still in the pool. Moreover, $U_{e_k}^{m,n}(t)$ is the price of an m-to-default of n firms where firm k has been excluded. Now, I find

Proposition 3 *Assume that firms cannot default simultaneously. Then an m-to-default contract satisfies the recursion*

$$U^{m,n}(t) = \frac{1}{m-1}\left(\sum_{k=1}^{m} U_{e_k}^{m-1,n-1}(t) - (n-m)U^{m-1,n}(t)\right).$$

The recursion gives a way to calculate an m-to-default contract as a portfolio of first-to-default contracts, which can be priced as in Chapter 2.

The result is based on an arbitrage argument, so any m-to-default contract can be priced from a set of first-to-default contracts.

To see the recursion, consider the time of the first default and let i be the defaulted firm. Then $U_{e_k}^{m-1,n-1}$ pays out a premium of $Y_i(\tau_i)$ if $i \neq k$, i.e., I receive $\frac{n-1}{m-1}Y_i(\tau_i)$. This is partly cancelled by the premium of $\frac{n-m}{m-1}Y_i(\tau_i)$ I have to pay for shorting $\frac{n-m}{m-1}$ of $U^{m-1,n}$. In total I receive $\frac{n-1-n+m}{m-1}Y_i(\tau_i) = Y_i(\tau_i)$. Until default number $m-1$ the situation is the same since all the contracts are still paying out premiums. After this time only the $m-1$ contracts

$U_{e_k}^{m-1,n-1}$ where k has already defaulted pay out dividends. At this time these contracts are all first-to-defaults out of the remaining $n-m+1$ firms. This is, at the time of the m'th default I receive the premium $\frac{m-1}{m-1}Y_j(\tau_j) = Y_j(\tau_j)$ where j is the m'th firm to default, and after this time no more premiums are paid out.

If one is interested in a payment Z at maturity conditioning on less than m defaults have occurred the recursion should be changed as follows

$$W^{m,n}(t) = \frac{1}{m-1}\left(\sum_{k=1}^{m} W_{e_k}^{m-1,n-1}(t) - (n-m+1)W^{m-1,n}(t)\right).$$

To see this, if less than $m-1$ defaults occur all the contracts pay out Z and I get

$$\frac{n-(n-m+1)}{m-1}Z = Z.$$

In case of exactly $m-1$ defaults I get $\frac{m-1}{m-1}Z = Z$ from the $m-1$ contracts where I excluded one of the defaulted firms. Here $W^{m,n}$ is the price of a contract that pays at time T in case of less than m defaults among the all n firms.

Chapter 3

Conclusion

I have introduced in this essay Collateralized Debt Obligations and the valuation model for pricing them. The setup presented here mostly provides a basic framework for modelling the default probability, which is the most important component in pricing CDOs. However, the valuation requires more than just the default probability and it can become very complex, depending on the structure of the collateral. There is ongoing research on how to price this product that will allow understanding the issue more thoroughly.

Bibliography

[1] A. Cifuentes and G. O'Connor. The Binomial Expansion Method Applied to CBO/CLO Analysis. Technical report, Moody's Investors Service, 1996.

[2] Darrell Duffie and Nicolae Garleanu. Risk and valuation of collateralized debt obligations. *Financial Analyst Journal*, 133:41–59, January/February 2001.

[3] Darrell Duffie, Jun Pan, and Kenneth Singleton. Transform Analysis and Asset Pricing for Affine Jump-Diffusions. *Econometrica*, 68(6):1343–1376, November 2000.

[4] Darrell Duffie and Kenneth Singleton. Modeling Term Structures of Defaultable Bonds. *Review of Financial Studies*, 12(4):687–720, November 1999.

[5] Brian Huge. *On Defaultable Claims and Credit Derivatives.* PhD thesis, University of Copenhagen, 2001.

[6] Domenico Picone. Collateralised Debt Obligations. Technical report, City University Business School, London, September 2002.

[7] Richard K. Skora. Correlation - the hidden risk in Collateralized Debt Obligations. Technical report, Skora & Company Inc., November 1998.

Appendix A

Default Probabilities and Pricing for the Affine Model

An assumption used often in developing dynamic asset pricing models is that the state vector X follows an affine jump-diffusion. In this setting all the drift vector, instantaneous covariance matrix, and jump intensities have affine dependence on the state vector.

Following [3] I derive the closed-form expression for the transform

$$\mathrm{E}[\exp(-\int_t^T R(X_s)ds)(v_0 + v_1 X_T)e^{u_0 + u_1 X_T} \mid \mathcal{F}_t]. \tag{A.1}$$

Here $R(X_t)$ is the stochastic discount rate used for computing present values of future cash flows. I consider it to be affine in X_t.

To price now a security subject to default, one can assume that the arrival of default is at a stochastic intensity λ and upon default the holder recovers a constant fraction $\overline{f}$ of the face value. Then, the initial price of a T-period zero-coupon bond is given by

$$\delta_t q_t + \overline{f} \int_0^t \delta_u \pi_u du. \tag{A.2}$$

Here $\delta_t = \mathrm{E}[\exp(-\int_0^t r_u du)]$ and $q_t = \mathrm{E}[\exp(-\int_0^t (\lambda_u) du)]$. The first term in (A.2) is the value of a claim that pays 1 contingent on survival to maturity T. We may view π_t as the price density of a claim that pays 1 if default occurs during $(t, t+dt)$. The second term in (A.2) is the price of any proceeds from default before T. Both these terms and the price density π_t can be computed in closed form using the transform mentioned in (A.1).

Let X be a Markov process solving the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t$$

where W is a Brownian motion and J is a pure jump process whose jumps have a fixed probability distribution ν and arrive with intensity $\{\lambda(X_t) : t \geq 0\}$.

Intuitively, this means that, conditional on the path of X, the jump times of J are the jump times of a Poisson process with time-varying intensity $\{\lambda(X_t) : t \geq 0\}$, and that the size of the jump of J at a jump time T is independent of $\{X_s : 0 \leq s < T\}$ and has the probability distribution ν.

The transition semi-group of the Markov process X has an infinitesimal generator $\mathcal{D}$ of the Levy type, defined at a bounded function f by

$$\mathcal{D}f = f_t + f_x\mu(x) + \frac{1}{2}f_{xx}(\sigma(x))^2 + \lambda(x)\int_{\mathbb{R}}(f(t,x+z) - f(t,x))d\nu(z). \tag{A.3}$$

In our setting $\mu(x) = \kappa(\theta - x)$, $\sigma(x) = \sigma\sqrt{x}$, $\lambda(x) = l$ and $\nu(z) = \dfrac{e^{-\frac{z}{\mu}}}{\mu}$.

The distribution of λ_t and the price of a security can be recovered now by inverting the transforms

$$\varphi(u, \lambda_t, t, T) = \mathrm{E}[\exp(-\int_t^T (p + q\lambda_s)ds)e^{u_0+u_1\lambda_T} \mid \mathcal{F}_t] \tag{A.4}$$

$$\psi(v, u, \lambda_t, t, T) = \mathrm{E}[\exp(-\int_t^T (p + q\lambda_s)ds)(v_0 + v_1\lambda_T)e^{u_0+u_1\lambda_T} \mid \mathcal{F}_t]. \tag{A.5}$$

This is known in closed form

$$\psi(v, u, x, t, T) = \varphi(u, x, t, T)(A(T - t) + B(T - t)x) \quad \text{and} \quad \varphi(u, x, t, T) = e^{\alpha(T-t)+\beta(T-t)x}.$$

Here α, β, A, B satisfy some complex-valued ODE's. The derivation of the differential equations is provided in Appendix B.

With this, we can compute the characteristic function $\varphi(\cdot)$ of λ_t given λ_0, defined by

$$\varphi(z) = \mathrm{E}[e^{iz\lambda_t}] = e^{\alpha(t)+\beta(t)\lambda_0} \tag{A.6}$$

by taking $u_0 = p = q = 0$, $u_1 = iz$ and corresponding complex $\alpha(\cdot), \beta(\cdot)$ functions.

The Laplace transform $L(\cdot)$ of λ_t is computed similarly

$$L(z) = \mathrm{E}[e^{-z\lambda_t}] = e^{\alpha(t)+\beta(t)\lambda_0} \tag{A.7}$$

by taking $u_0 = p = q = 0$, $u_1 = -z$ and the new pair $\alpha(\cdot), \beta(\cdot)$ associated.

Proof of Proposition 1:
It suffices to verify that the Laplace transform of Z coincides with that of a basic affine process with parameters $(\kappa, \theta, \sigma, \mu, l)$ and initial condition $Z(0) = X(0) + Y(0)$. That is, the sum of the Laplace transforms of λ_1 and λ_2 equals the transform of the process with parameters $(\kappa, \theta_1 + \theta_2, \sigma, \mu, l_1 + l_2)$.

I need to compute the Laplace transform of a basic affine process at an arbitrary time t, which is easily done according to equation (A.7) above and the explanation following it. In order for the set of ODEs to characterize the transform it is sufficient that $U_t = e^{\alpha(T-t)+\beta(T-t)X(t)}$ is a martingale.

Now it is a simple matter to verify that the sum of the Laplace transform of X and Y equals the transform of the basic affine process with parameters $(\kappa, \theta, \sigma, \mu, l)$ and initial condition $Z(0) = X(0) + Y(0)$, using the explicit solutions from Appendix B (see (B.4) and (B.5)).

The β_X for X is the same as β_Y for Y, since both solve the same ODE (B.4) with parameters $n = \kappa$, $\rho = -\sigma^2$, $q = 0$ and the initial condition $-z$. Furthermore, the sum of α_X for X and α_Y for Y satisfies an ODE as in (B.5) with $m = -\kappa(\theta_1 + \theta_2)$, $l = l_1 + l_2$, $p = 0$ and initial condition 0. In this way I obtain the Laplace transform for a basic affine process governed by parameters $(\kappa, \theta_1 + \theta_2, \sigma, \mu, l_1 + l_2)$ and initial condition $X(0) + Y(0)$ (this is since $\beta_X = \beta_Y$). The proof is complete now.

Appendix B

Transform Analysis for Affine Jump Diffusion State Vectors

1. Solution for The Basic Affine Model

An important role will be played by the characteristic function of an exponentially distributed random variable Z with mean μ. Since the density function is $d(z) = \frac{e^{-\frac{z}{\mu}}}{\mu}\mathbf{1}_{[0,\infty)}(z)$, one can get the formula for the characteristic function $\varphi_Z(t)$ as

$$\int_{\mathbf{R}} e^{itz} d(z)dz = \int_{\mathbf{R}_+} \frac{e^{itz-\frac{z}{\mu}}}{\mu} dz = \frac{1}{1-i\mu t}.$$

Now it's easy to get any moment of Z using the inversion formula for the characteristic function. Indeed,

$$\mathrm{E}[Z^n] = \frac{\varphi_Z(t)}{i^n} = n!\mu^n.$$

In order to derive (A.4) I introduce $U_t = \exp(-\int_0^t (p + q\lambda_s)ds)$ and $V_t = e^{\alpha(T-t)+\beta(T-t)\lambda_t}$. I want to choose α and β such that $\zeta_t = U_t \cdot V_t$ is a martingale. If this is the case, multiplying $\zeta_t = \mathrm{E}[\zeta_T \mid \mathcal{F}_t]$ by $\exp(-\int_0^t z\lambda_s ds)$, one can obtain the closed form of (A.4).

It suffices that the drift of the SDE satisfied by ζ_t is equal with 0.

I have $dU_t = -U_t(p + q\lambda_t)dt$ and $d\zeta_t = dU_t \cdot V_t + U_t \cdot dV_t + d[U,V]_t$.

The drift of the SDE satisfied by V_t is obtained from applying the $\mathcal{D}$ operator (A.3) to $f(t,x) = e^{\alpha(T-t)+\beta(T-t)x}$. This is

$$\begin{aligned} \mathcal{D} &= V_t \cdot (-\alpha'(T-t) - \beta'(T-t)\lambda_t) + V_t \cdot \beta(T-t)\kappa(\theta - \lambda_t) + \\ &+ V_t \cdot \frac{1}{2}\beta^2(T-t)\sigma^2\lambda_t + V_t \cdot l\frac{\mu\beta(T-t)}{1-\mu\beta(T-t)}. \end{aligned} \tag{B.1}$$

I will explain now how one can get the last term from $\mathcal{D}$ formula. I need to compute

$$l\int_{\mathbf{R}_+} (f(z+\lambda_t) - f(\lambda_t))\frac{e^{-\frac{z}{\mu}}}{\mu}dz = V_t \cdot l\int_{\mathbf{R}_+} (e^{\beta(T-t)z} - 1)\frac{e^{-\frac{z}{\mu}}}{\mu}dz.$$

Now using the properties of an exponentially distributed random variable presented in the beginning of this Appendix, one can get the last term in the above formula for $\mathcal{D}$.

Since $d[U,V]_t = 0$, it follows that the drift of $d\zeta_t$ is $-p - q\lambda_t + \mathcal{D} = 0$. The last equality can be viewed as a polynomial in λ_t and is satisfied if its coefficients are null, i.e.,

$$-p - \alpha'(T-t) + \kappa\theta\beta(T-t) + l\frac{\mu\beta(T-t)}{1-\mu\beta(T-t)} = 0 \tag{B.2}$$

$$-q - \beta'(T-t) - \kappa\beta(T-t) + \frac{1}{2}\sigma^2\beta^2(T-t) = 0. \tag{B.3}$$

For the basic affine model, the coefficients $\alpha(\cdot)$ and $\beta(\cdot)$ determining the solution of the survival-time distribution are given as solutions of the ordinary differential equations

$$\beta' = n\beta + \frac{1}{2}\rho\beta^2 + q \tag{B.4}$$

$$\alpha' = m\beta - l\frac{\mu\beta}{1-\mu\beta} + p \tag{B.5}$$

with boundary conditions $\alpha(0) = \beta(0) = 0$, for $n = \kappa$, $\rho = -\sigma^2$, $q = 1$, $m = -\kappa\theta$ and $p = 0$. More generally, for any constants p, q, u_0, and u_1, under technical integrability conditions we have

$$\mathrm{E}[\exp(-\int_t^T (p + q\lambda_s)ds)e^{u_0 + u_1\lambda(t)}] = e^{\alpha(T-t)+\beta(T-t)\lambda(t)}$$

where α, β solve the same ODEs as before with the more general boundary conditions $\alpha(0) = u_0$ and $\beta(0) = u_1$.

For $(p,q) = (0,-z)$ one can obtain the assertion made in the proof of Proposition 1 about the martingale U_t.

A similar procedure will be applied for ψ_t in deriving (A.5). The difference from the case of φ_t discussed above is the formula for V_t, given by

$$V_t = e^{\alpha(T-t)+\beta(T-t)\lambda_t}(A(T-t) + B(T-t)\lambda_t).$$

Using again the $\mathcal{D}$ operator with $f(t,x) = e^{\alpha(T-t)+\beta(T-t)x}(A(T-t)+B(T-t)x)$, the first three terms involved are straightforward to get

$$V_t \cdot ((-\alpha'(T-t) - \beta'(T-t)\lambda_t)(A(T-t) + B(T-t)\lambda_t) - A'(T-t) - B'(T-t)\lambda_t) \tag{B.6}$$

$$V_t \cdot (\beta(T-t)(A(T-t) + B(T-t)\lambda_t) + B(T-t)) \cdot \kappa(\theta - \lambda_t) \tag{B.7}$$

$$V_t \cdot (\beta^2(T-t)(A(T-t) + B(T-t)\lambda_t) + 2\beta(T-t)B(T-t)) \cdot \frac{1}{2}\sigma^2\lambda_t. \tag{B.8}$$

For the last one I have to compute

$$l\int_{\mathbf{R}_+} (f(z+\lambda_t) - f(\lambda_t))\frac{e^{-\frac{z}{\mu}}}{\mu}dz \tag{B.9}$$

for $f(t,x) = e^{\alpha(T-t)+\beta(T-t)x}(A(T-t) + B(T-t)x)$.

The integral from (B.9) can be rewritten as

$$
\begin{aligned}
& l\int_{\mathbf{R}_+} e^{\alpha(T-t)+\beta(T-t)(z+\lambda_t)}(A(T-t)+B(T-t)(z+\lambda_t))\frac{e^{-\frac{z}{\mu}}}{\mu}dz- \\
& \quad -l\int_{\mathbf{R}_+} e^{\alpha(T-t)+\beta(T-t)\lambda_t}(A(T-t)+B(T-t)\lambda_t)\frac{e^{-\frac{z}{\mu}}}{\mu}dz= \\
& \quad = V_t\cdot l\int_{\mathbf{R}_+}(e^{\beta(T-t)z}(A(T-t)+B(T-t)(z+\lambda_t))-(A(T-t)+B(T-t)\lambda_t))\frac{e^{-\frac{z}{\mu}}}{\mu}dz= \\
& \quad = V_t\cdot l\cdot(A(T-t)+B(T-t)\lambda_t)\cdot\int_{\mathbf{R}_+}(e^{\beta(T-t)z}-1)\frac{e^{-\frac{z}{\mu}}}{\mu}dz+ \\
& \quad +V_t\cdot l\cdot B(T-t)\cdot\int_{\mathbf{R}_+} e^{\beta(T-t)z}z\frac{e^{-\frac{z}{\mu}}}{\mu}dz.
\end{aligned}
$$

The final form of (B.9) is given again from the formulas characterizing an exponentially distributed random variable and presented in the beginning of this Appendix

$$
V_t\cdot l\cdot(A(T-t)+B(T-t)\lambda_t)\cdot\frac{\mu\beta(T-t)}{1-\mu\beta(T-t)}+V_t\cdot l\cdot B(T-t)\cdot\frac{\mu}{(1-\mu\beta(T-t))^2}.
$$

I will account that α and β satisfy a set of ODEs as in (B.4) and (B.5), for $(p,q)=(0,0)$ and initial conditions $\alpha(0)=u_0$ and $\beta_0=u_1$.

$$
\begin{aligned}
\beta' &= \kappa\beta-\frac{1}{2}\sigma^2\beta^2 \\
\alpha' &= -\kappa\theta\beta-l\frac{\mu\beta}{1-\mu\beta}.
\end{aligned}
$$

The drift of $d(U_t\cdot V_t)$ will be 0 if the coefficients of the quadratic form in λ_t obtained are 0. This will lead after simplifications to the following set of ODEs for $A(t)$ and $B(t)$.

$$
\begin{aligned}
B'(t) &= q+\kappa B(t)-\sigma^2\beta(t)B(t) && \text{(B.10)}\\
A'(t) &= p-\kappa\theta B(t)-\frac{l\mu B(t)}{(1-\mu\beta(t))^2}. && \text{(B.11)}
\end{aligned}
$$

The initial conditions for A and B are straightforward, $A(0)=v_0$ and $B(0)=v_1$.

2. Long-run mean and variance

To compute the long run mean (2.2) one should notice that the differential equation satisfied by $g(t)=\mathrm{E}[\lambda_t]$ is $g'(t)=\kappa(\theta-g(t))+l\mu$ with initial condition $g(0)=\lambda_0$. If I denote $\overline{m}=\theta+\frac{l\mu}{\kappa}$, the solution is given by

$$
g(t)=\overline{m}+(\lambda_0-\overline{m})e^{-\kappa t}.
$$

Taking the limit $\lim_{t\to\infty} g(t)$, the long-run mean will be equal with $\overline{m}$.

For the long-run variance (2.3) one should compute first the operator $\mathcal{D}$ for $f(t,x) = x^2$ and get the differential equation satisfied by the $h(t) = \mathrm{E}[\lambda_t^2]$:

$$h'(t) = (2\kappa\theta g(t) - 2\kappa h(t)) + \sigma^2 g(t) + l(2\mu^2 + 2\mu g(t)).$$

Here I used the formulas for the mean and variance of an exponentially distributed random variable. The initial condition is $h(0) = \lambda_0^2$ and the equation can be rewritten as

$$h'(t) = -2\kappa h(t) + 2l\mu^2 + (\overline{m} + (\lambda_0 - \overline{m})e^{-\kappa t})(\sigma^2 + 2\kappa\overline{m}).$$

The solution is given by

$$h(t) = \frac{2l\mu^2 + \overline{m}\sigma^2 + 2\kappa\overline{m}^2}{2\kappa} + \frac{(\lambda_0 - \overline{m})(\sigma^2 + 2\kappa\overline{m})}{\kappa} e^{-\kappa t} + Ce^{-2\kappa t}.$$

The constant C is chosen in such a way that $h(0) = \lambda_0^2$.

Now the long-run variance will be the limit $\lim_{t\to\infty}(h(t) - (g(t))^2)$ and further simplifications will yield the final answer $\dfrac{\sigma^2\overline{m}}{2\kappa} + \dfrac{l\mu^2}{\kappa}$.