Absolutely Continuous Spectrum for the Anderson Model on Trees

by

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Abstract

This dissertation is an examination of the absolutely continuous spectrum for the Anderson model on different types of trees. The text is divided into four chapter: an introduction, two main chapters and conclusions.

In Chapter 2 the existence of purely absolutely continuous spectrum is proven for the Anderson model on a Cayley tree, or Bethe lattice, of degree K. The method used, a geometric one, is based on some properties of the hyperbolic distance. It is a simplified generalization of a result for $K = 3$ given by R. Froese, D. Hasler and W. Spitzer.

In Chapter 3 a similar result is proven for a more general tree which has vertices of degrees 2 and 3 alternating in a periodic manner. The lack of symmetry changes the analysis, making it possible to eliminate one of the steps in the proof for the Cayley tree.

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Chapter 1

Introduction

1.1 Preliminaries

In quantum mechanics, the Schrödinger equation describes the change in the quantum state of a physical system. In the standard interpretation of quantum mechanics, the quantum state, also called a wave function or state vector, ψ_t, is the most complete description that can be given to a physical system. Solutions to Schrödinger's equation describe not only atomic and subatomic systems, atoms and electrons, but also macroscopic systems, possibly even the whole universe. The equation of a general quantum system is $\mathrm{i}\,\partial_t \psi_t = H\,\psi_t$, where H, the Hamiltonian, is a self-adjoint operator on a Hilbert space. We know, due to functional calculus, that its solution with initial condition ψ_0 is $\psi_t = e^{-\mathrm{i}tH}\psi_0$.

The total energy of a particle, in quantum mechanics, is expressed as the sum of operators corresponding to the kinetic and potential energies, in the form $H = T + V$ on the Hilbert space $L^2(\mathbb{R}^n)$. For such a system the kinetic term is the unbounded operator $T = -\frac{1}{2}\Delta = \frac{1}{2}\mathrm{p}^2$, with $\mathrm{p} = \mathrm{i}\nabla$. The potential V is a multiplication operator with a function on the configuration space, a function that depends on the application we want to consider. For random Schrödinger operators this function is a random variable.

The spectral analysis of H helps us determine some physical properties of the system. The spectrum, σ, of an operator has three components. Those components are pure point spectrum, σ_{pp}, singular continuous spectrum, σ_{sc} and absolutely continuous spectrum, σ_{ac}. The pure point spectrum corresponds to energy levels for which the system is generally (depending also on other properties of the model) an insulator and the absolutely continuous spectrum corresponds to energy levels for which the system is a conductor.

The study of random Schrödinger operators is an area of very active research in mathematical physics and mathematics. Since the replacement of a continuous system by a discrete one is a common approximation the physics literature, special attention has been give to the study of random Schrödinger operators on the discrete space $\mathbb{Z}^d$ with $d = 1, 2, \ldots$, the associated Hilbert space being $l^2(\mathbb{Z}^d)$. The ensemble of Hamiltonians of the form: $H_q = \Delta + k\,q(x)$, where $\Delta\psi(x) = \sum_{|y-x|=1} \psi(y)$ for $\psi \in l^2(\mathbb{Z}^d)$, $k \geq 0$ and $\{q(x)\}_{x\in\mathbb{Z}^d}$ are independent identically distributed random variables is known in the literature as *the Anderson model* [2]. The assumptions on the random variables are not well motivated, but they are useful for simplicity. It is certainly interesting (and in many cases a challenging problem) to relax these assumptions.

In his paper, *Absence of diffusion in certain random lattices* (1958), P. Anderson discovered one of the most striking quantum interference phenomena: particle localization due to disorder. Cited in 1977 for the Nobel prize in physics, the paper was fundamental for many subsequent developments in condensed matter theory. In particular, in the last 25 years the phenomenon of localization proved to be crucial for the understanding of the Quantum Hall effect, mesoscopic fluctuations in small conductors as well as some aspects of quantum chaotic behaviour. Random Schrödinger operators are an area of very active research in mathematical physics and mathematics. Here the main effort is to clarify the nature of the underlying spectrum. We will give a short presentation of both what physicists concluded and what mathematicians proved.

There is a qualitative difference between one dimensional disordered systems ($d = 1$) and higher dimensional ones ($d \geq 3$). For one dimensional disordered systems one expects that the whole spectrum is pure point. Thus, there is a complete system of eigenfunctions which decay exponentially at infinity. This phenomenon is called *Anderson localization* or *exponential localization* and it corresponds to low mobility of the electrons in our system. Therefore, one dimensional disordered systems (e.g. thin wires with impurities) should have low or even vanishing conductivity and an arbitrarily small disorder will change the total spectrum from absolutely continuous to pure point. This has been already proved for all energies for an ample class of disordered systems.

The physicists seem to believe that we have complete Anderson localization for $d = 2$ similar to the case $d = 1$. However, the pure point spectrum is expected to be less stable for $d = 2$. There are no mathematical proofs.

In dimension ($d \geq 3$) the physics of the system is more complicated. For small randomness, Anderson localization occurs near the band edges of the spectrum. Near any band edge a there is an inteval $[a, a + \delta]$, (respectively $[a - \delta, a]$) of pure point spectrum and the corresponding eigenfunctions are *exponentially localized* in the sense that they decay exponentially fast at infinity. Inside the bands the spectrum is expected to be absolutely continuous at small disorder. Since the corresponding (generalized) eigenfunctions are certainly not square integrable, one speaks of *extended states* or *Anderson delocalization* in this regime. The pure point spectrum expands when the randomness increases and the absolutely continuous part is expected to become smaller and smaller. The physics community believes in the existence of a phase transition from an insulating phase to a conducting phase. A transition point between these phases is called a *mobility edge*. At a certain level of disorder the absolutely continuous spectrum would disappear and we will be left with only pure point spectrum.

There are no mathematical results on the Anderson delocalization in $\ell^2(\mathbb{Z}^d)$, $d \geq 2$. There is no proof of existence of absolutely continuous spectrum for any of the models we have discussed so far. In particular it is not known whether there is a conducting phase or a mobility edge at all. However, there are results for continuum models on $L^2(\mathbb{R}^d)$, for $d \geq 2$. On $L^2(\mathbb{R}^2)$, A. Klein, O. Lenoble and P. Mueller proved the existence of dynamical delocalization at energies in the Landau bands of the randomly perturbed Landau Hamiltonian. Also, existence of absolutely continuous spectrum is known for trees. Anderson delocalization on trees is the focus of this disertation.

The first result on delocalization was obtained by Abel Klein in 1998. He proved the existence of purely absolutely continuous spectrum, under weak disorder, on the Bethe lattice (or Cayley tree). This is a graph ("lattice") without loops (hence a tree) with a fixed number of edges at every site, its degree. One considers the graph Laplacian on the Bethe lattice, which is analogously defined to the Laplacian on $\mathbb{Z}^d$ and an independently identically distributed potential on the sites of the graph. Using supersymmetric representations Klein showed that for small disorder the almost sure spectrum is purely absolutely continuous in an energy range which is contained in $\sigma_{ac}(\Delta)$ as shown in [6]. Moreover, the states in this energy range exhibit super-ballistic-transport behaviour [7].

Later, in 2005, Aizenman, Sims and Warzel presented a different method for establishing the persistence of some absolutely continuous spectrum under weak disorder. Their work does not address the question whether the ac spectrum is pure in the intervals under study. However, the results apply to more general situations.

During the same year, Froese, Hasler and Spitzer introduced a geometric method which proved the existence of purely absolutely continuous spectrum on a Bethe lattice of degree 3. Their result is similar to the one obtained by Klein in 1998, but it is only for a Bethe lattice of degree 3 whereas Klein's is for any degree $K \geq 3$.

The first paper of this manuscript brings a few simplifications to the method and generalizes it to the Bethe lattice of degree K. The second paper extends the method to a more general, less symmetrical, type of trees. The model in Chapter 2 is ergodic and therefore the spectral components are almost surely constant; the one in Chapter 3 is not but nevertheless, the absolutely continuous spectrum is identified almost surely.

1.2 General Outline

In both cases, the proofs follow some common steps; these steps will be explained in this section. Let us consider a tree $\mathbb{T}$; for simplicity we will use the symbol $\mathbb{T}$ for both our tree and its set of vertices. We denote by o its origin. For each $x \in \mathbb{T}$ we have at most one neighbor towards the root and one or more in what we refer to as the forward direction. We say that $y \in \mathbb{T}$ is in the future of $x \in \mathbb{T}$ if the path connecting y and the root runs through x. The subtree consisting of all the vertices in the future of x, with x regarded as its root, is denoted by $\mathbb{T}^x$. The Anderson Hamiltonian, H, on the Hilbert space $\ell^2(\mathbb{T}) = \left\{ \varphi : \mathbb{T} \to \mathbb{C} ; \sum_{x\in\mathbb{T}} |\varphi(x)|^2 < \infty \right\}$ is the operator of the form $H = \Delta + k\,q$ where:

1. The free Laplacian Δ is defined by

$$(\Delta\varphi)(x) = \sum_{y:d(x,y)=1} (\varphi(x) - \varphi(y)) \text{ , for all } \varphi \in \ell^2(\mathbb{T}) \,,$$

   with the distance d denoting the number of edges in the shortest (only) path between sites.

2. The operator q is a random potential,

$$(q\varphi)(x) = q(x)\varphi(x),$$

   where $\{q(x)\}_{x\in\mathbb{T}}$ is a family of independent, identically distributed real random variables with common probability distribution ν. We assume the $2(1 + p)$ moment, $\int |q|^{2(1+p)} d\nu$, is finite for some $p > 0$. The coupling constant k measures the disorder.

The goal is to prove the existence of absolutely continuous spectrum for this operator when k is small, more precisely, that there are intervals on which all the spectral measures associated to this Hamiltonian are absolutely continuous. The analysis herein focuses on the resolvent $(H - \lambda)^{-1}$ for $\lambda \in \mathbb{C}$.

The *spectrum* of H, denoted by $\sigma(H)$, is defined to be the set of $\lambda \in \mathbb{C}$ such that $(H - \lambda)^{-1}$, the *resolvent*, does not exist as a bounded operator. For a self-adjoint operator this spectrum is real. The matrix elements of the resolvent will often be referred to as the *Green functions* and denoted by $G_{(x,y)}(\lambda) := \langle\delta_x, (H - \lambda)^{-1}\delta_y\rangle$ and $G_x(\lambda)$ for the diagonal elements. Here δ_x is the Kronecker delta function.

The spectral measure μ_x associated to δ_x is absolutely continuous with respect to the Lebesgue measure on some finite interval E, if

$$\liminf_{\beta \searrow 0} \int_E \left|\langle \delta_x, (H-\lambda)^{-1}\delta_x\rangle\right|^{1+p} d\alpha < \infty \,,$$

for some fixed $p > 0$ and $\lambda = \alpha + \mathrm{i}\beta$. This result is proved, for reference, in the appendix of Chapter 3. Using Fatou's lemma and Fubini's theorem we have

$$\mathbb{E}\left(\liminf_{\beta \searrow 0} \int_E \left|\langle \delta_x, (H-\lambda)^{-1}\delta_x\rangle\right|^{1+p} d\alpha\right) \leq \liminf_{\beta \searrow 0} \int_E \mathbb{E}\left(\left|\langle \delta_x, (H-\lambda)^{-1}\delta_x\rangle\right|^{1+p}\right) d\alpha \,,$$

hence, for the existence of absolutely continuous spectrum in E it suffices to prove that

$$\sup_{\lambda \in R(E,\epsilon)} \mathbb{E}\left(\left|\left\langle \delta_x, (H-\lambda)^{-1}\delta_x\right\rangle\right|^{1+p}\right) < \infty \,, \tag{1.1}$$

where $R(E,\epsilon) = \{z \in \mathbb{C} : \mathrm{Re}(z) \in E, 0 < \mathrm{Im}(z) \leq \epsilon\}$ is a strip along the real axis. We first prove (1.1) at $x = o$, the origin of the tree and then extend it to all the other spectral measures. In (1.1) we have a supremum of the expected value of the $(1 + p)$ power of the absolute value of a Green function. We will first prove the inequality for a weight function w, instead of the absolute value, where

$$\mathrm{w}(z) = \frac{|z - z_\lambda|^2}{\mathrm{Im}(z)\mathrm{Im}(z_\lambda)} = 2\left(\cosh(\mathrm{dist}_{\mathbb{H}}(z, z_\lambda)) - 1\right) .$$

Up to constants, $\mathrm{w}(z)$ is the hyperbolic cosine of the hyperbolic distance from z to $z_\lambda = \left\langle \delta_0, (\Delta - \lambda)^{-1}\delta_0\right\rangle$, the Green function at the origin for the Laplacian. We use the inequality $|z| \leq 4\mathrm{w}(z)\mathrm{Im}(z_\lambda) + 2|z_\lambda|$ to finish the proof of (1.1).

For the free Laplacian we can determine intervals of absolutely continuous spectrum. We then prove the persistence of this spectrum for the perturbed Laplacian on closed subintervals, E.

As it can already be observed the matrix elements of the resolvent, the Green functions, will play an important role in this dissertation. We can derive a recursion formula for $G_0(\lambda)$ based on the forward Green functions.

Let H^x be the restriction of H to $\ell^2(\mathbb{T}^x)$. The *forward Green function* $G^x(\lambda)$ is defined to be the Green function for the truncated graph, given by $G^x(\lambda) = \left\langle \delta_x, (H^x - \lambda)^{-1}\delta_x\right\rangle$.

The recursion relation for the forward Green functions on any graph can be determined using Schur's formula (see [5]). In our case, since we only have trees, the forward Green function at some vertex $x \in \mathbb{T}$ depends only on the forward Green functions at the neighbouring sites in the future of x. For example, in the case of a binary tree if x_1 and x_2 are the forward neighbours of a vertex x, then $G^x(\lambda) = \phi(G^{x_1}, G^{x_2}, \lambda, q)$ where $\phi(z_1, z_2, \lambda, q) = \dfrac{-1}{z_1 + z_2 + \lambda - q}$. This recursion expression can be easily derived using resolvent properties.

For the free Laplacian, self similarity of the tree implies that the forward Green function at the origin is a fixed point for the transformation ϕ. Thus, the spectrum of the free Laplacian can be determined by calculating this fixed point.

Using the above mentioned recursion formula we have, for our example, $\mathrm{w}^{1+p}(G^x(\lambda)) = \mathrm{w}^{1+p}(\phi(G^{x_1}, G^{x_2}, \lambda, q)$. By taking expectation we obtain the probabilistic recursion $\mathbb{E}\left(\mathrm{w}^{1+p}(G^x(\lambda))\right) = \mathbb{E}\left(\mathrm{w}^{1+p}(\phi(G^{x_1}, G^{x_2}, \lambda, q))\right)$. The particular values for the forward Green functions at the vertices x, x_1 and x_2 are different but their probability distribution is the same and therefore $\mathbb{E}\left(\mathrm{w}^{1+p}(G^x(\lambda))\right) = \mathbb{E}\left(\mathrm{w}^{1+p}(G^{x_1}(\lambda))\right) = \mathbb{E}\left(\mathrm{w}^{1+p}(G^{x_2}(\lambda))\right)$. If we define $\mu_2(z_1, z_2, q, \lambda) = \dfrac{2\mathrm{w}^{1+p}(\phi(z_1, z_2, \lambda, q))}{\mathrm{w}^{1+p}(z_1) + \mathrm{w}^{1+p}(z_2)}$ we have the following:

$$\begin{aligned} &\mathbb{E}\left(\mathrm{w}^{1+p}(G^x(\lambda))\right) \\ &= \mathbb{E}\left(\mu_2\left(G^{x_1}(\lambda), G^{x_2}(\lambda), q, \lambda\right)\left(\frac{1}{2}\mathrm{w}^{1+p}(G^{x_1}(\lambda)) + \frac{1}{2}\mathrm{w}^{1+p}(G^{x_2}(\lambda))\right)\right). \end{aligned}$$

Now suppose we can prove

$$\mu_2(z_1, z_2, \lambda, q) \leq (1 - \epsilon)\chi_1(z_1, z_2) + \chi_2(z_1, z_2) \tag{1.2}$$

where $\chi_1(z_1, z_2)$ and $\chi_2(z_1, z_2)$ are cut-off functions and $\chi_2(z_1, z_2)$ is supported in a region where $\frac{1}{2}\mathrm{w}^{1+p}(z_1) + \frac{1}{2}\mathrm{w}^{1+p}(z_2) < C$ and $\epsilon > 0$ is small.

Then, $\mathbb{E}\left(\mathrm{w}^{1+p}(G^x(\lambda))\right) \leq (1-\epsilon)\,\mathbb{E}\left(w^{1+p}(G^x(\lambda)\right) + C$. This proves (1.1). The crucial estimate for the proofs is (1.2) and the main work in the thesis consists in proving this estimate (or a similar, more complicated one in Chapter 2) for the trees considered. The intermediate steps for achieving this are different depending on the choice of tree. In Chapter 2 we look at the Bethe lattice on which all nodes look alike. All these symmetries make the contraction properties of ϕ less obvious and two recursion steps are needed in the analysis. The tree in Chapter 3 has a little bit less symmetry and therefore the analysis requires only one recursion step.

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Chapter 2

Absolutely Continuous Spectrum for the Anderson Model on a Cayley Tree [1]

2.1 Introduction

One of the most important open problems in the field of random Schrödinger operators is to prove the existence of absolutely continuous spectrum for weak disorder in the Anderson model [2] in three and higher dimensions. The first result in this direction is Abel Klein's, for random Schrödinger operators acting on a tree, or Bethe lattice, of any degree larger than 2. Klein [6] proves that for weak disorder, almost all potentials will produce absolutely continuous spectrum. This means that there must be many potentials on a tree for which the corresponding Schrödinger operator has absolutely continuous spectrum without there being an obvious reason, such as periodicity or decrease at infinity. Later on, different other proofs were given to the same result (see [4] and [1]). The goal of this chapter is to generalize the geometric method in [4] from a Bethe lattice of degree 3 to one of any degree $M+1$, with $M \geq 2$.

2.2 The Model and the Results

A Bethe lattice (or Cayley tree), $\mathbb{B}$, is a connected infinite graph with no closed loops and a fixed degree (number of nearest neighbors) at each vertex (site or point), x. The distance between two sites x and y will be denoted by $d(x, y)$ and is equal to the length of the shortest (only) path connecting x and y.

[1] A version of this chapter will be submitted for publication. Halasan, F. *Absolutely Continuous Spectrum for the Anderson Model on a Cayley Tree*.

The Anderson model on the Bethe lattice is given by the random Hamiltonian

$$H = \Delta + k\,q$$

on the Hilbert space $\ell^2(\mathbb{B}) = \{\varphi : \mathbb{B} \to \mathbb{C}\,;\, \sum_{x\in\mathbb{B}} |\varphi(x)|^2 < \infty\}$. The (centered) Laplacian Δ is defined by

$$(\Delta\varphi)(x) = \sum_{y:\, d(x,y)=1} \varphi(y)$$

and has spectrum $\sigma(\Delta) = [-2\sqrt{M}, 2\sqrt{M}]$. The operator q is a random potential, with $q(x)$, $x \in \mathbb{B}$, being independent, identically distributed real random variables with common probability distribution ν. We assume the $2(1+p)$ moment, $\int |q|^{2(1+p)} d\nu$, is finite for some $p > 0$. The coupling constant k measures the disorder.

As mentioned above, the existence of purely absolutely continuous spectrum for the Anderson model on the Bethe lattice was first proved, in a different manner, by Klein in 1998. Given any closed interval E contained in the interior of the spectrum of Δ on the Bethe lattice, he proved that for small disorder, H has purely absolutely continuous spectrum in some interval E with probability one, and its integrated density of states is continuously differentiable on the interval (he only needed a finite second moment, whereas we have a finite 2(1+p) moment in our model). We prove a similar result in this chapter. A key point is the definition of a weight function appearing in the proofs. This definition is motivated by hyperbolic geometry.

Theorem 2.1. *For any E, with $0 < E < 2\sqrt{M}$ and H defined above, there exists $k(E) > 0$ such that for all $0 < |k| < k(E)$ the spectrum of H is purely absolutely continuous in $[-E, E]$ with probability one, i.e., we have almost surely*

$$\sigma_{\text{ac}} \cap [-E, E] = [-E, E]\ , \quad \sigma_{\text{pp}} \cap [-E, E] = \emptyset\ , \quad \sigma_{\text{sc}} \cap [-E, E] = \emptyset\ .$$

Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ denote the complex upper half plane. For convenience, we fix an arbitrary site in $\mathbb{B}$ to be the origin and denote it by 0. For each $x \in \mathbb{B}$ we have at most one neighbour towards the root and two or more in what we refer to as the forward direction. We say that $y \in \mathbb{B}$ is in the future of $x \in \mathbb{B}$ if the path connecting y and the root runs through x. Let $x \in \mathbb{B}$ be an arbitrary vertex, the subtree consisting of all the vertices in the future of x, with x regarded as its root, is denoted by $\mathbb{B}^x$. We will write H^x for H when restricted to $\mathbb{B}^x$ and set $G^x(\lambda) = \langle \delta_x, (H^x - \lambda)^{-1}\delta_x \rangle$ the Green function for the truncated graph. G^x is called the forward Green function.

Proposition 2.2. *For any $\lambda \in \mathbb{H}$ we have*

$$G(\lambda) = \left\langle \delta_0, (H-\lambda)^{-1}\delta_0 \right\rangle = -\left(\sum_{x:\, d(x,0)=1} G^x(\lambda) + \lambda - k\, q(0) \right)^{-1} \tag{2.1}$$

and, for any site $x \in \mathbb{B}$,

$$G^x(\lambda) = -\left(\sum_{y:\, d(y,x)=1,\, y\in\mathbb{B}^x} G^y(\lambda) + \lambda - k\, q(x) \right)^{-1} . \tag{2.2}$$

Proof. We will prove (2.1); (2.2) is proven in exactly the same way. Let us write $H = \tilde{H} + \Gamma$, where

$$\tilde{H} \;=\; k\, q(0) \oplus \left(\bigoplus_{x:\, d(x,0)=1} H^x \right)$$

is the direct sum corresponding to the decomposition $\mathbb{B} = \{0\} \cup \left(\bigcup_{x:\, d(x,0)=1} \mathbb{B}^x \right)$. The operator Γ has matrix elements $\langle \delta_x, \Gamma\delta_0 \rangle = \langle \delta_0, \Gamma\delta_x \rangle = 1$ if $d(x,0) = 1$, with all other matrix elements being 0. The resolvent identity gives

$$(\tilde{H} - \lambda)^{-1} \;=\; (H-\lambda)^{-1} + (\tilde{H}-\lambda)^{-1}\Gamma\,(H-\lambda)^{-1} .$$

Also,

$$(\tilde{H} - \lambda)^{-1} \;=\; (k\, q(0) - \lambda)^{-1} \oplus \left(\bigoplus_{x:\, d(x,0)=1} (H^x - \lambda)^{-1} \right) .$$

Thus

$$\left\langle \delta_0, (\tilde{H}-\lambda)^{-1}\delta_0 \right\rangle \;=\; \left\langle \delta_0, (H-\lambda)^{-1}\delta_0 \right\rangle + \left\langle \delta_0, (\tilde{H}-\lambda)^{-1}\Gamma\,(H-\lambda)^{-1}\delta_0 \right\rangle .$$

Hence

$$G(\lambda) \;=\; (q(0)-\lambda)^{-1} - (k\, q(0)-\lambda)^{-1} \sum_{x:\, d(x,0)=1} \left\langle \delta_x, (H-\lambda)^{-1}\delta_0 \right\rangle , \tag{2.3}$$

The resolvent formula also implies that for each x with $d(x,0)=1$,

$$\left\langle \delta_x, (H-\lambda)^{-1}\delta_0 \right\rangle = -G^x(\lambda)\, G(\lambda)\,. \tag{2.4}$$

(2.2) follows from (2.3) and (2.4). □

The recursion relation for $G^x(\lambda)$ that we just proved leads us to the following transformation

$$\phi : \mathbb{H}^M \times \mathbb{R} \times \mathbb{H} \to \mathbb{H}$$

defined by

$$\phi(z_1,...z_M,q,\lambda) = \frac{-1}{z_1+...+z_M+\lambda-q}\,. \tag{2.5}$$

It is easy to see the equivalence between (2.1) and (2.5). Let $q \equiv 0$. If $\mathrm{Im}(\lambda) > 0$, the transformation $z \mapsto \phi(z,...,z,0,\lambda)$ has a unique fixed point, z_λ, in the upper half plane, i.e. $\mathrm{Im}(z_\lambda) > 0$ (for details see Proposition 2.1, in [3]). Explicitly,

$$z_\lambda = \frac{-\lambda}{2M} + \frac{1}{M}\sqrt{(\lambda/2)^2 - M}\,,$$

where we will always make the choice $\mathrm{Im}\sqrt{\cdot} \geq 0$ (and $\sqrt{a} > 0$ for $a > 0$). This fixed point as a function of $\lambda \in \mathbb{H}$ extends continuously onto the real axis. This extension yields, for $\mathrm{Im}(\lambda) = 0$ and $|\lambda| < 2\sqrt{M}$, the fixed point

$$z_\lambda = -\frac{\lambda}{2M} + \frac{i}{2M}\sqrt{4M-\lambda^2}\,,$$

lying on an arc of the circle $|z| = 1/\sqrt{M}$. When $\mathrm{Im}(\lambda) = 0$ and $|\lambda| \leq E < 2\sqrt{M}$, the arc is strictly contained in the upper half plane. Thus, when λ lies in the strip

$$R(E,\epsilon) = \{z \in \mathbb{H} : \mathrm{Re}(z) \in [-E,E], 0 < \mathrm{Im}(z) \leq \epsilon\}$$

with $0 < E < 2\sqrt{M}$ and ϵ sufficiently small, $\mathrm{Im}(z_\lambda)$ is bounded below and $|z_\lambda|$ is bounded above by a positive constant.

In order to prove that the spectral measures are absolutely continuous we need to establish bounds for $\mathbb{E}(|G^x(\lambda)|^{1+p})$. Since z_λ equals $G^x(\lambda)$ for the case $q \equiv 0$ and any $x \in \mathbb{B}$, in order to prove the desired bounds we will use the weight function $\mathrm{w}(z)$ defined by

$$\mathrm{w}(z) = 2\left(\cosh(\mathrm{dist}_{\mathbb{H}}(z,z_\lambda)) - 1\right) = \frac{|z-z_\lambda|^2}{\mathrm{Im}(z)\mathrm{Im}(z_\lambda)}\,. \tag{2.6}$$

Up to constants, $\mathrm{w}(z)$ is the hyperbolic cosine of the hyperbolic distance from z to z_λ, provided $\lambda \in R(E,\epsilon)$ with $0 < E < 2\sqrt{M}$ and ϵ sufficiently small. This notation suppresses the λ dependence. In essence, we are looking at the hyperbolic cosine of the distance between $G^x(\lambda)$ for the free Laplacian and the one for the perturbed one, H. The goal is to prove that this quantity, which blows up on the boundary, stays mostly finite.

To prove a bound for $\mathbb{E}(\mathrm{w}^{1+p}(G^x(\lambda)))$ we will need to use (2.5), more than once, to express the forward Green function as a function of the forward Green functions at future nodes. As a result, the study of the following quantity becomes needed:

$$\mu_{3,p}(z_1 \ldots z_{2M-1}, q_1, q_2, \lambda) = \sum_{\sigma} \frac{\mathrm{w}^{1+p}(\phi(\phi(z_{\sigma_1} \ldots z_{\sigma_M}, q_1, \lambda), z_{\sigma_{M+1}} \ldots z_{\sigma_{2M-1}} q_2, \lambda))}{\mathrm{w}^{1+p}(z_1) + \ldots + \mathrm{w}^{1+p}(z_{2M-1})}$$

where σ are all cyclic permutations. We will state here the needed lemmas, but we will give the proofs later.

Lemma 2.3. *For any E, $0 < E < 2\sqrt{M}$ and any $0 < p < 1$, there exist positive constants ϵ, η_1, ϵ_0 and a compact set $\mathcal{M} \in \mathbb{H}^{2M-1}$ such that*

$$\mu_{3,p}|_{\mathcal{M}^c \times [-\eta_1, \eta_1]^2 \times R(E,\epsilon_0)} \leq 1 - \epsilon. \tag{2.7}$$

Here $\mathcal{M}^c$ denotes the complement $\mathbb{H}^{2M-1} \setminus \mathcal{M}$.

Lemma 2.4. *For any E, $0 < E < 2\sqrt{M}$ and any $0 < p < 1$, there exist positive constants ϵ_0, C and a compact set $\mathcal{M} \in \mathbb{H}^{2M-1}$ such that*

$$\mu_{3,p}|_{\mathcal{M}^c \times \mathbb{R}^2 \times R(E,\epsilon_0)} \leq C(1 + \sum_{i=1}^{2} |q_i|^{2(1+p)}). \tag{2.8}$$

Similarly, if we define

$$\mu'_{3,p}(z_1, \ldots, z_{M+1}) = \frac{\mathrm{w}(-(\sum_{i=1}^{M+1} z_i + \lambda - q)^{-1})^{1+p}}{\mathrm{w}(z_1)^{1+p} + \ldots + \mathrm{w}(z_{M+1})^{1+p}},$$

then

$$\mu'_{3,p}|_{\mathcal{M}^c \times \mathbb{R}^2 \times R(E,\epsilon_0)} \leq C(1 + |q|^{2(1+p)}).$$

Theorem 2.5. *Let x be a nearest neighbour of* 0*. For any E,* $0 < E < 2\sqrt{M}$ *and all* $0 < p < 1$*, there exists $k(E) > 0$ such that for all $0 < |k| < k(E)$ we have*

$$\sup_{\lambda \in R(E,\epsilon)} \mathbb{E}\left(\mathrm{w}^{1+p}(G^x(\lambda))\right) < \infty .$$

Proof. In order to prove that the above quantity is bounded we need a couple of preparatory steps.

Let η_1 and p be given by Lemma 2.3, and choose ϵ_0 and $\mathcal{M}$ that work in both Lemma 2.3 and Lemma 2.4. For $(z_1, \ldots, z_{2M-1}) \in \mathcal{M}^c$, we estimate

$$\begin{aligned}
&\int_{\mathbb{R}^2} \mu_{3,p}(z_1, \ldots, z_{2M-1}, k\,q_1, k\,q_2, \lambda) d\nu(q_1) d\nu(q_2) \\
&\leq (1-\epsilon) \int_{\left[\frac{-\eta_1}{k}, \frac{\eta_1}{k}\right]^2} d\nu(q_1) d\nu(q_2) + C \int_{\mathbb{R}^2 \setminus \left[\frac{-\eta_1}{k}, \frac{\eta_1}{k}\right]^2} (1 + \sum_{i=1}^{2} |k\,q_i|^{2(1+p)})\, d\nu(q_1) d\nu(q_2) \\
&\leq (1-\epsilon) + C \int_{\mathbb{R}^2 \setminus \left[\frac{-\eta_1}{k}, \frac{\eta_1}{k}\right]^2} d\nu(q_1) d\nu(q_2) + 2C|k|^{2(1+p)} M_{2(1+p)} \leq 1 - \epsilon/2
\end{aligned}$$

provided k is sufficiently small. Here $M_{2(1+p)}$ denotes the moment $\int |q|^{2(1+p)}\, d\nu(q)$.

The probability distributions for G and G^x on the hyperbolic plane are defined by $\rho_G(A) = \text{Prob}\{G(\lambda) \in A\}$ and $\rho(A) = \text{Prob}\{G^x(\lambda) \in A\}$. This implies

$$\begin{aligned}
&\rho(A) = \text{Prob}\{\phi(z_1 \ldots z_M, k\,q, \lambda) \in A\} = \text{Prob}\{(z_1 \ldots z_M, k\,q, \lambda) \in \phi^{-1}(A)\} \\
&= \int\limits_{\phi^{-1}(A)} d\rho(z_1) \ldots d\rho(z_M)\, d\nu(q) = \int\limits_{\mathbb{H}^M \times \mathbb{R}} \chi_A(\phi(z_1 \ldots z_M, k\,q, \lambda))\, d\rho(z_1) \ldots d\rho(z_M)\, d\nu(q)
\end{aligned}$$

which gives us that for any bounded continuous function $\mathrm{w}(z)$,

$$\int_{\mathbb{H}} \mathrm{w}(z) d\rho(z) = \int_{\mathbb{H}^M \times \mathbb{R}} \mathrm{w}(\phi(z_1, \ldots, z_M, k\,q, \lambda))\, d\rho(z_1)\, \ldots\, d\rho(z_M)\, d\nu(q).$$

Now we have all the ingredients needed to prove our theorem. Using the previous relation twice, for $\lambda \in R(E, \epsilon_0)$, we obtain:

$$\mathbb{E}\left(\mathrm{w}^{1+p}(G^x(\lambda))\right) = \int\limits_{\mathbb{H}} \mathrm{w}^{1+p}(z)\, d\rho(z)$$

$$= \int\limits_{\mathbb{H}^M\times\mathbb{R}} \mathrm{w}^{1+p}(\phi(z_1 \dots z_M, k\, q_1, \lambda))\, d\rho(z_1) \dots d\rho(z_M)\, d\nu(q_1)$$

$$= \int\limits_{\mathbb{H}^M\times\mathbb{R}^2} \mathrm{w}^{1+p}(\phi(\phi(z_1 \dots z_M, k\, q_1, \lambda), z_{M+1} \dots z_{2M-1}, k\, q_2, \lambda))$$

$$d\rho(z_1) \dots d\rho(z_{2M-1})\, d\nu(q_1) d\nu(q_2)$$

$$= \int\limits_{\mathbb{H}^M\times\mathbb{R}^2} \frac{1}{2M-1} \sum_{\sigma} \mathrm{w}^{1+p}(\phi(\phi(z_{\sigma_1} \dots z_{\sigma_M}, k\, q_1, \lambda), z_{\sigma_{M+1}} \dots z_{\sigma_{2M-1}}, k\, q_2, \lambda)))$$

$$d\rho(z_1) \dots d\rho(z_{2M-1})\, d\nu(q_1) d\nu(q_2)$$

$$= \frac{1}{2M-1} \int_{\mathcal{M}^c} \left(\int_{\mathbb{R}^2} \mu_{3,p}(z_1 \dots z_{2M-1}, k\, q_1, k\, q_2, \lambda)\, d\nu(q_1) d\nu(q_2) \right)$$

$$\times (\mathrm{w}^{1+p}(z_1) + \dots + \mathrm{w}^{1+p}(z_{2M-1}))\, d\rho(z_1) \dots d\rho(z_{2M-1}) + C$$

$$\leq (1-\epsilon/2) \int_{\mathbb{H}} \mathrm{w}^{1+p}(z)\, d\rho(z) + C = (1-\epsilon/2)\, \mathbb{E}\left(\mathrm{w}^{1+p}(G^x(\lambda)\right) + C\,.$$

where C is some finite constant, only depending on the choice of $\mathcal{M}$.
Note: We used the fact that

$$\int_{\mathbb{H}} \mathrm{w}^{1+p}(z) d\rho(z) = \frac{1}{2M-1} \int_{\mathbb{H}^{2M-1}} \left(\mathrm{w}^{1+p}(z_1) + \dots + \mathrm{w}^{1+p}(z_{2M-1})\right) d\rho(z_1) \dots d\rho(z_{2M-1})$$

This implies that for all $\lambda \in R(E, \epsilon_0)$,

$$\mathbb{E}\left(\mathrm{w}^{1+p}(G^x(\lambda))\right) \leq \frac{2C}{\epsilon}\,.$$

□

Theorem 2.6. *Let $x \in \mathbb{B}$. Under the hypotheses of Theorem 2.5,*

$$\sup_{\lambda \in R(E,\epsilon)} \mathbb{E}\left(\left| \langle \delta_x, (H-\lambda)^{-1} \delta_x \rangle \right|^{1+p} \right) < \infty$$

for some $\epsilon > 0$.

Proof. It is an immediate consequence of Theorem 2.5 and the following inequality:

$$|z| \leq 4\mathrm{Im}(s)\frac{|z-s|^2}{\mathrm{Im}(z)\mathrm{Im}(s)} + 2|s| \,. \tag{2.9}$$

The inequality clearly holds for $|z| \leq 2|s|$. In the complementary case, we have $|z| > 2|s|$ and thus $|z-s| \geq ||z|-|s|| \geq |s|$, implying $|z|\mathrm{Im}(z) \leq |z|^2 \leq 2|z-s|^2 + 2|s|^2 \leq 4|z-s|^2$. This proves (2.9).

Using (2.9) with $s = z_\lambda$ yields that for $\lambda \in R(E,\epsilon)$, $|z| \leq 4\mathrm{w}(z) + C$, where C depends only on E and ϵ.

To finish the proof we need to transfer the estimate from ρ to ρ_G and therefore prove the inequality for $x = 0$. By symmetry it extends to any vertex $x \in \mathbb{B}$. In the proof of the following estimate we need the elementary fact that for $z_1 \ldots z_{M+1} \in \mathcal{M}$, $\mathrm{w}^{1+p}\left(\left(\sum_{i=1}^{M+1} z_i + \lambda - q\right)^{-1}\right) \leq C\left(1+|q|^{2(1+p)}\right)$. Let R denote $R(E,\epsilon)$, then

$$\begin{aligned}
&\sup_{\lambda\in R}\mathbb{E}\left(\left|\left\langle\delta_0,(H-\lambda)^{-1}\delta_0\right\rangle\right|^{1+p}\right) = \sup_{\lambda\in R}\int_{\mathbb{H}}|z|^{1+p}\,d\rho_G(z)\\
&\leq C_1\sup_{\lambda\in R}\int_{\mathbb{H}}\mathrm{w}^{1+p}(z)\,d\rho_G(z) + C_2\\
&= C_1\sup_{\lambda\in R}\int_{\mathbb{H}^{M+1}\times\mathbb{R}}\mathrm{w}^{1+p}\left(\left(\sum_{i=1}^{M+1} z_i + \lambda - k\,q\right)^{-1}\right)d\rho(z_1)\ldots d\rho(z_{M+1})\,d\nu(q) + C_2\\
&\leq C_1\sup_{\lambda\in R}\int_{\mathcal{M}^c\times\mathbb{R}}\mu'_{3,p}(z_1,\ldots,z_{M+1},k\,q,\lambda)\times(\mathrm{w}^{1+p}(z_1)+\ldots+\mathrm{w}^{1+p}(z_{M+1}))\\
&\qquad\qquad d\rho(z_1)\ldots d\rho(z_{M+1})\,d\nu(q) + C_2'\\
&\leq C\int_{\mathbb{H}\times\mathbb{R}}(1+|k\,q|^{2(1+p)})\mathrm{w}^{1+p}(z)\,d\rho(z)\,d\nu(q) + C_2 \leq C\int_{\mathbb{H}}\mathrm{w}^{1+p}(z)\,d\rho(z) + C_3\\
&= C\,\mathbb{E}\left(\mathrm{w}^{1+p}(G^x(\lambda))\right) + C_3 \leq C_4\,,
\end{aligned}$$

where C, C_1, C_2,C_3 and C_4 are positive constants. □

As it was proven in [5] (or in the next chapter), this theorem implies the main result of this chapter: **Theorem 2.1.** *For any E, with $0 < E < 2\sqrt{M}$, there exists $k(E) > 0$ such that for all $0 < |k| < k(E)$ the spectrum of H is purely absolutely continuous in $[-E,E]$ with* *probability one, i.e., we have almost surely*

$$\sigma_{\rm ac} \cap [-E,E] = [-E,E]\ , \quad \sigma_{\rm pp} \cap [-E,E] = \emptyset\ , \quad \sigma_{\rm sc} \cap [-E,E] = \emptyset\ .$$

2.3 Analysis of μ_2 and Proofs of Lemmas

For the proofs of our technical lemmas we need to analyse a quantity, μ_2, which will prove to play a significant role in the expression for $\mu_{3,p}$. We define μ_2 by

$$\mu_2(z_1 \ldots z_M, q, \lambda) = \frac{M\,\mathrm{w}(\phi(z_1 \ldots z_M, q, \lambda))}{\mathrm{w}(z_1) + \ldots + \mathrm{w}(z_M)}$$

as a function from $\mathbb{H}^M \backslash \{(z_\lambda, \ldots, z_\lambda)\} \times \mathbb{R} \times R \to \mathbb{R}$. In this section $R = R(E, \epsilon)$, for some $0 < E < 2\sqrt{M}$ and $\epsilon > 0$.

Proposition 2.7. *For all* $z_1, \ldots, z_M \in \mathbb{H}^M \backslash \{(z_\lambda, \ldots, z_\lambda)\}$ *and* $\lambda \in R$*,*

$$\mu_2(z_1, \ldots, z_M, 0, \lambda) < 1\ .$$

Proof. For $z, s \in \mathbb{H}$ set

$$\mathrm{c}(s,z) = 2(\cosh(\mathrm{dist}_{\mathbb{H}}(s,z)) - 1) = \frac{|s-z|^2}{\mathrm{Im}(s)\mathrm{Im}(z)}\ .$$

Note that $z \mapsto \mathrm{c}(s,z)$ is strictly convex. This can be seen for example by noting that its Hessian has strictly positive eigenvalues. Also, for $s = z_\lambda$, $\mathrm{c}(z_\lambda, z) = \mathrm{w}(z)$. The transformation $\phi'(z) = -1/(z+\lambda)$ is a hyperbolic contraction (see [3], Proposition 2.1) and since $\phi'(z_1 + \ldots + z_M) = \phi(z_1 \ldots z_M, 0, \lambda)$ we have $\phi'(Mz_\lambda) = z_\lambda$. This implies

$$\begin{aligned}
&\mathrm{dist}_{\mathbb{H}}(\phi'(Mz_\lambda), \phi'(z_1 + \ldots + z_M)) < \mathrm{dist}_{\mathbb{H}}(Mz_\lambda, z_1 + \ldots + z_M) \Leftrightarrow \\
&\cosh(\mathrm{dist}_{\mathbb{H}}(\phi'(Mz_\lambda), \phi'(z_1 + \ldots + z_M))) < \cosh(\mathrm{dist}_{\mathbb{H}}(Mz_\lambda, z_1 + \ldots + z_M)) \Leftrightarrow \\
&c(z_\lambda, \phi(z_1, \ldots, z_M, 0, \lambda)) < c(Mz_\lambda, z_1 + \ldots + z_M) = c\left(z_\lambda, \frac{(z_1 + \ldots + z_M)}{M}\right) \\
&\qquad \le \frac{1}{M} \sum_{i=1}^{M} c(z_\lambda, z_i)\,,
\end{aligned}$$

hence

$$\frac{Mc(z_\lambda, \phi(z_1, \ldots, z_M, 0, \lambda))}{\sum_{i=1}^{M} c(z_\lambda, z_i)} < 1$$

Also, from Proposition 2.1 [3], if $\mathrm{Im}(\lambda) = 0$ then ϕ' is a hyperbolic isometry. Therefore

$$\begin{aligned} c(\phi'(Mz_\lambda), \phi'(z_1 + \ldots + z_M)) &= c(Mz_\lambda, z_1 + \ldots + z_M) \\ &= c\left(z_\lambda, \frac{z_1 + \ldots + z_M}{M}\right) \leq \frac{1}{M} \sum_{i=1}^{M} c(z_\lambda, z_i) \end{aligned}$$

If $\mathrm{Im}(\lambda) = 0$, then $\mu_2(z, \ldots, z, 0, \lambda) = 1$. If $\mathrm{Im}(\lambda) > 0$, since ϕ' is a hyperbolic contraction, $\mu_2(z, \ldots, z, 0, \lambda) = 1$ iff $z_1 = \ldots = z_M = z_\lambda$. □

Since in our lemmas we will use a compactification argument, we need to understand the behavior of $\mu_2(z_1, \ldots, z_M, q, \lambda)$ as $z_1, \ldots, z_M$ approach the boundary of $\mathbb{H}$ and λ approaches the real axis. Thus, it is natural to introduce the compactification $\overline{\mathbb{H}}^M \times \mathbb{R} \times \overline{R}$. Here $\overline{R}$ denotes the closure and $\overline{\mathbb{H}}$ is the compactification of $\mathbb{H}$ obtained by adjoining the boundary at infinity. (The word compactification is not quite accurate here because of the factor $\mathbb{R}$, but we will use the term nevertheless.)

The boundary at infinity is defined as follows. We cover the upper half plane model of the hyperbolic plane $\mathbb{H}$ with the atlas $\mathcal{A} = \{(U_i, \psi_i)_{i=1,2}\}$. We have $U_1 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0, |z| < C\}$, $\psi_1(z) = z$, $U_2 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0, |z| > C\}$ and $\psi_2(z) = -1/z = w$. The boundary at infinity consists of the sets $\{\mathrm{Im}(z) = 0\}$ and $\{\mathrm{Im}(w) = 0\}$ in the respective charts. The compactification $\overline{\mathbb{H}}$ is the upper half plane with the boundary at infinity adjoined. We will use $i\infty$ to denote the point where $w = 0$.

With this convention, μ_2 is defined in the interior of the compactification $\overline{\mathbb{H}}^M \times \mathbb{R} \times \overline{R}$ and we want to know how it behaves near the boundary. It turns out that in the coordinates introduced above, μ_2 is a rational function. For the majority of points on the boundary the denominator does not vanish in the limit and μ_2 has a continuous extension. There are, however, points where both numerator and denominator vanish and at these singular points the limiting value of μ_2 depends on the direction of approach. By blowing up the singular points, it would be possible to define a compactification to which μ_2 extends continuously. However, this is more than we need for our analysis. We will do a partial resolution of the singularities of μ_2 and then extend μ_2 to an upper semi-continuous function on the resulting compactification.

The reciprocal of the function w(z), $\chi(z) = \dfrac{1}{w(z)} = \dfrac{\text{Im}(z)\text{Im}(z_\lambda)}{|z - z_\lambda|^2}$ is a boundary defining function for $\mathbb{H}$. This means that in each of the two charts above, χ is positive near infinity and vanishes exactly to first order on the boundary at infinity. Further more, we can express μ_2 as follows:

$$\mu_2(z_1 \ldots z_M, q, \lambda) = \frac{M}{\chi(\phi(z_1 \ldots z_M, q, \lambda))[\frac{1}{\chi(z_1)} + \ldots + \frac{1}{\chi(z_M)}]}$$

or

$$\mu_2(z_1 \ldots z_M, q, \lambda) = \frac{M\chi(z_1)\ldots\chi(z_M)}{\chi(\phi(z_1 \ldots z_M, q, \lambda))[\chi(z_1)\ldots\chi(z_{M-1}) + \ldots + \chi(z_2)\ldots + \chi(z_M)]}$$

Since

$$\chi(\phi(z_1 \ldots z_M, q, \lambda)) = \frac{\text{Im}(\phi(z_1 \ldots z_M, q, \lambda))}{|z_\lambda - \phi(z_1 \ldots z_M, q, \lambda)|^2} = \frac{\text{Im}(z_1 + \ldots + z_M + \lambda)}{|z_\lambda(z_1 + \ldots + z_M) + \lambda z_\lambda - qz_\lambda + 1|^2}$$

we obtain

$$\mu_2(z_1 \ldots z_M, q, \lambda) = \frac{M \prod_{i=1}^{M} \chi(z_i)|z_\lambda \sum_{i=1}^{M} z_i + \lambda z_\lambda - qz_\lambda + 1|^2}{[\sum_{j=1}^{M} \prod_{\substack{i=1 \\ i\neq j}}^{M} \chi(z_i)][\sum_{i=1}^{M} \chi(z_i)|z_i - z_\lambda|^2 + \text{Im}(\lambda)]} \tag{2.10}$$

We will now describe our compactification of $\mathbb{H}^M \times \mathbb{R} \times R$. Start with $\overline{\mathbb{H}}^M \times \mathbb{R} \times \overline{R}$. Our blow-up consists of writing $\chi(z_1), \ldots, \chi(z_M)$ in polar co-ordinates. Thus we introduce new variables r_1 and β_i and impose the equations $\chi(z_1) = r_1\beta_1, \ldots, \chi(z_M) = r_1\beta_M$ and $\beta_1^2 + \ldots + \beta_M^2 = 1$. The blown up space, $\mathcal{K}$, is the variety in $\overline{\mathbb{H}}^M \times \mathbb{R} \times \overline{R} \times \mathbb{R}^{M+1}$ containing all points $(z_1, \ldots, z_M, q, \lambda, r_1, \beta_1, \ldots, \beta_M)$ that verify the blow-up constraints. The topology is the one given by the local description as a closed subset of Euclidean space. The set $\mathcal{K}\backslash\partial_\infty\mathcal{K}$ can be identified with $\mathbb{H}^M \times \mathbb{R} \times \overline{R}$. After the first blow-up, μ_2 becomes

$$\mu_2 = \frac{M \prod_{i=1}^{M} \beta_i |z_\lambda \sum_{i=1}^{M} z_i + \lambda z_\lambda - qz_\lambda + 1|^2}{[\sum_{j=1}^{M} \prod_{\substack{i=1 \\ i\neq j}}^{M} \beta_i][\sum_{i=1}^{M} \beta_i |z_i - z_\lambda|^2 + \text{Im}(\lambda)/r_1]} \tag{2.11}$$

We can extend μ_2 to an upper semi-continuous function on $\mathcal{K}$ by defining, for points $k \in \partial_\infty \mathcal{K}$,

$$\mu_2(k) = \limsup_{\substack{k_n \to k \\ k_n \in \mathcal{K} \setminus \partial_\infty \mathcal{K}}} \mu_2(k_n) \,.$$

Here $k_n = (z_{1,n}, z_{2,n}, \dots, z_{M,n}, q_{1,n}, \lambda_n)$ and it converges to k in $\mathcal{K}$.

Let us define Σ to be the subset of $\mathcal{K}$ where $\mu_2 = 1$ and let $\mathcal{K}_0$ denote the subset of $\partial_\infty \mathcal{K}$ where $\lambda \in (-2\sqrt{M}, 2\sqrt{M})$ and $q = 0$. For the analysis of μ_3 we need the following lemma:

Lemma 2.8. *Let* $\Gamma = \{k \in \mathcal{K} : k = (z_1, \dots, z_M, 0, \lambda, 0, \beta, \dots, \beta)\} \subset \mathcal{K}$*, it contains points in* $\mathcal{K}$ *with* $\beta_1 = \ldots = \beta_M = \beta$*. Then,*

$$\Gamma \cap \Sigma \cap \mathcal{K}_0 = \{k \in \mathcal{K}_0 : k = (z, \dots, z, 0, \lambda, 0, \beta, \dots, \beta)\} \,.$$

Proof. Let us first derive an upper bound μ_2^* for μ_2.
For $k = (z_1, \dots, z_M, 0, \lambda, r_1, \beta_1, \dots \beta_M) \in \mathcal{K} \setminus \partial_\infty \mathcal{K}$ we have

$$\begin{aligned}
\mu_2(k) &= \frac{M\, \mathrm{w}(\phi(z_1, \dots, z_M, q, \lambda))}{\sum_{i=1}^{M} \mathrm{w}(z_i)} = \frac{M\, \mathrm{c}(z_\lambda, \phi(z_1, \dots, z_M, q, \lambda))}{\sum_{i=1}^{M} \mathrm{c}(z_\lambda, z_i)} \\
&\le \frac{M\, \mathrm{c}(M z_\lambda, z_1 + \ldots + z_M)}{\sum_{i=1}^{M} c(z_\lambda, z_i)} = \frac{M\, \mathrm{w}(\frac{1}{M} \sum_{i=1}^{M} z_i)}{\sum_{i=1}^{M} \mathrm{w}(z_i)}.
\end{aligned}$$

Therefore we can define

$$\mu_2^*(k) = \frac{M\, \mathrm{w}(\frac{1}{M} \sum_{i=1}^{M} z_i)}{\sum_{i=1}^{M} \mathrm{w}(z_i)} = \frac{\prod_{i=1}^{M} \beta_i |\sum_{i=1}^{M} (z_i - z_\lambda)|^2}{[\sum_{j=1}^{M} \prod_{\substack{i=1 \\ i \neq j}}^{M} \beta_i][\sum_{i=1}^{M} \beta_i |z_i - z_\lambda|^2]} \tag{2.12}$$

Clearly $\mu_2 \le \mu_2^*$, with equality when λ is real.

Let $k \in \Gamma \cap \Sigma \cap \mathcal{K}_0$. If k is a point of continuity for μ_2^* then $\mu_2^*(k) = 1$. At a point of continuity k,

$$1 = \mu_2(k) = \limsup_{\substack{k_n \to k \\ k_n \in M \setminus \partial_\infty M}} \mu_2(k_n) \le \limsup_{\substack{k_n \to k \\ k_n \in M \setminus \partial_\infty M}} \mu_2^*(k_n) \le 1 \,.$$

The last inequality holds because at a point of continuity, the lim sup is actually a limit which can be evaluated in any order. If we take the limit in λ and q first, we may use the fact that for $\lambda \in (-2\sqrt{M}, 2\sqrt{M})$, $\mu_2 = \mu_2^*$. Proposition 2.7 proves that the limit in z_i is at most 1, which implies $\mu_2^*(k) = 1$ at the points of continuity.

Since we do not need to know the entire behavior of μ_2 at the boundary, we will concentrate only on the situations needed in the analysis of μ_3. Therefore we need two cases to consider:

Case I: Let $k \in \Gamma \cap \Sigma \cap \mathcal{K}_0$ such that $z_1, \ldots, z_M \in \partial_\infty \overline{\mathbb{H}}$ and $z_i \neq i\infty$ for all $i = 1, \ldots, M$. This is a point of continuity and we have:

$$\mu_2^*(k) = \frac{|\sum_{i=1}^{M}(z_i - z_\lambda)|^2}{M \sum_{i=1}^{M}|z_i - z_\lambda|^2}.$$

By the triangle inequality and the Cauchy Schwarz inequality,

$$|\sum_{i=1}^{M}(z_i - z_\lambda)|^2 \leq \left[\sum_{i=1}^{M}|z_i - z_\lambda|\right]^2 \leq M\left[\sum_{i=1}^{M}|z_i - z_\lambda|^2\right].$$

The first inequality turns into equality if $z_i - z_\lambda$ have the same argument for all i and the second one if $z_i - z_\lambda$ are equal in absolute values. Therefore, $\mu_2^* = 1$ iff all z_i are equal.

Case II: Let $k \in \Gamma \cap \Sigma \cap \mathcal{K}_0$, $z_1 = \ldots = z_a = i\infty$, and $z_{a+1}, \ldots, z_M$ are real, for some a, $1 < a < M$. Suppose (k_n) is a sequence that realizes the lim sup in the definition of $\mu_2(k)$.

$$\mu_2^*(k_n) = \frac{|\sum_{i=1}^{a}(z_i - z_\lambda) + \sum_{i=a+1}^{M}(z_i - z_\lambda)|^2}{M \sum_{i=1}^{M}|z_i - z_\lambda|^2} \leq \frac{|\sum_{i=1}^{a}(z_i - z_\lambda) + \sum_{i=a+1}^{M}(z_i - z_\lambda)|^2}{M \sum_{i=1}^{M}|z_i - z_\lambda|^2}.$$

The second term in the numerator stays finite in the limit and therefore, obviously
$\mu_2^*(k) \leq \dfrac{a}{M}$. □

We end this section with the proofs of our previous lemmas, Lemma 2.3 and Lemma 2.4.

Proof of Lemma 2.3: In order to simplify the notation, let us define
$Z = (z_1, \ldots, z_{2M-1})$, $Q = (q_1, q_2)$, $\xi_\sigma(Z, Q, \lambda) = (z_{\sigma_1}, \ldots, z_{\sigma_M}, q_1, \lambda)$,
$\tau_\sigma(Z, Q, \lambda) = (\phi(\xi_\sigma(Z, Q, \lambda)), z_{\sigma_{M+1}}, \ldots, z_{\sigma_{2M-1}}, q_2, \lambda)$ and

$$\nu_i = \frac{\mathrm{w}(z_i)}{\mathrm{w}(z_1) + \ldots + \mathrm{w}(z_{2M-1})}.$$

Extend $\mu_{3,p}$ to an upper semi-continuous function on $\overline{\mathbb{H}}^{2M-1} \times \mathbb{R}^2 \times \overline{R}$ by setting, at points Z_0, Q_0, λ_0 where it is not already defined,

$$\mu_{3,p}(Z_0, Q_0, \lambda_0) \quad = \quad \limsup_{Z\to Z_0, Q\to Q_0, \lambda\to\lambda_0} \mu_{3,p}(Z, Q, \lambda), \ .$$

The points Z, Q and λ are approaching their limits in the topology of $\overline{\mathbb{H}}^{2M-1} \times \mathbb{R}^2 \times \overline{R}$. To prove the lemma it is enough to show that

$$\mu_{3,p}(Z, Q, \lambda) \quad < \quad 1$$

for (Z, Q, λ) in the compact set $\partial_\infty \overline{\mathbb{H}}^{2M-1} \times \{0\}^2 \times [-E, E]$, since this implies that for some $\epsilon > 0$, the upper semi-continuous function $\mu_{3,p}(Z, Q, \lambda)$ is bounded by $1-2\epsilon$ on the set, and by $1 - \epsilon$ in some neighborhood. We have

$$\begin{aligned}
\mu_{3,p}(Z, Q, \lambda) &= \sum_\sigma \frac{\mathrm{w}^{1+p}(\phi(\tau_\sigma(Z, Q, \lambda)))}{\mathrm{w}^{1+p}(z_1) + \ldots + \mathrm{w}^{1+p}(z_{2M-1})} \\
&= \sum_\sigma \left(\frac{\mathrm{w}(\phi(\tau_\sigma(Z, Q, \lambda)))}{\mathrm{w}(z_1) + \ldots + \mathrm{w}(z_{2M-1})} \right)^{1+p} \frac{1}{\nu_1^{1+p} + \ldots + \nu_{2M-1}^{1+p}} = \\
&= \sum_\sigma \left[\mu_2(\tau_\sigma) \left(\frac{1}{M^2} \mu_2(\xi_\sigma)(\nu_{\sigma_1} + \ldots + \nu_{\sigma_M}) + \frac{1}{M}(\nu_{\sigma_{M+1}} + \ldots + \nu_{\sigma_{2M-1}}) \right) \right]^{1+p} \cdot \\
&\qquad \cdot \frac{1}{\nu_1^{1+p} + \ldots + \nu_{2M-1}^{1+p}}.
\end{aligned}$$

Define $\chi(z_1) = \dfrac{1}{\mathrm{w}(z_1)} = R_1\Omega_1, \ldots, \chi(z_{2M-1}) = \dfrac{1}{\mathrm{w}(z_{2M-1})} = R_1\Omega_{2M-1}$, where R_1, Ω_1, $\Omega_2, \ldots, \Omega_{2M-1}$ are defined functions of Z with the property $\Omega_1^2 + \ldots + \Omega_{2M-1}^2 = 1$. Notice that for any cyclic permutation σ,

$$\nu_{\sigma_l} \quad = \quad \frac{\prod_{\substack{j=1 \\ j\neq l}}^{2M-1} \Omega_{\sigma_j}}{\sum_{i=1}^{2M-1} \left[\prod_{\substack{j=1 \\ j\neq 1}}^{2M-1} \Omega_j \right]} \tag{2.13}$$

In the analysis of $\mu_2(\xi_\sigma)$ we use the blow-up with coordinates $r_{1\sigma}(\xi_\sigma)$ and $\beta_{\sigma_j}(\xi_\sigma)$ where $j = 1, \ldots, M$ and in the analysis of $\mu_2(\tau_\sigma)$ we use the blow-up with coordinates

$r_{2\sigma}(\tau_\sigma)$ and $\beta_{\sigma_j}(\tau_\sigma)$ where $j = M, \ldots, 2M-1$. Therefore we have the following relations:

$R_1\Omega_{\sigma_j} = r_{1\sigma}\beta_{\sigma_j}(\xi_\sigma)$ when $j = 1, \ldots, M$

$R_1 F = \chi(\phi(\xi_\sigma)) = r_{2\sigma}\beta_{\sigma_1}(\tau_\sigma)$

$R_1\Omega_{\sigma_j} = r_{2\sigma}\beta_{\sigma_j}(\tau_\sigma)$ when $j = M+1, \ldots, 2M-1$

where

$$F = \frac{\chi(\phi(\xi_\sigma))}{R_1} = \frac{r_{1\sigma} M \prod_{i=1}^{M} \beta_{\sigma_i}}{R_1 \mu_2(\xi_\sigma(Z,Q,\lambda)) \sum_{j=1}^{M} \prod_{\substack{i=1\\ i\neq j}}^{M} \beta_{\sigma_i}} = \frac{M\Omega_{\sigma_1} \prod_{i=2}^{M} \beta_{\sigma_i}}{\mu_2(\xi_\sigma(Z,Q,\lambda)) \sum_{j=1}^{M} \prod_{\substack{i=1\\ i\neq j}}^{M} \beta_{\sigma_i}}.$$

Consequently

$\Omega^2_{\sigma_j} = \beta^2_{\sigma_j}(\xi_\sigma)(\Omega^2_{\sigma_1} + \ldots + \Omega^2_{\sigma_M})$ for $j = 1, \ldots, M$

$\Omega^2_{\sigma_j} = \beta^2_{\sigma_j}(\tau_\sigma)(F + \Omega^2_{\sigma_{M+1}} + \ldots + \Omega^2_{\sigma_{2M-1}})$ for $j = M, \ldots, 2M-1$.

Suppose that $\mu_{3,p}(Z, Q, \lambda) = 1$ for some $(Z, Q, \lambda) \in \partial_\infty \overline{\mathbb{H}}^{2M-1} \times \{0\}^2 \times [-E, E]$. Then there must exist a sequence (Z_n, Q_n, λ_n) with $Z_n \longrightarrow Z$ in $\overline{\mathbb{H}}^{2M-1}$, $Q_n \longrightarrow (0,0)$ and $\lambda_n \longrightarrow \lambda \in [-E, E]$ such that

$$\lim \mu_{3,p}(Z_n, Q_n, \lambda_n) = 1.$$

From now on Z and λ will denote the limiting values of the sequences Z_n and λ_n. Similarly, we will denote by ν_i and Ω_i the limits of $\nu_i(Z_n)$ and $\Omega_i(Z_n)$.

We claim that

$$\nu_1 = \ldots = \nu_{2M-1} = \frac{1}{2M-1}. \tag{2.14}$$

This follows from the expression for $\mu_{3,p}(Z, Q, \lambda)$, the bound for μ_2 and the convexity of $x \mapsto x^{1+p}$:

$$
\begin{aligned}
1 &= \mu_{3,p}(Z, Q, \lambda) \\
&= \sum_{\sigma} \left[\mu_2(\tau_\sigma) \left(\frac{1}{M^2} \mu_2(\xi_\sigma)(v_{\sigma_1} + \ldots + v_{\sigma_M}) + \frac{1}{M}(v_{\sigma_{M+1}} + \ldots + v_{\sigma_{2M-1}}) \right) \right]^{1+p} \cdot \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \frac{1}{v_1^{1+p} + \ldots + v_{2M-1}^{1+p}} \\
&\leq \sum_{\sigma} \left[\left(\frac{1}{M^2}(v_{\sigma_1} + \ldots + v_{\sigma_M}) + \frac{1}{M}(v_{\sigma_{M+1}} + \ldots + v_{\sigma_{2M-1}}) \right) \right]^{1+p} \frac{1}{v_1^{1+p} + \ldots + v_{2M-1}^{1+p}} \\
&\leq \sum_{\sigma} \left(\frac{1}{M^2}(v_{\sigma_1}^{1+p} + \ldots + v_{\sigma_M}^{1+p}) + \frac{1}{M}(v_{\sigma_{M+1}}^{1+p} + \ldots + v_{\sigma_{2M-1}}^{1+p}) \right) \frac{1}{v_1^{1+p} + \ldots + v_{2M-1}^{1+p}} = 1,
\end{aligned}
$$

so the inequalities must actually be equalities. Since $p > 0$, strict convexity implies that equality only holds if $v_1 = \ldots = v_{2M-1}$. Since their sum is 1, their common value must be $\frac{1}{2M-1}$.

By going to a subsequence, we may assume that $\Omega_i(Z_n)$ converge. Then (2.13) and (2.14) imply that their limiting values along the sequence must be

$$
\Omega_1 = \ldots = \Omega_{2M-1} = \frac{1}{\sqrt{2M-1}}. \tag{2.15}
$$

One consequence is that

$$
z_i \in \partial_\infty \overline{\mathbb{H}} \tag{2.16}
$$

for $i = 1, \ldots, 2M-1$.

Now consider the values of $\xi_\sigma(Z_n, Q_n, \lambda_n)$ and $\tau_\sigma(Z_n, Q_n, \lambda_n)$. Since these values vary in a compact region in $\mathcal{M}$ we may, again by going to a subsequence, assume that they converge in $\mathcal{M}$ to values which we will denote ξ_σ and τ_σ. Using (2.14) and the bound $\mu_2 \leq 1$, we find that

$$
\begin{aligned}
1 &= \lim_{n\to\infty} \sum_{\sigma} \left[\mu_2(\tau_\sigma(Z_n, Q_n, \lambda_n)) \left(\frac{\mu_2(\xi_\sigma(Z_n, Q_n, \lambda_n)) + M - 1}{M(2M-1)} \right) \right]^{1+p} (2M-1)^p \\
&\leq \frac{1}{2M-1} \sum_{\sigma} \left[\frac{1}{M} \mu_2(\tau_\sigma)\,(\mu_2(\xi_\sigma) + M - 1) \right]^{1+p} \leq 1.
\end{aligned}
$$

This implies that for every σ occurring in the sum we have $\mu_2(\xi_\sigma) = \mu_2(\tau_\sigma) = 1$. Therefore, using (2.16) we conclude that for each σ, ξ_σ and τ_σ lie in the set Σ given by Lemma 2.8.

Now consider the coordinates β_{σ_i}, $i = 1, \dots, M$ for the point ξ_σ. These are the limiting values of $\beta_{\sigma_i}(z_{\sigma_1}, \dots, z_{\sigma_M})$ along our sequence. Since $\Omega^2_{\sigma_j} = \beta^2_{\sigma_j}(\Omega^2_1 + \dots + \Omega^2_M)$ and $\Omega_i = \frac{1}{\sqrt{2M-1}}$, we have $\beta_{\sigma_i} = \frac{1}{\sqrt{M}}$ for $i = 1, \dots, M$. Going back to the analysis of μ_2, Lemma 2.8, we conclude that the $\overline{\mathbb{H}}$ coordinates of ξ_σ, namely the limiting values of $z_{\sigma_1}, \dots, z_{\sigma_M}$ must be equal. Since this is true for every cyclic permutation, we conclude that

$$z = z_1 = z_2 = \dots = z_{2M-1} \in \partial_\infty \overline{\mathbb{H}}.$$

We have two distinct cases:

• If $z \in \mathbb{R}$ then $\phi(z_{\sigma_1}, \dots, z_{\sigma_M}, q, \lambda) \longrightarrow \phi(z, \dots, z, 0, \lambda) = \frac{-1}{Mz+\lambda}$. From the analysis of μ_2, Case I, the only way $\tau_\sigma = (\phi(z, \dots, z, 0, \lambda), z, \dots, z)$ can lie in Σ is if $\phi(z, \dots, z, 0, \lambda) = z$ which would imply $z = z_\lambda$ and this cannot happen since $z_\lambda \notin \partial_\infty \overline{\mathbb{H}}$.

• If $z = i\infty$ then $\phi(z_{\sigma_1}, \dots, z_{\sigma_M}, q, \lambda) \longrightarrow 0$ therefore $\tau_\sigma \longrightarrow (0, i\infty, \dots, i\infty)$. Since $\Omega^2_{\sigma_j} = \beta^2_{\sigma_j}(\tau_\sigma)(F + \Omega^2_{\sigma_{M+1}} + \dots + \Omega^2_{\sigma_{2M-1}})$ for $j = M, \dots, 2M-1$ and $F = \frac{1}{\sqrt{2M-1}}$ in the limiting case, $\beta_{\sigma_j}(\tau_\sigma)$ are equal. Going back to the analysis of μ_2, Case II, we conclude that $\mu_2(\tau_\sigma) < 1$.

Therefore, $\mu_{3,p}(Z, Q, \lambda) < 1$. □

Proof of Lemma 2.4: Each term in the sum appearing in $\mu_{3,p}$ can be estimated

$$\frac{\mathrm{w}^{1+p}(\phi(\cdots\cdots))}{\mathrm{w}^{1+p}(z_1) + \dots + \mathrm{w}^{1+p}(z_{2M-1})} = \frac{(\mathrm{w}(z_1) + \dots + \mathrm{w}(z_{2M-1}))^{1+p}}{\mathrm{w}^{1+p}(z_1) + \dots + \mathrm{w}^{1+p}(z_{2M-1})} \cdot \cdot \left(\frac{\mathrm{w}(\phi(\cdots\cdots))}{\mathrm{w}(z_1) + \dots + \mathrm{w}(z_{2M-1})} \right)^{1+p} \leq (2M-1)^p \left(\frac{\mathrm{w}(\phi(\cdots\cdots))}{\mathrm{w}(z_1) + \dots + \mathrm{w}(z_{2M-1})} \right)^{1+p},$$

where $\phi(\cdots\cdots)$ denotes $\phi(\phi(z_{\sigma_1}, \dots, z_{\sigma_M}, q_{\sigma_1}, \lambda), z_{\sigma_{M+1}}, \dots, z_{\sigma_{2M-1}}, q_{\sigma_2}, \lambda)$. Therefore it is enough to prove

$$\frac{\mathrm{w}(\phi(\cdots\cdots))}{\mathrm{w}(z_1) + \dots + \mathrm{cd}(z_{2M-1})} \leq C(1 + \sum_{i=1}^{2} |q_i|^2)\,.$$

Let $\phi(\cdots)$ denote $\phi(z_{\sigma_1},\dots,z_{\sigma_M},q_1,\lambda)$. We have

$$\frac{\mathrm{w}(\phi(\cdots\cdots))}{\mathrm{w}(z_1)+\ldots+\mathrm{w}(z_{2M-1})} = \frac{\left|1+z_\lambda\left(\phi(\ldots)+\sum_{i=M+1}^{2M-1} z_{\sigma_i}+\lambda-q_{\sigma_2}\right)\right|^2}{\mathrm{Im}\left[\phi(\ldots)+\sum_{i=M+1}^{2M-1} z_{\sigma_i}+\lambda\right]}\cdot\frac{1}{\sum_{i=1}^{2M-1}\frac{|z_i-z_\lambda|^2}{\mathrm{Im}(z_i)}} =$$

$$= \frac{|\sum_{i=1}^{M} z_{\sigma_i}+\lambda-q_{\sigma_1}+z_\lambda(-1+(\sum_{i=1}^{M} z_{\sigma_i}+\lambda-q_{\sigma_1})(\sum_{i=M+1}^{2M-1} z_{\sigma_i}+\lambda-q_{\sigma_2}))|^2}{\mathrm{Im}(\sum_{i=1}^{M} z_{\sigma_i}+\lambda)+\mathrm{Im}(\sum_{i=M+1}^{2M-1} z_{\sigma_i}+\lambda)|\sum_{i=1}^{M} z_{\sigma_i}+\lambda-q_{\sigma_1}|^2}\cdot\frac{1}{\sum_{i=1}^{2M-1}\frac{|z_i-z_\lambda|^2}{\mathrm{Im}(z_i)}}$$

$$\le C\left(\frac{1}{\mathrm{Im}(\sum_{i=M+1}^{2M-1} z_{\sigma_i})}+\frac{|-1+(\sum_{i=M+1}^{2M-1} z_{\sigma_i}+\lambda-q_{\sigma_2})(\sum_{i=1}^{M} z_{\sigma_i}+\lambda-q_{\sigma_1})|^2}{\mathrm{Im}(\sum_{i=1}^{M} z_{\sigma_i})+\mathrm{Im}(\sum_{i=M+1}^{2M-1} z_{\sigma_i})|\sum_{i=1}^{M} z_{\sigma_i}+\lambda-q_{\sigma_1}|^2}\right)\cdot\frac{1}{\sum_{i=1}^{2M-1}\frac{|z_i-z_\lambda|^2}{\mathrm{Im}(z_i)}}$$

$$\le C\left(\frac{1}{\mathrm{Im}(\sum_{i=M+1}^{2M-1} z_{\sigma_i})}+2\left(\frac{1}{\mathrm{Im}(\sum_{i=1}^{M} z_{\sigma_i})}+\frac{|\sum_{i=M+1}^{2M-1} z_{\sigma_i}+\lambda-q_{\sigma_2}|^2|\sum_{i=1}^{M} z_{\sigma_i}+\lambda-q_{\sigma_1}|^2}{\mathrm{Im}(\sum_{i=1}^{M} z_{\sigma_i})+\mathrm{Im}(\sum_{i=M+1}^{2M-1} z_{\sigma_i})|\sum_{i=1}^{M} z_{\sigma_i}+\lambda-q_{\sigma_1}|^2}\right)\right)\cdot\frac{1}{\sum_{i=1}^{2M-1}\frac{|z_i-z_\lambda|^2}{\mathrm{Im}(z_i)}}$$

$$\le C\left(\frac{1}{\mathrm{Im}(\sum_{i=M+1}^{2M-1} z_{\sigma_i})}+2\left(\frac{1}{\mathrm{Im}(\sum_{i=1}^{M} z_{\sigma_i})}+\frac{|\sum_{i=M+1}^{2M-1} z_{\sigma_i}+\lambda-q_{\sigma_2}|^2}{\mathrm{Im}(\sum_{i=M+1}^{2M-1} z_{\sigma_i})}\right)\right)\cdot\frac{1}{\sum_{i=1}^{2M-1}\frac{|z_i-z_\lambda|^2}{\mathrm{Im}(z_i)}}.$$

Choose the compact set $\mathcal{M}$ so that $\sum_{i=1}^{2M-1}|z_i-z_\lambda|^2/\mathrm{Im}(z_i)\ge C>0$ for some constant C and $(z_1,\dots,z_{2M-1})\in\mathcal{M}^c$. Then we can estimate each term depending on whether z_{σ_i} is close to z_λ.

If all z_{σ_i} are sufficiently close to z_λ, then $\mathrm{Im}(z_{\sigma_i})$ is bounded below and $|z_{\sigma_i}|$ is bounded above by a constant. Thus

$$\mathrm{Im}\left(\sum_{i=M+1}^{2M-1} z_{\sigma_i}\right)\sum_{i=1}^{2M-1}|z_i-z_\lambda|^2/\mathrm{Im}(z_i)\ge\mathrm{Im}\left(\sum_{i=M+1}^{2M-1} z_{\sigma_i}\right)C\ge C'>0\,,$$

$$\mathrm{Im}\left(\sum_{i=1}^{M} z_{\sigma_i}\right)\sum_{i=1}^{2M-1}|z_i-z_\lambda|^2/\mathrm{Im}(z_i)\ge\mathrm{Im}\left(\sum_{i=1}^{M} z_{\sigma_i}\right)C\ge C'>0$$

and

$$\Big|\sum_{i=M+1}^{2M-1} z_{\sigma_i} + \lambda - q_{\sigma_2}\Big|^2 \le \left(\Big|\sum_{i=M+1}^{2M-1} z_{\sigma_i} + \lambda\Big| + |q_{\sigma_2}|\right)^2 \le \left(\Big|\sum_{i=M+1}^{2M-1} z_{\sigma_i} + \lambda\Big|^2 + 1\right)\left(|q_{\sigma_2}|^2 + 1\right)$$
$$\le \left(\left(\Big|\sum_{i=M+1}^{2M-1} z_{\sigma_i}\Big| + |\lambda|\right)^2 + 1\right)\left(|q_{\sigma_2}|^2 + 1\right) \le \left(\Big|\sum_{i=M+1}^{2M-1} z_{\sigma_i}\Big|^2 \left(|\lambda|^2 + 1\right) + 1\right)\left(|q_{\sigma_2}|^2 + 1\right)$$

$\le C\left(\Big|\sum_{i=M+1}^{2M-1} z_{\sigma_i}\Big|^2 + 1\right)\left(|q_{\sigma_2}|^2 + 1\right) \le C\left(1 + |q_{\sigma_2}|^2\right)$, so we are done.

If all z_{σ_i} are far from z_λ, $\mathrm{Im}(\sum_{i=M+1}^{2M-1} z_{\sigma_i}) \sum_{i=1}^{2M-1} |z_i - z_\lambda|^2/\mathrm{Im}(z_i) \ge \sum_{i=M+1}^{2M-1} |z_{\sigma_i} - z_\lambda|^2 \ge \frac{1}{M-2}|\sum_{i=M+1}^{2M-1} (z_{\sigma_i} - z_\lambda)|^2 \ge C(1 + |\sum_{i=M+1}^{2M-1} z_{\sigma_i}|^2)$ so that $|\sum_{i=M+1}^{2M-1} z_{\sigma_i} + \lambda - q_{\sigma_2}|^2 \Big/ \left(\mathrm{Im}(\sum_{i=M+1}^{2M-1} z_{\sigma_i}) \sum_{i=1}^{2M-1} |z_i - z_\lambda|^2/\mathrm{Im}(z_i)\right) \le C(1 + |q_{\sigma_2}|^2)$ in this case too.
Also,

$$\mathrm{Im}(\sum_{i=1}^{M} z_{\sigma_i}) \sum_{i=1}^{2M-1} |z_i - z_\lambda|^2/\mathrm{Im}(z_i) \ge \sum_{i=1}^{M} |z_{\sigma_i} - z_\lambda|^2 \ge C(1 + |\sum_{i=1}^{M} z_{\sigma_i}|^2)\,.$$

If at least one z_{σ_j} is not close to z_λ for $j = 1, \ldots, M$, the first term is still bounded. If at least one z_{σ_j} is close to z_λ for $j = M+1, \ldots, 2M-1$, then the second term is finite and

$$\mathrm{Im}(\sum_{i=M+1}^{2M-1} z_{\sigma_i}) \sum_{i=1}^{2M-1} |z_i - z_\lambda|^2/\mathrm{Im}(z_i) \ge C + |z_{\sigma_j} - z_\lambda|^2 \ge C(C + |z_{\sigma_j}|^2)\,.$$

Therefore

$$\frac{\Big|\sum_{i=M+1}^{2M-1} z_{\sigma_i} + \lambda - q_{\sigma_2}\Big|^2}{\mathrm{Im}\left(\sum_{i=M+1}^{2M-1} z_{\sigma_i}\right)\sum_{i=1}^{2M-1} |z_i - z_\lambda|^2/\mathrm{Im}(z_i)} \le C\frac{\left(C_1 + |z_{\sigma_j}|^2\right)\left(1 + |q_{\sigma_2}|^2\right)}{C_2 + |z_{\sigma_j}|^2} \le C\left(1 + |q_{\sigma_2}|^2\right).$$

The estimates for $\mu'_{3,p}$ are very similar. We omit the details. □

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Chapter 3

Absolutely Continuous Spectrum for the Anderson Model on Some Tree-like Graphs [2]

3.1 Introduction

Random Schrödinger Operators are used as models for disordered quantum mechanical systems. In particular, the Anderson Model was introduced to describe the motion of a quantum-mechanical electron in a crystal with impurities. For this model, the states corresponding to an absolutely continuous spectrum describe mobile electrons. Thus, an interval of absolutely continuous spectrum is an energy range in which the material is a conductor.

An outstanding open problem, the extended states conjecture, is to prove existence of absolutely continuous spectrum for the lattice $\mathbb{Z}^d$ with $d > 2$. Until now, it is only for the Bethe lattice that this has been established. A first result on the topic was obtained by A. Klein, [6], in 1998; he proved that for weak disorder, on the Bethe lattice, there exists absolutely continuous spectrum for almost all potentials. More recently, Aizenman, Sims and Warzel proved similar results for the Bethe lattice using a different method (see [1]). Their method establishes the persistence of absolutely continuous spectrum under weak disorder and also in the presence of a periodic background potential. During the same time, Froese, Hasler and Spitzer introduced a geometric method for proving the existence of absolutely continuous spectrum on graphs (see [3]). In their second paper on the topic, [4], they proved delocalization for the Bethe lattice of degree 3 using this geometric approach.

In this work, we provide a version of the geometric method on a more general class of trees.

[2] A version of this chapter has been submitted for publication. Halasan, F. *Absolutely Continuous Spectrum for the Anderson Model on Some Tree-like Graphs*.

Statement of the Main Result

We prove the existence of purely absolutely continuous spectrum for the Anderson Model on a tree-like graph, $\mathbb{T}$, defined as follows (see Figure 4.1).

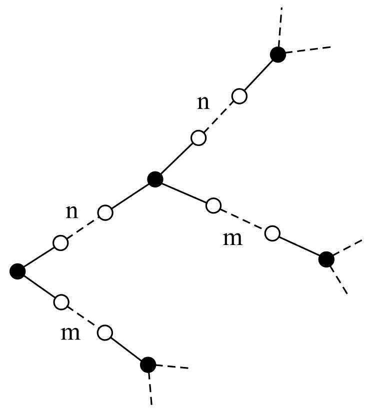

Figure 3.1: The $\mathbb{T}$ tree

Definition 3.1. Let $\mathbb{B}$ be an infinite full binary tree in which each node has degree 3 except for the origin, which has degree 2. Let us call its nodes principal nodes and denote by o its origin. For the origin and each principal node there are two edges leading away from the origin. Choose one of them and call it the top edge and call the other the bottom edge. On each top edge, we add m distinct auxiliary nodes; similarly we add n, $m \neq n$, distinct auxiliary nodes on each bottom edge. Thus we obtain the tree $\mathbb{T}$ which has a set of principal nodes denoted by $\mathbb{T}_p$ and a set of auxiliary nodes denoted by $\mathbb{T}_a$.

The conclusions in this paper remain valid if we start with any k-nary tree. We present the binary case for simplicity. By excluding the $m = n$ case we break some of the symmetry in our tree; this asymmetry is used in Proposition 4, Section 3. The proof for the $m = n$ case would constitute a generalization of the Bethe lattice proof presented in [5] and be considerably longer.

Using the terminology established in [1], we will use the symbol $\mathbb{T}$ for both our tree graph and its set of vertices. For each $x \in \mathbb{T} = \{o\} \cup \mathbb{T}_p \cup \mathbb{T}_a$ we have at most one neighbor towards the root and two in what we refer to as the forward direction. We say that $y \in \mathbb{T}$ is in the future of $x \in \mathbb{T}$ if the path connecting y and the root runs through x. The subtree consisting of all the vertices in the future of x, with x regarded as its root, is denoted by $\mathbb{T}^x$.

The Anderson Model on $\mathbb{T}$ is given by the random Hamiltonian, H, on the Hilbert space $\ell^2(\mathbb{T}) = \left\{ \varphi : \mathbb{T} \to \mathbb{C}\,;\ \sum_{x\in\mathbb{T}} |\varphi(x)|^2 < \infty \right\}$. This operator is of the form

$$H = \Delta + k\,q$$

where:

1. The free Laplacian Δ is defined by

$$(\Delta\varphi)(x) = \sum_{y:d(x,y)=1} (\varphi(x) - \varphi(y))\ ,\ \text{for all}\, \varphi \in \ell^2(\mathbb{T})\,,$$

   where the distance d denotes the number of edges between sites.

2. The operator q is a random potential,

$$(q\varphi)(x) = q(x)\varphi(x),$$

   where $\{q(x)\}_{x\in\mathbb{T}}$ is a family of independent, identically distributed real random variables with common probability distribution ν. We assume the $2(1+p)$ moment, $\int |q|^{2(1+p)} d\nu$, is finite for some $p > 0$. The coupling constant k measures the disorder.

Our main theorem states that the above defined Anderson model exhibits purely absolutely continuous spectrum for low disorder.

Theorem 3.2. *Let F be the open interior of the absolutely continuous spectrum of Δ (this spectrum depends on m and n) with a finite set of values, S, removed. For any closed subinterval E, $E \subset F$, there exists $k(E) > 0$ such that for all $0 < |k| < k(E)$ the spectrum of H is purely absolutely continuous in E with probability one.*

Remarks.
1). The finite set S will be properly identified in Proposition 3.7.
2). The actual definition we use for F is $F := \{\lambda \in \mathbb{R} : z_\lambda \in \mathbb{C}, \mathrm{Im}(z_\lambda) > 0\} \setminus S$ where $z_\lambda = \langle \delta_o, (\Delta - \lambda)^{-1}\delta_o \rangle$ (δ_o is the indicator function at the origin). Defined like this, F is the support of the absolutely continuous component of the spectral measure of Δ for δ_o without the special values contained in S. Following the ideas in Lemma 3.9 (Section 3.4), i.e. rearranging the tree and deriving a formula for the Green function at the new origin, we can prove that the set $\{\lambda \in \mathbb{R} : z_\lambda \in \mathbb{C}, \mathrm{Im}(z_\lambda) > 0\}$ is, in fact, the support of the pure absolutely continuous spectrum for the Laplacian Δ.
Let $\delta_x \in \ell^2(\mathbb{T})$ be the indicator function supported at the site $x \in \mathbb{T}$ and let $R(E, \epsilon) = \{z \in \mathbb{C} : \mathrm{Re}(z) \in E, 0 < \mathrm{Im}(z) \leq \epsilon\}$ be a strip along the real axis, for E defined in the previous

theorem. The following theorem together with the criterion from Section 3.4 gives us the proof of Theorem 3.2.

Theorem 3.3. *Under the hypothesis of the previous theorem, we have*

$$\sup_{\lambda \in R(E,\epsilon)} \mathbb{E}\left(\left|\left\langle \delta_x, (H-\lambda)^{-1}\delta_x \right\rangle\right|^{1+p}\right) < \infty \,,$$

for all sufficiently small $p > 0$, some $\epsilon > 0$ and all $x \in \mathbb{T}$.

Proof of Theoreom 3.2. Let us consider $\lambda = \alpha + i\beta$. Using Fatou's lemma, Fubini's theorem and Theorem 3.3, we obtain

$$\begin{aligned} &\mathbb{E}\left(\liminf_{\beta \searrow 0} \int_E \left|\langle \delta_x, (H-\lambda)^{-1}\delta_x \rangle\right|^{1+p} d\alpha\right) \\ &\leq \liminf_{\beta \searrow 0} \int_E \mathbb{E}\left(\left|\langle \delta_x, (H-\lambda)^{-1}\delta_x \rangle\right|^{1+p}\right) d\alpha < \infty \,. \end{aligned}$$

Therefore we must have

$$\liminf_{\beta \searrow 0} \int_E \left|\langle \delta_x, (H-\lambda)^{-1}\delta_x \rangle\right|^{1+p} d\alpha < \infty \,,$$

with probability one. Since $\langle \delta_x, (H-\lambda)^{-1}\delta_x \rangle$ is the Stieltjes transform of the measure $d\mu_x$, it follows from Proposition 3.8, Section 5 that the restriction of μ_x to E is purely absolutely continuous with probability one. In other words, the spectral measure for H corresponding to δ_x, for any $x \in \mathbb{T}$, is purely absolutely continuous in E with probability one. Therefore the operator H has purely absolutely continuous spectrum on E. □

3.2 Outline of the Proof

Let $G_x(\lambda) = \left\langle \delta_x, (H-\lambda)^{-1}\delta_x \right\rangle$ denote the diagonal matrix element of the resolvent at some arbitrary vertex $x \in \mathbb{T}$, often referred to as the Green function. Our goal is to find bounds for these Green functions. We first do so for $G_o(\lambda)$ and then extend the bound to all diagonal terms.

Let H^x be the restriction of H to $\ell^2(\mathbb{T}^x)$. The forward Green function $G^x(\lambda)$ is defined to be the Green function for the truncated graph, given by

$$G^x(\lambda) = \left\langle \delta_x, (H^x-\lambda)^{-1}\delta_x \right\rangle \,.$$

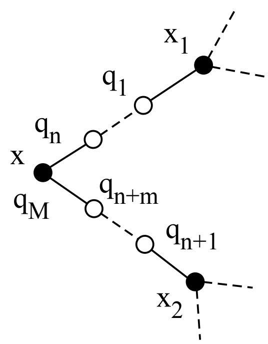

Figure 3.2: The nodes in the recurrence relation for the forward Green function.

The forward Green function $G^x(\lambda)$, for x a principal node, can be expressed recurrently as a function depending on the forward Green function for the two forward principal nodes and all the random potentials in between. Thus the recurrence relation, which can be derived using resolvent properties, has the form

$$G^x(\lambda) = \phi(G^{x_1}(\lambda), G^{x_2}(\lambda), q_1 \dots q_M, \lambda), \tag{3.1}$$

where

$$\phi : \mathbb{H}^2 \times \mathbb{R}^M \times \mathbb{H} \to \mathbb{H}$$

is defined by

$$\phi(z_1, z_2, q_1 \dots q_M, \lambda) = \frac{-1}{\phi_n(z_1, q_1 \dots q_n, \lambda) + \phi_m(z_2, q_{n+1} \dots q_{n+m}, \lambda) + \lambda - q_M} \tag{3.2}$$

with

$$\begin{aligned} \phi_0(z, \lambda) &= z \\ \phi_1(z, q_1, \lambda) &= \frac{-1}{z + \lambda - q_1 + 1} \\ &\dots \\ \phi_n(z, q_1 \dots q_n, \lambda) &= \frac{-1}{\phi_{n-1}(z, q_1 \dots q_{n-1}, \lambda) + \lambda - q_n + 1} \end{aligned}$$

and $M = m + n + 1$. The nodes x, x_1, x_2 and the potentials $q_1, \cdots, q_M$ involved in the recurrence (3.1) are shown in Figure 3.2. Because the origin has degree 2, the recurrence relation for $G^o(\lambda)$ is given by $G^o(\lambda) = \phi(G^{x_1}(\lambda), G^{x_2}(\lambda), q_1 \dots q_M, \lambda + 1)$.

In the above definition $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ is the complex upper half plane.

Notice that since H is random, at each $x \in \mathbb{T}$, the forward Green function $G^x(\lambda)$ is an $\mathbb{H}$-valued random variable. We notice that since the random potential is i.i.d., at any $x \in \mathbb{T}_p$, $G^x(\lambda)$ has the same probability distribution denoted by ρ. The Green function at the origin, $G_o(\lambda) = G^o(\lambda)$, has probability distribution denoted by ρ_o.

The transformations ϕ_n and ϕ_m, in the recursion formula, are compositions of fractional linear transformations, hence fractional linear transformations themselves. This implies ϕ is a rational function whose numerator and denominator have degree 2. If $\mathrm{Im}(\lambda) > 0$, the map $z \mapsto \phi(z, z, 0, \ldots, 0, \lambda)$ is an analytic map from $\mathbb{H}$ to $\mathbb{H}$, a hyperbolic contraction. Let z_λ denote its unique fixed point in the upper half plane, a solution to the cubic equation $z = \phi(z, z, 0, \ldots, 0, \lambda)$. The set $\{\lambda : \mathrm{Im}(\lambda) = 0 \text{ and } z_\lambda \in \mathbb{H}\}$ is a reunion of two disjoint open intervals on the real axis. Thus, for $q \equiv 0$, $G^x(\lambda) = z_\lambda$ for all $x \in \mathbb{T}_p$ and $G^o(\lambda) = \phi(z_\lambda, z_\lambda, 0 \ldots 0, \lambda + 1)$. The map $\mathbb{H} \ni \lambda \mapsto z_\lambda$ extends continuously onto the real axis. Therefore we define $F := \{\lambda : \mathrm{Im}(\lambda) = 0 \text{ and } z_\lambda \in \mathbb{H}\} \setminus S$ where, as mentioned before, S is defined in Proposition 3.7. We should note again that the set $F \cup S$ is the support of the absolutely continuous component of the spectral measure for δ_o, for the free Laplacian. The set $\{z_\lambda\}_{\lambda \in E}$ is a compact curve strictly contained in $\mathbb{H}$. Thus, when λ lies in the strip

$$R(E, \epsilon) = \{z \in \mathbb{H} : \mathrm{Re}(z) \in E, 0 < \mathrm{Im}(z) \leq \epsilon\}$$

with $E \subset F$ closed and ϵ sufficiently small, $\mathrm{Im}(z_\lambda)$ is bounded below and $|z_\lambda|$ is bounded above.

To prove absolutely continuous spectrum, we need the bound on $|G_x|^{1+p}$ stated in Theorem 2. To get this bound we first prove that $\mathrm{w}^{1+p}(G_x)$ is bounded, where w is a weight function defined as follows:

$$\mathrm{w}(z) = 2(\cosh(\mathrm{dist}_{\mathbb{H}}(z, z_\lambda)) - 1) = \frac{|z - z_\lambda|^2}{\mathrm{Im}(z)\mathrm{Im}(z_\lambda)} .$$

Up to constants, $\mathrm{w}(z)$ is the hyperbolic cosine of the hyperbolic distance from z to z_λ, the Green function at the root for Δ. We have dropped the λ-dependence from the notation.

Our proof relies on a pair of lemmas about the following quantity:

$$\mu_p(z_1, z_2, q_1 \ldots q_M, \lambda) = \frac{\mathrm{w}^{1+p}(\phi(z_1, z_2, q_1 \ldots q_M, \lambda)) + \mathrm{w}^{1+p}(\phi(z_2, z_1, q_1 \ldots q_M, \lambda))}{\mathrm{w}^{1+p}(z_1) + \mathrm{w}^{1+p}(z_2)},$$

for $z_1, z_2 \in \mathbb{H}^2$, $q_1, \ldots, q_M \in \mathbb{R}$ and $\lambda \in R(E, \epsilon)$.

Lemma 3.4. *For any closed subinterval E, $E \subset F$ and all sufficiently small* $0 < p < 1$, *there exist positive constants ϵ, η_1, ϵ_0 and a compact set $\mathcal{K} \subset \mathbb{H}^2$ such that*

$$\mu_p|_{\mathcal{K}^c \times [-\eta_1, \eta_1]^M \times R(E,\epsilon_0)}(z_1, z_2, q_1 \ldots q_M, \lambda) \quad \leq \quad 1 - \epsilon. \tag{3.3}$$

Here $\mathcal{K}^c$ denotes the complement $\mathbb{H}^2 \setminus \mathcal{K}$.

Lemma 3.5. *For any closed subinterval E, $E \subset F$ and any $0 < p < 1$, there exist positive constants ϵ_0, C and a compact set $\mathcal{K} \subset \mathbb{H}^2$ such that*

$$\mu_p|_{\mathcal{K}^c \times \mathbb{R}^M \times R(E,\epsilon_0)}(z_1, z_2, q_1 \ldots q_M, \lambda) \quad \leq \quad C \prod_{i=1}^{M} (1 + |q_i|^{2(1+p)}) \,. \tag{3.4}$$

Given these two lemmas we can prove that the decay of the probability distribution function of the forward Green function at infinity is preserved as $\mathrm{Im}(\lambda)$ becomes small, provided that ν has a finite moment of order $2(1 + p)$. Using Lemma 3.4 and Lemma 3.5 we prove Theorem 3.6 below, the last ingredient needed in the proof.

Theorem 3.6. *For any closed subinterval E, $E \subset F$, there exists $k(E) > 0$ such that for all $0 < |k| < k(E)$ we have*

$$\sup_{\lambda \in R(E,\epsilon)} \mathbb{E}\left(\mathrm{w}^{1+p}(G^x(\lambda))\right) < \infty \,,$$

for all $x \in \mathbb{T}_p$.

Proof. Let η_1 and p be given by Lemma 3.4, and choose ϵ_0 and $\mathcal{K}$ that work in both Lemma 3.4 and Lemma 3.5. For any $(z_1, z_2) \in \mathcal{K}^c$ and $\lambda \in R(E, \epsilon)$, we estimate

$$\begin{aligned}
&\int_{\mathbb{R}^M} \mu_p(z_1, z_2, kq_1 \ldots kq_M, \lambda) d\nu(q_1) \ldots d\nu(q_M) \\
&\leq (1 - \epsilon) \int_{[-\frac{\eta_1}{k}, \frac{\eta_1}{k}]^M} d\nu(q_1) \ldots d\nu(q_M) + C \int_{\mathbb{R}^M \setminus [-\frac{\eta_1}{k}, \frac{\eta_1}{k}]^M} \prod_{i=1}^{M} \left(1 + |k\, q_i|^{2(1+p)}\right) d\nu(q_1) \ldots \\
&\qquad\qquad d\nu(q_M) \\
&\leq 1 - \epsilon/2,
\end{aligned}$$

provided $|k|$ is sufficiently small.

The probability distributions on the hyperbolic plane are defined by

$$\rho(A) = \mathrm{Prob}\{G^x(\lambda) \in A\} \,,$$

where x is any site in $\mathbb{T}_p$. The recursion formula for the Green function implies that the distributions $d\rho$ are related by

$$\begin{aligned}\rho(A) &= \text{Prob}\{\phi(z_1, z_2, k\,q_1 \ldots k\,q_M, \lambda) \in A\} = \text{Prob}\{(z_1, z_2, k\,q_1 \ldots k\,q_M, \lambda) \in \phi^{-1}(A)\} \\ &= \int_{\phi^{-1}(A)} d\rho(z_1)\, d\rho(z_2)\, d\nu(q_1) \ldots d\nu(q_M)) \\ &= \int_{\mathbb{H}^2\times\mathbb{R}^M} \chi_A(\phi(z_1, z_2, k\,q_1 \ldots k\,q_M, \lambda))\, d\rho(z_1)\, d\rho(z_2)\, d\nu(q_1) \ldots d\nu(q_M)\end{aligned}$$

which gives us that for any bounded continuous function $f(z)$

$$\int_{\mathbb{H}} f(z) d\rho(z) = \int_{\mathbb{H}^2\times\mathbb{R}^M} f(\phi(z_1, z_2, k\,q_1 \ldots k\,q_M, \lambda))\, d\rho(z_1)\, d\rho(z_2)\, d\nu(q_1) \ldots d\nu(q_M).$$

Using this relation, for $\lambda \in R(E, \epsilon_0)$, we obtain

$$\begin{aligned}&\mathbb{E}\left(\mathrm{w}^{1+p}(G^x(\lambda))\right) = \int_{\mathbb{H}} \mathrm{w}^{1+p}(z)\, d\rho(z) \\ &= \int_{\mathbb{H}^2\times\mathbb{R}^M} \mathrm{w}^{1+p}(\phi(z_1, z_2, k\,q_1 \ldots k\,q_M, \lambda))\, d\rho(z_1)\, d\rho(z_2)\, d\nu(q_1) \ldots d\nu(q_M) \\ &= \int_{\mathbb{H}^2\times\mathbb{R}^M} \frac{1}{2}(\mathrm{w}^{1+p}(\phi(z_1, z_2, k\,q_1 \ldots k\,q_M, \lambda)) + \mathrm{w}^{1+p}(\phi(z_2, z_1, k\,q_1 \ldots k\,q_M, \lambda)))\, d\rho(z_1) \\ &\qquad\qquad d\rho(z_2)\, d\nu(q_1) \ldots d\nu(q_M) \\ &= \frac{1}{2}\int_{\mathcal{K}^c} \left(\int_{\mathbb{R}^M} \mu_p(z_1, z_2, k\,q_1 \ldots k\,q_M, \lambda)\, d\nu(q_1)\ \ldots\ d\nu(q_M)\right) \times (\mathrm{w}^{1+p}(z_1) + \mathrm{w}^{1+p}(z_2)) \\ &\qquad\qquad d\rho(z_1)\, d\rho(z_2) + C \\ &\le (1 - \epsilon/2) \int_{\mathbb{H}} \mathrm{w}^{1+p}(z)\, d\rho(z) + C = (1 - \epsilon/2)\, \mathbb{E}\left(\mathrm{w}^{1+p}(G^x(\lambda))\right) + C,\end{aligned}$$

where C is some finite constant, only depending on the choice of $\mathcal{K}$. This implies that for all $\lambda \in R(E, \epsilon_0)$,

$$\mathbb{E}\left(\mathrm{w}^{1+p}(G^x(\lambda))\right) \le \frac{2C}{\epsilon}\ .$$

□

Proof of Theorem 2. It is an immediate consequence of Theorem 3.6, Lemma 3.9 and the following inequality which holds for any two complex numbers z and s in $\mathbb{H}$:

$$|z| \le 4\text{Im}(s)\frac{|z - s|^2}{\text{Im}(z)\text{Im}(s)} + 2|s|\ . \tag{3.5}$$

The inequality clearly holds for $|z| \leq 2|s|$. In the complementary case, we have $|z| > 2|s|$ and thus $|z - s| \geq ||z| - |s|| \geq |s|$, implying

$$|z|\mathrm{Im}(z) \leq |z|^2 \leq 2\left(|z-s|^2 + |s|^2\right) \leq 4|z-s|^2$$

and further $|z| \leq 4|z - s|^2/\mathrm{Im}(z)$. This proves (3.5).

By using (3.5) with $s = z_\lambda$ we obtain that for $\lambda \in R(E, \epsilon)$

$$|z| \leq 4\mathrm{w}(z) + C\,,$$

where C depends only on E and ϵ.

Lemma 3.9 extends Theorem 3.6 to all $x \in \mathbb{T}$ and due to the previous inequality the statement of Theorem 3.3 follows. □

3.3 Proofs of Lemma 3.4 and Lemma 3.5

In this section we will prove the bounds for μ_p stated in Lemma 3.4 and Lemma 3.5. In order to do so we extend μ_p, define some quantities to simplify the calculations and prove Proposition 3.7. We prove Lemma 3.4 with the use of Proposition 3.7 and then prove Lemma 3.5.

Since in our lemmas we will use a compactification argument, we need to understand the behavior of $\mu_p(z_1, z_2, q_1 \ldots q_M, \lambda)$ as z_1, z_2 approach the boundary of $\mathbb{H}$ and λ approaches the real axis. Thus, it is natural to introduce the compactification $\overline{\mathbb{H}}^2 \times \mathbb{R}^M \times \overline{R}$. Here $\overline{R}$ denotes the closure and $\overline{\mathbb{H}}$ is the compactification of $\mathbb{H}$ obtained by adjoining the boundary at infinity. (The word compactification is not quite accurate here because of the factor $\mathbb{R}$, but we will use the term nevertheless.)

The boundary at infinity is defined as follows. We cover the upper half plane model of the hyperbolic plane $\mathbb{H}$ with the atlas $\mathcal{A} = \{(U_i, \psi_i)_{i=1,2}\}$. We have $U_1 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0, |z| < C\}$, $\psi_1(z) = z$, $U_2 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0, |z| > C\}$ and $\psi_2(z) = -1/z = u$. The boundary at infinity consists of the sets $\{\mathrm{Im}(z) = 0\}$ and $\{\mathrm{Im}(u) = 0\}$ in the respective charts. The compactification $\overline{\mathbb{H}}$ is the upper half plane with the boundary at infinity adjoined. We will use $i\infty$ to denote the point where $u = 0$.

We defined μ_p for $z_1, z_2 \in \mathbb{H}^2$ and $\lambda \in R(E, \epsilon)$, and now we extend μ_p to an upper semi-continuous function on $\overline{\mathbb{H}}^2 \times \mathbb{R}^M \times \overline{R}$ by defining it as

$$\mu_p(z_{1,0}, z_{2,0}, q_1 \dots q_M, \lambda_0) \quad = \quad \limsup_{z_1 \to z_{1,0}, z_2 \to z_{2,0}, \lambda \to \lambda_0} \mu_p(z_1, z_2, q_1 \dots q_M, \lambda)\ ,$$

at points $(z_{1,0}, z_{2,0})$ and λ_0 where it is not already defined. Here, the points (z_1, z_2) and λ are approaching their limits in the topology of $\overline{\mathbb{H}}^2 \times \overline{R}$. For computational purposes we define the following quantities:

$$\begin{aligned}
A_n &= (1 + \lambda - q_1 + z_1)(1 + \lambda - q_2 + \phi_1(z_1, q_1, \lambda)) \dots \\
&\qquad\qquad\qquad\qquad (1 + \lambda - q_n + \phi_{n-1}(z_1, q_1 \dots q_{n-1}, \lambda)) \\
A_m &= (1 + \lambda - q_{n+1} + z_1)(1 + \lambda - q_{n+2} + \phi_1(z_1, q_{n+1}, \lambda)) \dots \\
&\qquad\qquad\qquad\qquad (1 + \lambda - q_{M-1} + \phi_{m-1}(z_1, q_{n+1} \dots q_{M-2}, \lambda)) \\
C_n &= (1 + \lambda + z_\lambda)(1 + \lambda + \phi_1(z_\lambda, 0 \dots 0, \lambda)) \dots (1 + \lambda + \phi_{n-1}(z_\lambda, 0 \dots 0, \lambda))
\end{aligned}$$

and similarly, if we replace z_1 with z_2 we obtain B_n and B_m. C_m is defined analogously to C_n with m factors in the product instead of n. If we expand the expressions for A_n, B_n and C_n, respectively A_m, B_m and C_m, defined above we can see that they are linear polynomials in the z_i variable (i can be 1, 2 or λ). It is also worth mentioning that $C_n \neq 0$, respectively $C_m \neq 0$, and if $\mathrm{Im}(z_i) > 0$ then A_n, A_m, B_n, B_m are also different from 0. For more properties of these quantities see Section 3.5.
For the proof of Lemma 3.4 we need the following result:

Proposition 3.7. *For all $z_1, z_2 \in \partial_\infty \overline{\mathbb{H}}^2$ and $\lambda \in E$,*

$$\mu_0(z_1, z_2, 0 \dots 0, \lambda) \quad < \quad 1\ . \tag{3.6}$$

Here E is any closed interval with $E \subset int(F \setminus S)$.

Remark. In the case $m = n$, μ_0 is symmetric in z_1 and z_2 and equals 1 at some points on the boundary. To then prove our desired result we would need to go back one more step in our recurrence formula and analyse a more complicated version of μ_p.
Proof of Proposition 3.7. Let us assume $n > m$. For $z_1, z_2 \in \mathbb{H}^2 \backslash (z_\lambda, z_\lambda)$ we write $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$. Using these conventions, the triangle inequality and some simplifications

we have

$$\mathrm{w}(\phi(z_1, z_2, 0\ldots 0, \lambda)) = \frac{|(z_1 - z_\lambda)(B_m C_m) + (z_2 - z_\lambda)(A_n C_n)|^2}{(y_1|B_m|^2 + y_2|A_n|^2)(|C_m|^2 + |C_n|^2)\mathrm{Im}(z_\lambda)} \tag{3.7}$$
$$\leq \frac{(|z_1 - z_\lambda||B_m C_m| + |z_2 - z_\lambda||A_n C_n|)^2}{(y_1|B_m|^2 + y_2|A_n|^2)(|C_m|^2 + |C_n|^2)\mathrm{Im}(z_\lambda)}$$

and a similar inequality for $\mathrm{w}(\phi(z_2, z_1, 0\ldots 0, \lambda))$. These inequalities give us

$$\mu_0(z_1, z_2, 0\ldots 0, \lambda) \quad \leq \quad N/D \tag{3.8}$$

with

$$N = \Big((|z_1 - z_\lambda||B_m C_m| + |z_2 - z_\lambda||A_n C_n|)^2(y_2|A_m|^2 + y_1|B_n|^2) + (|z_2 - z_\lambda||A_m C_m| + |z_1 - z_\lambda||B_n C_n|)^2(y_1|B_m|^2 + y_2|A_n|^2)\Big)y_1 y_2 \tag{3.9}$$

and

$$D = (|C_m|^2 + |C_n|^2)(y_2|A_m|^2 + y_1|B_n|^2)(y_1|B_m|^2 + y_2|A_n|^2)(|z_1 - z_\lambda|^2 y_2 + |z_2 - z_\lambda|^2 y_1)\,. \tag{3.10}$$

It is easy to check that $N/D \leq 1$ for $z_1, z_2 \in \mathbb{H}^2 \setminus (z_\lambda, z_\lambda)$, but we do not need this since the statement of our proposition only refers to the boundary $\partial(\overline{\mathbb{H}}^2) = \partial(\overline{\mathbb{H}})\times\partial(\overline{\mathbb{H}})\cup\partial(\overline{\mathbb{H}})\times\mathbb{H}\cup\mathbb{H}\times\partial(\overline{\mathbb{H}})$ where $\partial(\overline{\mathbb{H}}) = \mathbb{R}\cup\{i\infty\}$. We know $\mu_0 \leq N/D \leq 1$, so we need to prove that at least one inequality is strict on the boundary. A few cases are to be considered:

Case I: Both z_1 and z_2 are on the real axis. Let $(z_{1,i}, z_{2,i}, \lambda_i) \to (z_1, z_2, \lambda)$ be a sequence that realizes the lim sup in the definition of μ_0. Notice that since $y_{1,i} \to 0$ and $y_{2,i} \to 0$, $\lim_{i\to\infty} N = \lim_{i\to\infty} D = 0$ so the limit of N/D may depend on the direction in which $z_{1,i}$ and $z_{2,i}$ approach $z_1$and z_2 . All the following variables will in fact be sequences determined by $(z_{1,i}, z_{2,i}, \lambda_i)$. We will sometimes suppress the index i for simplicity. In order to deal with this undetermined case we use a blow-up, more precisely we write y_1 and y_2 in the following form:

$$y_1 = r_1\omega_1$$
$$y_2 = r_1\omega_2$$

with $\omega_1^2 + \omega_2^2 = 1$ and $r_1 > 0$, all functions of z_1 and z_2. By going to a subsequence if needed, assume ω_1 and ω_2 converge as $i \to \infty$. After cancelling a factor r_1, N and D in (3.8) become

$$N = \Big((|z_1 - z_\lambda||B_m C_m| + |z_2 - z_\lambda||A_n C_n|)^2(\omega_2|A_m|^2 + \omega_1|B_n|^2) + (|z_2 - z_\lambda||A_m C_m| + |z_1 - z_\lambda||B_n C_n|)^2(\omega_1|B_m|^2 + \omega_2|A_n|^2)\Big)\omega_1\omega_2\,, \tag{3.11}$$

$$D = (|C_m|^2 + |C_n|^2)(\omega_2|A_m|^2 + \omega_1|B_n|^2)(\omega_1|B_m|^2 + \omega_2|A_n|^2)(|z_1 - z_\lambda|^2\omega_2 + |z_2 - z_\lambda|^2\omega_1)\,. \tag{3.12}$$

Let us first look at the points on the boundary where D has a non vanishing limit. The points where $D \to 0$ will need extra blow-ups and will be analysed afterwards.

We first show that $N/D \leq 1$ which is equivalent to proving the polynomial

$$\begin{aligned} P(X,Y) =& X^2\omega_2\Big(\omega_1\omega_2|A_m B_m C_n|^2 + \omega_1\omega_2|A_n B_n C_m|^2 + \omega_2^2|A_n A_m|^2(|C_m|^2 + |C_n|^2)\Big) + \\ & Y^2\omega_1\Big(\omega_1\omega_2|A_m B_m C_n|^2 + \omega_1\omega_2|A_n B_n C_m|^2 + \omega_1^2|B_n B_m|^2(|C_m|^2 + |C_n|^2)\Big) - \\ & 2XY\omega_1\omega_2|C_m C_n|\left(|A_n B_m|(\omega_2|A_m|^2 + \omega_1|B_n|^2) + |A_m B_n|(\omega_1|B_m|^2 + \omega_2|A_n|^2)\right) \end{aligned}$$

being positive; here $X = |z_1 - z_\lambda|$ and $Y = |z_2 - z_\lambda|$. It is easy to see that $P(X,Y) \geq 0$ since its discriminant has the form $\left(|A_m B_m||C_n|^2 - |A_n B_n||C_m|^2\right)^2\left(\omega_2|A_m|^2 + \omega_1|B_n|^2\right)$ $\Big(\omega_1|B_m|^2 + \omega_2|A_n|^2\Big)$.

Let us now assume $\mu_0 = 1$, so that $\mu_0 = N/D = 1$, and prove that the number of λ values for which this can happen is finite. The condition $\mu_0 = N/D$, which means equality in (3.7), is equivalent to the existence of p_1, p_2, s_1, s_2 positive real numbers and γ and δ reals such that

$$\begin{aligned} B_m C_m(z_1 - z_\lambda) &= p_1 e^{\mathrm{i}\delta}, \\ A_n C_n(z_2 - z_\lambda) &= p_2 e^{\mathrm{i}\delta}, \\ A_m C_m(z_2 - z_\lambda) &= s_1 e^{\mathrm{i}\gamma}, \\ B_m C_m(z_1 - z_\lambda) &= s_2 e^{\mathrm{i}\gamma}, \end{aligned}$$

which implies

$$A_m B_m C_m^2 (z_1 - z_\lambda)(z_2 - z_\lambda) = p_1 s_1 e^{\mathrm{i}(\delta+\gamma)},$$
$$A_n B_n C_n^2 (z_1 - z_\lambda)(z_2 - z_\lambda) = p_2 s_2 e^{\mathrm{i}(\delta+\gamma)},$$

and therefore $p_1 s_1 A_n B_n C_n^2 = p_2 s_2 A_m B_m C_m^2$. This equality can be true iff 1). both sides are 0 or 2). we have only non-zero terms which means, since A_n, A_m, B_n, B_m are all real, $(C_n/C_m)^2$ must be real. Let us look at each of these two scenarios in detail.

1). There are a few ways in which the right hand side of our equality can vanish.

a) $p_1 = 0$; this implies $B_m = 0$. Now, $N/D = 1$ iff the discriminant mentioned above is 0 which can happen if:

- $|A_n| = 0$ which means we are in the case $D = 0$ discussed later;
- $\omega_2 = 0$, we are again in the case $D = 0$,
- $\omega_1 = 0$, $|A_m| = 0$ we are in the case $D = 0$,
- $\omega_2 = 0$, $|B_n| = 0$ we are in the case $D = 0$,
- $|B_n| = 0$; in this case $B_m = B_n = 0$ and according to Lemma 3.10 this can happen for at most a finite number of λ values which will be included in S.

b) $s_1 = 0$; this implies $A_m = 0$ and the analysis will be almost identical to the one in a).

c) $A_n = 0$; this implies $p_2 = 0$ and we are in a similar case to a).

d) $B_n = 0$; this implies $s_2 = 0$ and we are in a similar case to b).

We should also notice $C_n \neq 0$ and $C_m \neq 0$.

2). $\left(\frac{C_n}{C_m}\right)^2 \in \mathbb{R}$. We have two possibilities:

- $\frac{C_n}{C_m} \in \mathbb{R}$ which according to Lemma 3.10 can be true for at most a finite number of λ values which will be included in S,
- $\frac{C_n}{C_m} = r\,\mathrm{i}$, $r \in \mathbb{R}$, which according to Lemma 3.10 can be true for at most a finite number of λ values which will be included in S.

The points where $D \to 0$ have to be analysed separately. There are a few ways in which our denominator can vanish. The first and the last term in the expression for D cannot be zero, it is only the two middle factors that can become 0. The following situations arise:

Scenario 1: $(\omega_2|A_m|^2 + \omega_1|B_n|^2) \nrightarrow 0$ and $(\omega_1|B_m|^2 + \omega_2|A_n|^2) \to 0$. This situation can happen if:

- $\omega_2 \to 0$ and $|B_m|^2 \to 0$, or
- $|A_n|^2 \to 0$ and $\omega_1 \to 0$. Since the analysis of these two cases is almost identical, we will only look at this second one. We need to consider a blow-up:

$$\begin{aligned} |A_n|^2 &= r_2 \sin(\alpha) \\ \omega_1 &= r_2 \cos(\alpha) \end{aligned}$$

with $r_2 > 0$ and $\alpha \in [0, \pi/2]$ functions of z_1, z_2 and λ. With this new blow-up we have

$$N = \Big((|z_1 - z_\lambda||B_m C_m| + |z_2 - z_\lambda|(r_2 \sin(\alpha))^{1/2}|C_n|)^2(\omega_2|A_m|^2 + r_2 \cos(\alpha)|B_n|^2) \\ + (|z_2 - z_\lambda||A_m C_m| + |z_1 - z_\lambda||B_n C_n|)^2(r_2 \cos(\alpha)|B_m|^2 + \omega_2 r_2 \sin(\alpha))\Big) r_2 \sin(\alpha)\omega_2$$

and

$$D = (|C_m|^2 + |C_n|^2)(\omega_2|A_m|^2 + r_2 \cos(\alpha)|B_n|^2)(r_2 \cos(\alpha)|B_m|^2 + \omega_2 r_2 \sin(\alpha)) \\ \cdot (|z_1 - z_\lambda|^2\omega_2 + |z_2 - z_\lambda|^2 r_2 \cos(\alpha)) \,.$$

By going to a subsequence if needed we can assume that $r_{2,i}$, $B_{m,i}$, $C_{m,i}$, $C_{n,i}$, $\omega_{2,i}$, α_i converge to 0, $\overline{B}_m$, $\overline{C}_m$, $\overline{C}_n$, 1, $\overline{\alpha}$ respectively (recall that $B_{m,i}$ is a linear polynomial in $z_{2,i}$). In this situation we find

$$\mu_0 \le \frac{|\overline{B}_m|^2|\overline{C}_m|^2 \cos(\overline{\alpha})}{(|\overline{C}_m|^2 + |\overline{C}_n|^2)(\sin(\overline{\alpha}) + |\overline{B}_m|^2 \cos(\overline{\alpha}))} < 1 \,.$$

- The last case under this scenario is $|A_n|^2 \to 0$ and $|B_m|^2 \to 0$. After a blow-up of the form

$$\begin{aligned} |A_n|^2 &= r_3 \cos(\beta) \\ |B_m|^2 &= r_3 \sin(\beta) \end{aligned}$$

with $r_3 > 0$ and $\beta \in [0, \pi/2]$ functions of z_1, z_2 and λ, we have

$$N = \Big((|z_1 - z_\lambda|(\sin(\beta))^{1/2}|C_m| + |z_2 - z_\lambda|(\cos(\beta))^{1/2}|C_n|)^2(\omega_2|A_m|^2 + \omega_1|B_n|^2) + \\ (|z_2 - z_\lambda||A_m C_m| + |z_1 - z_\lambda||B_n C_n|)^2(\omega_1 \sin(\beta) + \omega_2 \cos(\beta))\Big)\omega_1\omega_2\,,$$

$$D = (|C_m|^2 + |C_n|^2)(\omega_2|A_m|^2 + \omega_1|B_n|^2)(\omega_1 \sin(\beta) + \omega_2 \cos(\beta)) \\ (|z_1 - z_\lambda|^2\omega_2 + |z_2 - z_\lambda|^2\omega_1)\,.$$

Now, the new expression for D would vanish only if $\omega_1 = 0$ and $\cos(\beta) = 0$, or $\sin(\beta) = 0$ and $\omega_2 = 0$ respectively. This means we have the cases $\omega_1 = 0$ and $|A_n|^2 = 0$, or $|B_m|^2 = 0$ and $\omega_2 = 0$ which were already discussed. Otherwise, the expression N/D is well defined and by arguments similar to the ones before strictly less than unity.

Scenario 2: $(\omega_2|A_m|^2 + \omega_1|B_n|^2) \to 0$ and $(\omega_1|B_m|^2 + \omega_2|A_n|^2) \nrightarrow 0$. This can happen if:

- $\omega_2 \to 0$ and $|B_n|^2 \to 0$, or
- $|A_m|^2 \to 0$ and $\omega_1 \to 0$ or
- $|A_m|^2 \to 0$ and $|B_n|^2 \to 0$.

Since this scenario is very much the same as the previous one we will not discuss it any further.

Scenario 3: $(\omega_2|A_m|^2 + \omega_1|B_n|^2) \to 0$ and $(\omega_1|B_m|^2 + \omega_2|A_n|^2) \to 0$. This situation can happen if:

- $\omega_2 \to 0$, $|B_n|^2 \to 0$ and $|B_m|^2 \to 0$, or
- $\omega_1 \to 0$, $|A_n|^2 \to 0$ and $|A_m|^2 \to 0$. Again, due to the symmetry of our expression, it is enough to look at this second case. We need a blow-up of the form

$$\begin{aligned} \omega_1 &= r_3\,\gamma_1 \\ |A_n|^2 &= r_3\,\gamma_2 \\ |A_m|^2 &= r_3\,\gamma_3 \end{aligned}$$

with $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$ and $r_3 > 0$, functions of z_1, z_2 and λ.

$$N = \Big((|z_1 - z_\lambda||B_m C_m| + |z_2 - z_\lambda|(r_3\gamma_2)^{1/2}|C_n|)^2(\omega_2\gamma_3 + \gamma_1|B_n|^2) + \\ (|z_2 - z_\lambda|(r_3\gamma_2)^{1/2}|C_m| + |z_1 - z_\lambda||B_n C_n|)^2(\gamma_1|B_m|^2 + \omega_2\gamma_2)\Big)\gamma_1\omega_2\,,$$

$$D = (|C_m|^2 + |C_n|^2)(\omega_2\gamma_3 + \gamma_1|B_n|^2)(\gamma_1|B_m|^2 + \omega_2\gamma_2) \\ (|z_1 - z_\lambda|^2\omega_2 + |z_2 - z_\lambda|^2(r_3\gamma_2)^{1/2})\,.$$

By going to a subsequence if needed we can assume that $r_{3,i}$, $B_{m,i}$, $C_{m,i}$, $C_{n,i}$, $\omega_{2,i}$, $\gamma_{1,i}$, $\gamma_{2,i}$, $\gamma_{3,i}$ converge to 0, $\overline{B}_m$, $\overline{C}_m$, $\overline{C}_n$, 1, $\overline{\gamma_1}$, $\overline{\gamma_2}$, $\overline{\gamma_3}$ respectively. In this situation we obtain

$$\mu_0 \leq \frac{\Big(|\overline{B}_m\overline{C}_m|^2(\overline{\gamma_3} + \overline{\gamma_1}|\overline{B}_n|^2) + |\overline{B}_n\overline{C}_n|^2(\overline{\gamma_1}|\overline{B}_m|^2 + \overline{\gamma_2})\Big)\overline{\gamma_1}}{(|\overline{C}_m|^2 + |\overline{C}_n|^2)(\overline{\gamma_3} + \overline{\gamma_1}|\overline{B}_n|^2)(\overline{\gamma_1}|\overline{B}_m|^2 + \overline{\gamma_2})} < 1\,,$$

provided the denominator does not vanish. If $\overline{\gamma}_1 = \overline{\gamma}_2 = 0$, $\overline{\gamma}_1 = \overline{\gamma}_3 = 0$, $\overline{\gamma}_3 = |\overline{B}_n|^2 = 0$ or/and $|\overline{B}_m|^2 = \overline{\gamma}_2 = 0$ extra blow-ups are needed, but the limiting value for N/D stays strictly less than 1. As an example, if we consider the extra blow-up given by $\overline{\gamma_3} = r_4\gamma_3$, $\overline{\gamma_1} = r_4\gamma_1$, $\gamma_1^2 + \gamma_3^2 = 1$ and $r_4 > 0$ we have

$$\mu_0 \leq \frac{|\overline{B}_n\overline{C}_n|^2\gamma_1}{(|\overline{C}_m|^2 + |\overline{C}_n|^2)(\gamma_3 + \gamma_1|\overline{B}_n|^2)} < 1\,.$$

- $|A_m|^2 \to 0$, $|B_n|^2 \to 0$, $|B_m|^2 \to 0$ and $|A_m|^2 \to 0$. After a needed blow-up the expressions will look similar to the ones in (3.11) and (3.12), but in the blown-up variables.

Case II: Both z_1 and z_2 are i∞. Let $(z_{1,j}, z_{2,j}, \lambda_j) \in \overline{\mathbb{H}^2} \times \overline{R}$ be a sequence that realizes the lim sup in the definition of μ_0. We sometimes suppress the index j for simplicity. We consider the change of variables, $u_1 = -\frac{1}{z_1}$, $u_2 = -\frac{1}{z_2}$ and $u_\lambda = -\frac{1}{z_\lambda}$; now, both u_1 and u_2 approach 0. With these new variables, N and D from (3.8) are given by

$$N = \Big((|u_\lambda - u_1||u_2u_\lambda B_m C_m| + |u_\lambda - u_2||u_1u_\lambda A_n C_n|)^2(\mathrm{Im}(u_2)|u_1A_m|^2 + \mathrm{Im}(u_1)|u_2B_n|^2) \\ + (|u_\lambda - u_2||u_1u_\lambda A_m C_m| + |u_\lambda - u_1||u_2u_\lambda B_n C_n|)^2 \\ \cdot (\mathrm{Im}(u_1)|u_2B_m|^2 + \mathrm{Im}(u_2)|u_1A_n|^2)\Big)\mathrm{Im}(u_1)\mathrm{Im}(u_2),$$

$$D = \left(|C_m|^2 + |C_n|^2\right)|u_\lambda|^2 \left(\mathrm{Im}(u_2)|u_1 A_m|^2 + \mathrm{Im}(u_1)|u_2 B_n|^2\right)$$
$$\cdot \left(\mathrm{Im}(u_1)|u_2 B_m|^2 + \mathrm{Im}(u_2)|u_1 A_n|^2\right)\left(|u_\lambda - u_1|^2 \mathrm{Im}(u_2) + |u_\lambda - u_2|^2 \mathrm{Im}(u_1)\right).$$

Since both sequences $u_{1,j}$ and $u_{2,j}$ are approaching 0, we can write $u_{1,j} = r_j \cos(\gamma_j)e^{ix_j}$ and $u_{2,j} = r_j \sin(\gamma_j)e^{iy_j}$, with $\gamma_j, x_j, y_j \in [0, \pi/2]$. By going to a subsequence if needed and recalling that $A_{n,j}$, $A_{m,j}$, $B_{n,j}$ and $B_{m,j}$ are linear polynomial is z_1 and z_2, we can assume that r_j, $u_{1,j}A_{n,j}$, $u_{1,j}A_{m,j}$, $u_{2,j}B_{n,j}$, $u_{2,j}B_{m,j}$, $u_{\lambda,j}C_{n,j}$, $u_{\lambda,j}C_{m,j}$, γ_j, x_j, y_j converge to 0, $\overline{A}_n$, $\overline{A}_m$, $\overline{B}_n$, $\overline{B}_m$, $\overline{C}_n$, $\overline{C}_m$, $\overline{\gamma}$, $\overline{x}$, $\overline{y}$ respectively. After cancelling the common factor of r_j and $|u_{\lambda,j}|$ in the above expressions for N and D and taking the limit we get

$$N = \Big((|\overline{B}_m\overline{C}_m| + |\overline{A}_n\overline{C}_n|)^2(\sin(\gamma)\sin(y)|\overline{A}_m|^2 + \cos(\gamma)\sin(x)|\overline{B}_n|^2)$$
$$+ (|\overline{A}_m\overline{C}_m| + |\overline{B}_n\overline{C}_n|)^2(\sin(\gamma)\sin(y)|\overline{A}_n|^2 + \cos(\gamma)\sin(x)|\overline{B}_m|^2)\Big)$$
$$\cdot \sin(x)\sin(y)\sin(\gamma)\cos(\gamma),$$

$$D = (|\overline{C}_m|^2 + |\overline{C}_n|^2)(\sin(\gamma)\sin(y)|\overline{A}_m|^2 + \cos(\gamma)\sin(x)|\overline{B}_n|^2)$$
$$(\sin(\gamma)\sin(y)|\overline{A}_n|^2 + \cos(\gamma\sin(x)|\overline{B}_m|^2)(\sin(x)\cos(\gamma) + \sin(y)\sin(\gamma)).$$

If we compare this with *Case I* and consider $|z_1 - z_\lambda| = |z_2 - z_\lambda|$, $y_1 = \cos(\gamma)\sin(x)$ and $y_2 = \sin(\gamma)\sin(y)$ we can see that we are in a similar situation to the one in *Case I*.

Case III: $z_1 \in \mathbb{R}$ and $z_2 = i\infty$, respectively $z_2 \in \mathbb{R}$ and $z_1 = i\infty$. We consider again a sequence that realizes the lim sup in the definition of μ_0 and we use the same change of variables for z_2, as before. Since $z_1 \to \mathbb{R}$ and $u_2 \to 0$ we can write $u_2 = re^{iy_2}$ with $r > 0$ and $\mathrm{Im}(z_1) = y_1$. After we cancel in both N and D a factor of r^6 and $|u_\lambda|^2$ we have

$$N = \Big((|z_1 - z_\lambda||u_2 B_m u_\lambda C_m| + |re^{iy_2} - u_\lambda||A_n C_n|)^2(r\sin(y_2)|A_m|^2 + y_1|u_2 B_n|^2) \tag{3.13}$$
$$+ (|re^{iy_2} - u_\lambda||A_m u_\lambda C_m| + |z_1 - z_\lambda||u_2 B_n u_\lambda C_n|)^2(y_1|u_2 B_m|^2 + r\sin(y_2)|A_n|^2)\Big) r y_1 \sin(y_2),$$

$$D = (|u_\lambda C_m|^2 + |u_\lambda C_n|^2)\left(r\sin(y_2)|A_m|^2 + y_1|u_2 B_n|^2\right)\left(y_1|u_2 B_m|^2 + r\sin(y_2)|A_n|^2\right)$$
$$\left(|z_1 - z_\lambda|^2 r\sin(y_2)|u_\lambda|^2 + |re^{iy_2} - u_\lambda|^2 y_1\right). \tag{3.14}$$

If we compare it with *Case I* and consider $|z_1 - z_\lambda| = |(z_1 - z_\lambda)u_\lambda|$, $|z_2 - z_\lambda| = |re^{iy_2} - u_\lambda|$, $y_1 = y_1$, $y_2 = r\sin(y_2)$, $|A_n| = |A_n|$, $|A_m| = |A_m|$, $|B_n| = |u_2 B_n|$ and $|B_m| = |u_2 B_m|$ we can see that the blow-ups needed are similar to the ones in *Case I* and we can conclude $N/D < 1$.

Case IV: $z_1 \in \mathbb{H}$ and $z_2 \in \mathbb{R}$, respectively $z_2 \in \mathbb{H}$ and $z_1 \in \mathbb{R}$. As before, we take a sequence that realizes the lim sup in the definition of μ_0. If we look at the expressions for D given by (3.10) we can see that we can have the following three undetermined cases: $y_2 \to 0$ and $|B_n|^2 \to 0$, similarly $y_2 \to 0$ and $|B_m|^2 \to 0$, or $y_2 \to 0$, $|B_n|^2 \to 0$ and $|B_m|^2 \to 0$. The analysis of these blow-up cases can be done in a similar manner with the one from *Case I* and we can conclude that N/D is strictly less than 1.

Case V: $z_1 \in \mathbb{H}$ and $z_2 = \mathrm{i}\infty$, respectively $z_2 \in \mathbb{H}$ and $z_1 = \mathrm{i}\infty$. We take a sequence that realizes the lim sup and we consider the same change of variables as in *Case III*. With the same notations $u_2 = re^{iy_2}$ with $r > 0$ and $\mathrm{Im}(u_1) = y_1$ we obtain the same expressions for N and D as in (3.13) and (3.14). With similar blow-ups with the ones in *Case III* we can conclude that also in this last case the limiting value for N/D is strictly less than 1. □

Proof of Lemma 3.4. To prove the lemma it is enough to show that

$$\mu_p(Z, Q, \lambda) \quad < \quad 1$$

for (Z, Q, λ) in the compact set $\partial_\infty(\overline{\mathbb{H}}^2) \times \{0\}^M \times E$, since this implies that for some $\epsilon > 0$, the upper semi-continuous function $\mu_p(Z, Q, \lambda)$ is bounded by $1 - 2\epsilon$ on the set, and by $1 - \epsilon$ in some neighborhood.

Let us rewrite μ_p in terms of μ_0.

$$\begin{aligned}\mu_p(Z, Q, \lambda) &= \frac{\mathrm{w}^{1+p}(\phi(z_1, z_2, q_1 \ldots q_M, \lambda)) + \mathrm{w}^{1+p}(\phi(z_2, z_1, q_1 \ldots q_M, \lambda))}{\mathrm{w}^{1+p}(z_1) + \mathrm{w}^{1+p}(z_2)} \\ &\leq \left(\frac{\mathrm{w}(\phi(z_1, z_2, q_1 \ldots q_M, \lambda)) + \mathrm{w}(\phi(z_2, z_1, q_1 \ldots q_M, \lambda))}{\mathrm{w}(z_1) + \mathrm{w}(z_2)}\right)^{1+p} \cdot \\ &\qquad \cdot \frac{1}{\nu_1^{1+p} + \nu_2^{1+p}},\end{aligned}$$

where

$$\nu_i = \frac{\mathrm{w}(z_i)}{\mathrm{w}(z_1) + \mathrm{w}(z_2)}, \ \text{ for } \ \mathrm{i} = 1,\ 2\,.$$

Since we are concentrating on the boundary of $\overline{\mathbb{H}}^2$, we need the following blow-up

$$\chi(z_1) = \frac{1}{\mathrm{w}(z_1)} = R_1\Omega_1$$
$$\chi(z_2) = \frac{1}{\mathrm{w}(z_2)} = R_1\Omega_2$$

where R_1, Ω_1 and Ω_2 are defined as functions of z_1 and z_2 with the property $\Omega_1^2 + \Omega_2^2 = 1$. Using the result in Proposition 3.7 we have

$$\mu_p|_{\partial_\infty(\overline{\mathbb{H}})^2\times\{0\}^M\times E} < \left(\mu_0|_{\partial_\infty(\overline{\mathbb{H}}^2)\times\{0\}^M\times E}\right)^{1+p} \frac{(\Omega_1+\Omega_2)^{1+p}}{\Omega_1^{1+p}+\Omega_2^{1+p}} \leq (1-\epsilon)2^p < 1\,,$$

for sufficiently small p. □

Proof of Lemma 3.5. Each term in the sum appearing in μ_p can be estimated

$$\begin{aligned}\frac{\mathrm{w}^{1+p}(\phi(z_1,z_2,q_1\ldots q_M,\lambda))}{\mathrm{w}^{1+p}(z_1)+\mathrm{w}^{1+p}(z_2)} &= \frac{(\mathrm{w}(z_1)+\mathrm{w}(z_2))^{1+p}}{\mathrm{w}^{1+p}(z_1)+\mathrm{w}^{1+p}(z_2)}\left(\frac{\mathrm{w}(\phi(z_1,z_2,q_1\ldots q_M,\lambda))}{\mathrm{w}(z_1)+\mathrm{w}(z_2)}\right)^{1+p}\\ &\leq 2^p\left(\frac{\mathrm{w}(\phi(z_1,z_2,q_1\ldots q_M,\lambda))}{\mathrm{w}(z_1)+\mathrm{w}(z_2)}\right)^{1+p}.\end{aligned}$$

Now it is enough to prove that $\frac{\mathrm{w}(\phi(z_1,z_2,q_1\ldots q_M,\lambda))}{\mathrm{w}(z_1)+\mathrm{w}(z_2)} \leq C\prod_{i=1}^{M}(1+|q_i|^2)$, since this bounds each term in μ_p by the desired quantity. With the notations introduced at the beginning of this section, $\phi_n(z_1,q_1\ldots q_n,\lambda) = -\frac{A_{n-1}}{A_n}$ and applying Cauchy-Schwarz inequality twice we get

$$\begin{aligned}&\frac{\mathrm{w}(\phi(z_1,z_2,q_1\ldots q_M,\lambda))}{\mathrm{w}(z_1)+\mathrm{w}(z_2)}\\ &= \frac{|1+z_\lambda\phi_n(z_1,q_1\ldots q_n,\lambda)+z_\lambda\phi_m(z_2,q_{n+1}\ldots q_{n+m},\lambda)+z_\lambda(\lambda-q_M)|^2}{\mathrm{Im}(\phi_n(z_1,q_1\ldots q_n,\lambda))+\mathrm{Im}(\phi_m(z_2,q_{n+1}\ldots q_{n+m},\lambda))+\mathrm{Im}(\lambda)}\cdot\frac{1}{\sum_{i=1}^{2}\frac{|z_i-z_\lambda|^2}{\mathrm{Im}(z_i)}}\\ &\leq \frac{|A_nB_m - z_\lambda A_{n-1}B_m - z_\lambda B_{m-1}A_n + A_nB_mz_\lambda(\lambda-q_M)|^2}{\mathrm{Im}(z_1)|B_m|^2+\mathrm{Im}(z_2)|A_n|^2}\cdot\frac{1}{\sum_{i=1}^{2}\frac{|z_i-z_\lambda|^2}{\mathrm{Im}(z_i)}}\\ &\leq (1+|z_\lambda|^2)\Big(\frac{|A_nB_m|^2}{\mathrm{Im}(z_1)|B_m|^2+\mathrm{Im}(z_2)|A_n|^2} + (|A_nB_m|^2+|A_{n-1}B_m+A_nB_{m-1}|^2)\cdot\\ &\qquad\qquad \frac{(1+|q_M-\lambda|^2)}{\mathrm{Im}(z_1)|B_m|^2+\mathrm{Im}(z_2)|A_n|^2}\Big)\cdot\frac{1}{\sum_{i=1}^{2}\frac{|z_i-z_\lambda|^2}{\mathrm{Im}(z_i)}}\\ &\leq (1+|z_\lambda|^2)\left(\frac{|A_n|^2(2+|q_M-\lambda|^2)}{\mathrm{Im}(z_1)} + 2(1+|q_M-\lambda|^2)\left(\frac{|A_{n-1}|^2}{\mathrm{Im}(z_1)}+\frac{|B_{m-1}|^2}{\mathrm{Im}(z_2)}\right)\right)\cdot\frac{1}{\sum_{i=1}^{2}\frac{|z_i-z_\lambda|^2}{\mathrm{Im}(z_i)}}.\end{aligned}$$

Since ϕ_n is a fractional linear transformation with coefficients given by the product matrix $\prod_{i=1}^{n} \begin{pmatrix} 0 & -1 \\ 1 & 1+\lambda-q_i \end{pmatrix}$, A_n, the denominator of ϕ_n, is a linear polynomial in z whose coefficients can be bounded above. We get $|A_n| \le n|1+z_1| \prod_{\substack{|1+\lambda-q_i|\ge 1 \\ i=1}}^{n} |1+\lambda-q_i|$ which implies, $|A_n|^2 \le C\left(1+|z_1|^2\right)\prod_{i=1}^{n}\left(1+|q_i|^2\right)$.

Going back to our inequality we have

$$\begin{aligned} \frac{\mathrm{w}(\phi(z_1,z_2,q_1\ldots q_M,\lambda))}{\mathrm{w}(z_1)+\mathrm{w}(z_2)} \le C\Big(& C_1(1+|q_M|^2)\prod_{i=1}^{n}(1+|q_i|^2)\frac{1+|z_1|^2}{\mathrm{Im}(z_1)} \\ &+ C_2(1+|q_M|^2)\prod_{i=1}^{n-1}(1+|q_i|^2)\frac{1+|z_1|^2}{\mathrm{Im}(z_1)} \\ &+ C_3(1+|q_M|^2)\prod_{k=n+1}^{n+m-1}(1+|q_i|^2)\frac{1+|z_2|^2}{\mathrm{Im}(z_2)}\Big)\cdot\frac{1}{\sum_{i=1}^{2}\frac{|z_i-z_\lambda|^2}{\mathrm{Im}(z_i)}}. \end{aligned}$$

Choose the compact set $\mathcal{K}$ such that $\sum_{i=1}^{2}|z_i-z_\lambda|^2/\mathrm{Im}(z_i) \ge C > 0$ for some constant C and $(z_1,z_2)\in\mathcal{K}^c$. Then we can estimate each term depending on whether z_i is close to z_λ. If z_j, $j=1,2$, is sufficiently close, then $\mathrm{Im}(z_j)$ is bounded below and $|z_j|$ is bounded above by a constant. Thus

$$\mathrm{Im}(z_j)\sum_{i=1}^{2}|z_i-z_\lambda|^2/\mathrm{Im}(z_i) \ge \mathrm{Im}(z_j)C \ge C' > 0$$

and $1+|z_j|^2 \le C$, so we are done. If z_j, $j=1,2$, is far from z_λ,

$$\mathrm{Im}(z_j)\sum_{i=1}^{2}|z_i-z_\lambda|^2/\mathrm{Im}(z_i) \ge |z_j-z_\lambda|^2 \ge C(1+|z_j|^2)$$

so $1+|z_j|^2/\left(\mathrm{Im}(z_j)\sum_{i=1}^{2}|z_i-z_\lambda|^2/\mathrm{Im}(z_i)\right) \le C$ again. □

3.4 Additional Results

This section contains two theorems on absolutely continuous spectrum. The first one gives a sufficient condition for a measure to be absolutely continuous with respect to the Lebesgue measure on an interval and the latter gives a sufficient condition for a random Schrödinger operator to exhibit purely absolutely continuous spectrum on some interval.

3.4.1 A Criterion for Absolutely Continuous Spectrum

Let μ be a finite measure on $\mathbb{R}$; its Stieltjes (or Borel) transform F is given by

$$F(z) = \int \frac{d\mu(t)}{t - z}$$

for $z = x + iy$ with $y > 0$. The following criterion has been proven in [8] for lim sup and we reproduce it here for lim inf.

Proposition 3.8. *Let* (a, b) *be a finite interval and let* $p > 0$*. Suppose*

$$\liminf_{y\to 0} \int_a^b |F(x + \mathrm{i}y)|^{1+p} dx < \infty\,.$$

Then μ *is absolutely continuous with respect to the Lebesgue measure on* (a, b)*.*

Proof. Since $\liminf_{y\to 0} \int_a^b |F(x + \mathrm{i}y)|^{1+p} dx < \infty$, there exists a sequence $y_n \to 0$ such that $\sup_n \int_a^b |F(x + \mathrm{i}y_n)|^{1+p} dx < C$, where C is some constant. Define $d\mu_{y_n}(x) = \pi^{-1}\mathrm{Im}(F(x + \mathrm{i}y_n))dx$. Then by [7], $d\mu_{y_n} \to d\mu$ weakly, as $n \to \infty$. That is, for f a continuous function of compact support we have $\lim_{n\to\infty} \int f(x)d\mu_{y_n}(x) = \int f(x)d\mu(x)$. Let f be a continuous function supported on (a, b), then

$$\begin{aligned}
\left|\int_a^b f(x)d\mu(x)\right| &= \lim_{n\to\infty} \left|\int_a^b f(x)d\mu_{y_n}(x)\right| \\
&= \lim_{n\to\infty} \pi^{-1} \left|\int_a^b f(x)\mathrm{Im}(F(x + \mathrm{i}y_n))dx\right| \\
&\le \lim_{n\to\infty} \left(\|f\|_{1+1/p}\|\mathrm{Im}(F(x + \mathrm{i}y_n))\|_{1+p}\right) \\
&\le C\|f\|_{1+1/p}\,.
\end{aligned}$$

This implies that $d\mu(x) = g(x)dx$ for some $g \in L^{1+p}$. □

3.4.2 Bounds on the Green Function at an Arbitrary Site

The following lemma proves that assuming we have a bound for the forward Green functions $G^x(\lambda)$ for all $x \in \mathbb{T}_p$, we can obtain a bound for all the diagonal matrix elements $G_x(\lambda)$, $x \in \mathbb{T}$, of the Green function.

Lemma 3.9. *Let F be the open interior of the absolutely continuous spectrum of Δ. Suppose that for any $x \in \mathbb{T}_p$*

$$\sup_{\lambda \in R(E,\epsilon)} \mathbb{E}\left(\mathrm{w}^{1+p}\left(\left\langle \delta_x, (H^x - \lambda)^{-1} \delta_x \right\rangle\right)\right) < \infty \ ,$$

for some closed subinterval $E \subset F$, $\epsilon > 0$ and $0 < p < 1$. Then, for every $x \in \mathbb{T}$, we also have

$$\sup_{\lambda \in R(E,\epsilon)} \mathbb{E}\left(\mathrm{w}^{1+p}\left(\left\langle \delta_x, (H - \lambda)^{-1} \delta_x \right\rangle\right)\right) < \infty \ .$$

Proof. Suppose we pick an arbitrary node x_0 in $\mathbb{T}$ and we consider its corresponding diagonal matrix element of the Green function for the whole tree $\mathbb{T}$, $G_{x_0}(\lambda) = \left\langle \delta_{x_0}, (H - \lambda)^{-1} \delta_{x_0} \right\rangle$. We rearrange the nodes, if needed, such that x_0 becomes the origin of the tree. For this origin, we have $G_{x_0}(\lambda) = G^{x_0}(\lambda)$. Looking at the vertices in the future of x_0, we can see that after a finite number of steps, on each branch,the future tree will be a copy of the original tree. Let us denote by x_i the nodes where such a copy starts. An example of such a rearrangement is illustrated in the picture below.
We know from the hypothesis that

$$\sup_{\lambda \in R(E,\epsilon)} \mathbb{E}\left(\mathrm{w}^{1+p}\left(\left\langle \delta_{x_i}, (H^{x_i} - \lambda)^{-1} \delta_{x_i} \right\rangle\right)\right) < \infty \ .$$

Starting with these nodes and using the recurrence formula for the forward Green function we can work our way back to the origin, and show that the inequality holds at each intermediate node between an x_i and x_0.

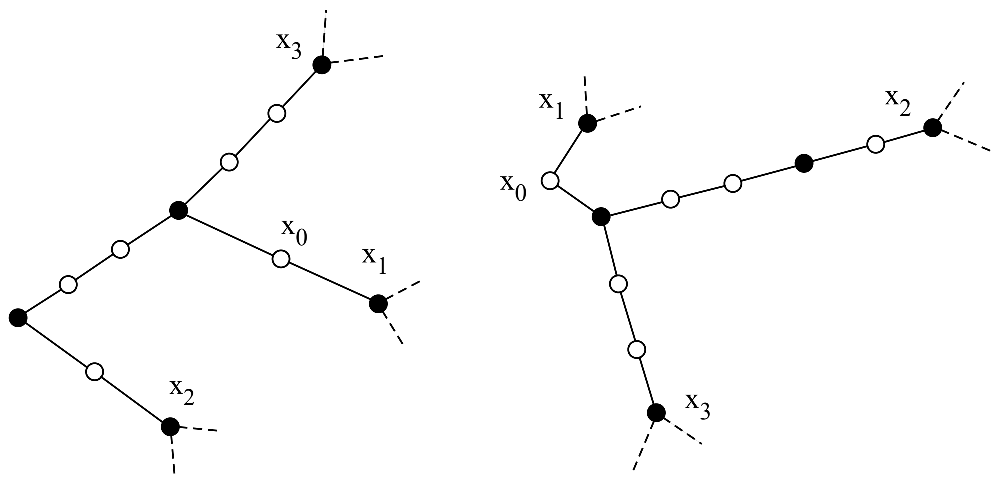

Figure 3.3: Rearrangement of a tree.

Let y be such a node, forward of x_0 and before x_i. Let ρ_j be the probability distribution of $G^{y_j}(\lambda)$, where y_j is a neighbor of y in its forward direction. We assume inductively that

$$\mathbb{E}\left(\mathrm{w}^{1+p}\left(\left\langle \delta_{y_j}, (H^{y_j}-\lambda)^{-1}\delta_{y_j}\right\rangle\right)\right) = \int_{\mathbb{H}} \mathrm{w}^{1+p}(z_j)\, d\rho_j(z_j) < \infty\,.$$

The functions that define the recurrence formula for the forward Green functions are fractional linear transformations and depend on the connectivity number of the node where the forward Green function is computed.

1. Assume $y \in \mathbb{T}_a \cup \{o\}$ with ρ' the probability distribution of $G^y(\lambda)$ and $\mathcal{K}$ a compact set in $\mathbb{H}$ such that z_λ is in the interior of $\mathcal{K}$:

$$\begin{aligned}\mathbb{E}\left(\mathrm{w}^{1+p}\left(G^y(\lambda)\right)\right) &= \int_{\mathbb{H}} \mathrm{w}^{1+p}(z)\, d\rho'(z) = \int_{\mathbb{H}\times\mathbb{R}} \mathrm{w}^{1+p}\left(\frac{-1}{z_1+\lambda-q+1}\right) d\rho_1(z_1)\, d\nu(q)\\ &= \frac{1}{2}\int_{\mathcal{K}^c}\left(\int_{\mathbb{R}} \frac{\mathrm{w}^{1+p}\left(-1/(z_1+\lambda-q+1)\right)}{\mathrm{w}^{1+p}(z_1)}\, d\nu(q)\right)\times \mathrm{w}^{1+p}(z_1)\, d\rho_1(z_1) + C\,.\end{aligned}$$

The quantity $\mu = \displaystyle\int_{\mathbb{R}} \frac{\mathrm{w}^{1+p}\left(-1/(z_1+\lambda-q+1)\right)}{\mathrm{w}^{1+p}(z_1)}\, d\nu(q)$ does not need to be less than 1, but only bounded outside the compact set $\mathcal{K}$. Using the inequalities from the proof of Lemma 3.5,

$$\mu \le \int_{\mathbb{R}}\left(\frac{\mathrm{Im}(z_1)}{\mathrm{Im}(z_1)+\mathrm{Im}(z_\lambda)}\,\frac{\left(1+|z_\lambda|^2\right)\left(1+C\left(1+|z_1|^2\right)\left(1+|kq|^2\right)\right)}{|z_1-z_\lambda|^2}\right)^{1+p} d\nu(q)$$

which is bounded on $\mathcal{K}^c$. We can therefore conclude,

$$\mathbb{E}\left(\mathrm{w}^{1+p}\left(G^{y}(\lambda)\right)\right) \leq C' \int_{\mathbb{H}} \mathrm{w}^{1+p}(z_1)\, d\rho_1(z_1) + C = C'\, \mathbb{E}\left(\mathrm{w}^{1+p}\left(G^{y_j}(\lambda)\right)\right) + C\,.$$

Hence $\sup_{\lambda\in R(E,\epsilon)} \mathbb{E}\left(\mathrm{w}^{1+p}\left(G^{y}(\lambda)\right)\right) < \infty$.

2. Assume $y \in \mathbb{T}_p$ with ρ'' the probability distribution of $G^y(\lambda)$ and $\mathcal{K}$ a compact set in $\mathbb{H}^2$ such that (z_λ, z_λ) is in the interior of $\mathcal{K}$:

$$\begin{aligned}
&\mathbb{E}\left(\mathrm{w}^{1+p}\left(G^{y}(\lambda)\right)\right) = \int_{\mathbb{H}} \mathrm{w}^{1+p}(z)\, d\rho''(z) \\
&= \int_{\mathbb{H}^2\times\mathbb{R}} \mathrm{w}^{1+p}\left(\frac{-1}{z_1+z_2+\lambda-kq}\right) d\rho_1(z_1)\, d\rho_2(z_2)\, d\nu(q) \\
&= \frac{1}{2}\int_{\mathcal{K}^c}\left(\int_{\mathbb{R}} \frac{2\,\mathrm{w}^{1+p}\left(-1/(z_1+z_2+\lambda-q)\right)}{\mathrm{w}^{1+p}(z_1)+\mathrm{w}^{1+p}(z_2)}\, d\nu(q)\right) \times (\mathrm{w}^{1+p}(z_1)+ \\
&\qquad\qquad + \mathrm{w}^{1+p}(z_2))\, d\rho_1(z_1)\, d\rho_2(z_2) + C \\
&\leq C''\left(\int_{\mathbb{H}} \mathrm{w}^{1+p}(z_1)\, d\rho_1(z_1) + \int_{\mathbb{H}} \mathrm{w}^{1+p}(z_2)\, d\rho_2(z_2)\right) + C \\
&\leq C''\left(\mathbb{E}\left(\mathrm{w}^{1+p}\left(G^{y_1}(\lambda)\right)\right) + \mathbb{E}\left(\mathrm{w}^{1+p}\left(G^{y_2}(\lambda)\right)\right)\right) + C\,.
\end{aligned}$$

Hence $\sup_{\lambda\in R(E,\epsilon)} \mathbb{E}\left(\mathrm{w}^{1+p}\left(G^{y}(\lambda)\right)\right) < \infty$. The q integral, outside the compact set $\mathcal{K}$, is bounded by arguments similar to the ones in the proof of Lemma 3.

When we reach the origin x_0, we know the inequality holds at all other nodes. The recurrence relation for the origin is slightly different than everywhere else, due to our definition of the Laplacian. The argument that proves this final step is nevertheless almost identical to the one above. □

3.5 On a recursion relation

At the beginning of Section 3 we introduced quantities A_i, B_i and C_i. For $q \equiv 0$, they all are recursions of the following form

$$\begin{cases} R_0(z) = 1 \\ R_1(z) = 1 + \lambda + z \\ R_{n+1}(z) = (1+\lambda)R_n(z) - R_{n-1}(z) \end{cases}$$

or, in a matrix form

$$\begin{bmatrix} R_{n+1}(z) \\ R_n(z) \end{bmatrix} = \begin{bmatrix} 1+\lambda & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} R_n(z) \\ R_{n-1}(z) \end{bmatrix} = \begin{bmatrix} 1+\lambda & -1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} R_1(z) \\ 1 \end{bmatrix}.$$

We can observe that $R_n(z)$ has the following general form, depending on λ, $R_n(z)$ = (Pol. of degree $(n-1)$ in λ) $\cdot z+$ (Pol. of degree n in λ). For $\lambda \neq -3, 1$ we have the following diagonal form

$$\begin{bmatrix} R_{n+1}(z) \\ R_n(z) \end{bmatrix} = \frac{1}{\det} \begin{bmatrix} 1 & 1 \\ \mu_2 & \mu_1 \end{bmatrix} \begin{bmatrix} \mu_1^n & 0 \\ 0 & \mu_2^n \end{bmatrix} \begin{bmatrix} \mu_1 & -1 \\ -\mu_2 & 1 \end{bmatrix} \begin{bmatrix} R_1(z) \\ 1 \end{bmatrix}$$

where $\mu_{1,2} = \frac{1+\lambda}{2} \pm \sqrt{\left(\frac{1+\lambda}{2}\right)^2 - 1}$ and $\det = 2\sqrt{\left(\frac{1+\lambda}{2}\right)^2 - 1}$.

The general formula for R_n is $R_n(z) = \frac{1}{\det}\left(\left(\mu_1^n - \mu_2^n\right)(1+\lambda+z) - \left(\mu_1^{n-1} - \mu_2^{n-1}\right)\right)$. Also, for $n > m$ we have

$$\begin{aligned} R_n(z) &= \frac{\mu_1^n - \mu_2^n}{\mu_1^m - \mu_2^m} R_m(z) + \frac{1}{\det}\left(\left(\mu_1^{m-1} - \mu_2^{m-1}\right)\frac{\mu_1^n - \mu_2^n}{\mu_1^m - \mu_2^m} - \left(\mu_1^{n-1} - \mu_2^{n-1}\right)\right), \\ &= \frac{\mu_1^n - \mu_2^n}{\mu_1^m - \mu_2^m} R_m(z) + \frac{1}{\det}\frac{(-1)^m (\mu_1 - \mu_2)\left(\mu_1^{n-m} - \mu_2^{n-m}\right)}{\mu_1^m - \mu_2^m}. \end{aligned}$$

Lemma 3.10. *The set of λ values for which either of the following identities is true is finite:*

(i) $R_n(z) = R_m(z) = 0$*;*

(ii) $\frac{R_n(z_\lambda)}{R_m(z_\lambda)} \in \mathbb{R}$*, where* $z_\lambda \in \mathbb{H}$ *is the fixed point introduced in the* Outline of the Proof*;*

(iii) $\frac{R_n(z_\lambda)}{R_m(z_\lambda)} = -r\,\mathrm{i}$*, where* $r \in \mathbb{R}$*.*

Proof. (i) Let us assume $R_m(z) = 0$. Since we know

$$R_n(z) = \frac{\mu_1^n - \mu_2^n}{\mu_1^m - \mu_2^m} R_m(z) + \frac{1}{\det}\frac{(-1)^m (\mu_1 - \mu_2)\left(\mu_1^{n-m} - \mu_2^{n-m}\right)}{\mu_1^m - \mu_2^m}, \tag{3.15}$$

$R_n(z) = 0$ iff $\frac{1}{\det}\frac{(-1)^m (\mu_1 - \mu_2)\left(\mu_1^{n-m} - \mu_2^{n-m}\right)}{\mu_1^m - \mu_2^m} = 0$. This identity is equivalent to

$$\mu_1^{n-m-1} + \mu_1^{n-m-2}\mu_2 + \ldots + \mu_1\mu_2^{n-m-2} + \mu_2^{n-m-1} = \mu_1^{m-1} + \mu_1^{m-2}\mu_2 + \ldots + \mu_1\mu_2^{m-2} + \mu_2^{m-1},$$

which is a polynomial of degree $\max\{n-m-1, m-1\}$ in λ.

(ii) Using (3.15) we can write

$$\frac{R_n(z_\lambda)}{R_m(z_\lambda)} = \frac{\mu_1^n - \mu_2^n}{\mu_1^m - \mu_2^m} + \frac{1}{\det}\frac{(-1)^m(\mu_1-\mu_2)\left(\mu_1^{n-m} - \mu_2^{n-m}\right)}{\left(\mu_1^m - \mu_2^m\right)R_m(z_\lambda)}. \tag{3.16}$$

The first term on the right hand side is a real number and since $R_m(z_\lambda) \notin \mathbb{R}$, the only way to obtain the desired conclusion is iff $\mu_1^{n-m} - \mu_2^{n-m} = 0$ which is equivalent to finding the roots of a polynomial of degree $n-m-1$ in λ.

(iii) Relation (3.16) becomes

$$\frac{R_n(z_\lambda)}{R_m(z_\lambda)} = \frac{\mu_1^n - \mu_2^n}{\mu_1^m - \mu_2^m} + \frac{1}{\det}\frac{(-1)^m(\mu_1-\mu_2)\left(\mu_1^{n-m} - \mu_2^{n-m}\right)\overline{R_m(z_\lambda)}}{\left(\mu_1^m - \mu_2^m\right)|R_m(z_\lambda)|^2}.$$

For condition (iii) to be true we need

$$\frac{\mu_1^n - \mu_2^n}{\mu_1^m - \mu_2^m} + \frac{1}{\det}\frac{(-1)^m(\mu_1-\mu_2)\left(\mu_1^{n-m} - \mu_2^{n-m}\right)\mathrm{Re}(R_m(z_\lambda))}{\left(\mu_1^m - \mu_2^m\right)|R_m(z_\lambda)|^2} = 0,$$

which is equivalent to

$$(\mu_1^n - \mu_2^n)|R_m(z_\lambda)|^2 + (-1)^m\left(\mu_1^{n-m} - \mu_2^{n-m}\right)\mathrm{Re}(R_m(z_\lambda)) = 0,$$

The condition resumes to finding the zeros of a polynomial in λ. □

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Chapter 4

Conclusions

Extended states, existence of absolutely continuous spectrum, was proved for Random Schrödinger operators on tree graphs. The first result was obtained by A. Klein, [6]; he proved that for weak disorder, on the Bethe lattice, there exists absolutely continuous spectrum for almost all potentials. More recently, Aizenman, Sims and Warzel proved similar results for the Bethe lattice and quantum graphs using a different method, [1] and [2]. Their method establishes the persistence of absolutely continuous spectrum under weak disorder (and also, for the Bethe lattice, in the presence of a periodic background potential). During the same time, Froese, Hasler and Spitzer introduced a geometric method for proving the existence of absolutely continuous spectrum on graphs, [3]. They used this approach to prove the existence of absolutely continuous spectrum for the Bethe lattice of degree 3 and for a tree with strong transverse correlations and large weighted loops, [4] and [5].

The results presented in this work prove the existence of purely absolutely continuous spectrum for the Anderson model on different types of trees. The main ideas behind our method are based on hyperbolic geometry and were first introduced by R. Froese, D. Hasler and W. Spitzer in [3].

Chapter 2 proves extended states for the Bethe lattice of any degree K. We generalized the result in [4] using a simplified version of the method. In [4] the quantity Z, a sum of forward Green functions is estimated, whereas we look directly at $G^x(\lambda)$, the forward Green function at a vertex x. This, plus some manipulation of the expressions makes the proof of the desired estimates shorter and easy to generalize. The main advantage of out method in comparison to [4] is that we do not need a full analysis of the quantity μ_2, defined in the introduction. Instead, we only need to look at the boundary points needed in the following estimates.

Chapter 3 deals with a more general tree where some of the symmetry is broken. The lack of symmetry changes the analysis, making it possible to eliminate one of the steps in the proof for the Cayley tree. The tree analysed, $\mathbb{T}$, has its principal nodes of degree 3 but the proof ca be extended to any degree K.

Since the main goal of all this analysis on trees is to ultimately be able to give an answer to the open problem of extended states for the Anderson Model on $\mathbb{Z}^d$, $d > 2$, the first step is to extend the method to graphs with loops. Our current project is extending the method to stacked trees. A stacked tree is defined as follows, we take a binary tree and an identical copy of it. We connect by an edge each vertex from the tree with its correspondent vertex on the copy and thus obtaining the stacked tree.

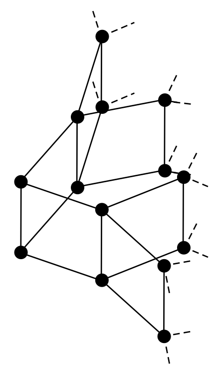

Figure 4.1: The stacked tree

Even though the loops in this graph are of a simpler nature than the ones encountered on $\mathbb{Z}^d$, it is still a very important step towards solving the main open problem.

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