MODELS FOR ELECTRICITY PRICES

by

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Abstract

The deregulation of electricity markets has led to higher uncertainty in electricity prices. Electricity, as a commodity, differs from other commodities and financial assets as it is not storable. Among the consequences of this non-storability restriction, the most conspicuous one is the presence of large spikes in electricity prices. Another outcome of the non-storability of electricity is that the classic no-arbitrage relationship between spot prices and forward prices no longer makes sense in electricity markets.

This essay focusses on two models for electricity prices: Barlow (2002) and Aid et al. (2009). Both models capture the distinct features exhibited by electricity prices. We review, in detail, each model and then explore the connection between them. We find that the model proposed by Barlow (2002) can be interpreted as a continuous version of the model by Aid et al. (2009). We use Barlow's model to price forwards and derivatives on electricity and briefly illustrate how to simulate prices using numerical procedures.

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Chapter 1

Introduction

Formerly, electricity prices were determined by regulatory authorities, often government controlled. There was not much uncertainty in the electricity prices since these were regulated to reflect the generation and distribution costs. However, in the early 1990s, some countries started to restructure their electricity markets by opening these to competition and leaving the determination of electricity prices to the principles of supply and demand. The evolution of electricity prices in these deregulated markets is characterized by much more uncertainty and many markets show extremely high volatility. As a consequence of the market liberalization, electricity and electricity derivatives, which enable consumers to hedge the risk prevalent in electricity markets, are now actively traded on power exchanges which are similar to financial stock exchanges.

However, there is a very striking difference between electricity markets and other commodities or financial markets. Electricity is a flow commodity and can neither be stored easily nor transported easily. Electricity can only be stored indirectly, for example, as water in a hydroelectric power station or as coal/oil in a thermal plant. There are transportation constraints on electricity due to capacity limits on transmission grids. Therefore, classical arbitrage theory which is based on storability and transportation breaks down in electricity markets. In particular, the relationship between spot and forward prices cannot be determined as clearly as it is done in financial markets.

In fact, electricity spot prices exhibit some distinct features, not observed in other commodities or financial markets. Therefore, in order to be able to price contracts on electricity satisfactorily, it becomes crucial to understand the special characteristics of electricity markets and electricity prices. In this essay, we will review two models for electricity prices which capture the distinct features of electricity markets. The first model is a simple diffusion model proposed by Barlow (2002). It relies on the principle of equilibrium between demand and supply. The second model has been recently proposed by Aid et al. (2009). It is attractive because the forward price derived from this model is analytically tractable. Aid et al. (2009) argue that even though electricity cannot be stored, fuels can be and therefore derivatives on electricity can be priced using no-arbitrage on the market of fuels.

This essay is organized as follows. In the rest of this introductory chapter, we outline some of the stylized facts exhibited by electricity prices, and give the definitions of some electricity derivatives. In Chapter 2, we describe the diffusion model for electricity spot prices proposed by Barlow (2002). In Chapter 3, we focus on the structural model for pricing electricity forwards proposed by Aid et al. (2009). We describe, in detail, the proposed model and how it is used to price forward contracts on electricity. Finally, in the last chapter, we explore the connection between the two models described in Chapters 2 and 3. We use Barlow's model to price forwards and plain vanilla options on electricity and explain how numerical procedures can be used to simulate these prices.

1.1 Stylized facts of electricity prices

In this section, we briefly outline some of the special features exhibited by electricity prices. It is essential to consider these features when modeling electricity prices, if we want to satisfactorily valuate financial claims on electricity spot prices. The stylized facts that we will discuss include seasonality, spikes and mean reversion.

Seasonality

Electricity prices show strong seasonal fluctuations. These arise firstly, because of the fluctuations in the level of human activity and secondly, because of seasonal climate changes. All electricity markets have daily peak hours, during which human activity is the highest (for example, in the early morning when people are getting ready for work, or in the evening, when people are making dinner and watching TV etc). Electricity prices usually also exhibit a weekly effect. This is attributed to the fact that major businesses are usually closed on weekends. Thus, the demand for electricity is lower and this translates to lower weekend electricity prices. On a larger scale, electricity prices demonstrate annual seasonal effects, which vary from place to place. For example, in countries with very cold winters and mild summers, the electricity consumption tends to be highest during the winters, whereas this is reversed in countries where the summers are hot and winters mild. In sum, electricity prices have at least three types of periodicities, namely, intra-daily, weekly and annual. Thus, models for electricity prices need to incorporate some kind of seasonal component to cater for these observed seasonal fluctuations.

Price Spikes

Another prominent feature of electricity prices is the presence of price spikes. It is not uncommon to have electricity prices rising by a factor of 10 or more within a few hours, and then falling back rapidly to price levels regarded as normal (Simonsen et al. 2004). These large price increases followed by rapid reversions to normal levels are called price spikes. They usually occur because of unexpected and considerable changes in demand which force electricity producers to resort to more expensive production technologies to satisfy the demand, thus hiking up the price. Another explanation for price spikes is the occurrence of extreme unexpected technical problems on the producers' side such as power plant outages or unpredicted problems on the transmission grids. These extreme events cause the maximum supply capacity to drop to levels close to the demand level, making the prices go up. As soon as the conditions which caused the demand to go up cease or as soon as the power outages are resolved, the electricity prices return back to their normal levels.

The presence of spikes is one of the reasons why, unlike in other financial and commodities markets, geometric Brownian motion cannot be used to model electricity spot prices. Failing to account for price spikes when modeling electricity spot prices can greatly increase the risk of derivative writers by underestimating the derivative premiums.

Mean Reversion

Electricity prices exhibit mean reversion. This is because of the basic fact that energy prices are driven by supply and demand. Whatever may be the causes for drastic changes in supply and demand, once the problems are resolved, prices will invariably revert towards an equilibrium level. Thus, electricity prices models will usually have some mean reverting property to capture the mean reverting behavior of electricity prices.

1.2 An overview of electricity derivatives

In this section, we give the definitions of some basic electricity derivatives. The definitions have been taken from Deng & Oren (2006).

Electricity Forwards

A forward contract is an agreement which obligates its holder to buy an asset at a future expiration time T for a fixed price F. F is known as the forward price and it is decided on the initial trade date. At expiration, the value of the forward contract is $S_T - F$ where S_T is the asset price at time T. Electricity forward contracts differ from regular forward contracts because electricity is a flow commodity and the delivery of electricity takes place over a period of time (delivery period) rather than at a specific time. The settlement price

S_T is usually calculated based on the average price of electricity over the delivery period.

Therefore, for an electricity forward with maturity $T_1 > 0$, delivery period $[T_1, T_2]$ with $T_1 < T_2$ and forward time-t price $F_t(T_1, T_2)$, the payoff is given by

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} P_T dT - F_t(T_1, T_2) \tag{1.1}$$

where P_T is the electricity spot price at time T.

Electricity Futures

Electricity futures contracts have the same payoff structure as electricity forwards. However, in contrast to electricity forward contracts, electricity futures are highly standardized in contract specifications, trading locations, transaction requirements and settlement procedures. Electricity futures are exclusively traded on organized exchanges, while electricity forwards are usually traded over-the-counter. The majority of electricity futures are settled by financial payments rather than physical delivery. In addition, credit risks in trading futures are much lower than those in trading forwards. This is because the gains and losses of electricity futures are paid out daily (the margin calls), as opposed to being cumulated and paid out in a lump sum at maturity time, as in trading forwards.

Electricity vanilla call and put options

Electricity call and put options offer their buyers the right, but not the obligation, to buy or sell a fixed amount of underlying electricity at a pre-determined strike price K by the option expiration time T. They have similar payoff structures as those of regular call and put options on financial securities and other commodities. The payoff of an electricity call option is given by $\max(P_T - K, 0)$ while, the payoff of an electricity put option is given by $\max(K - P_T, 0)$, where P_T is the electricity spot price at time T.

Chapter 2

A diffusion model for electricity spot prices

2.1 Introduction

In this chapter, we describe the diffusion model for electricity prices proposed by Barlow (2002). The model is counted among the so-called equilibrium models which explain the price process through state variables that are associated to supply and demand.

2.2 The model

Let P_t denote the electricity spot price at time t. Let $u_t : \mathbb{R} \longrightarrow [0, a_0]$ be the supply function at time t and $u_t(p)$ the supply of electricity if the price is p. Here, $a_0 := \sup_{p \geq 0} u_t(p)$ represents the maximum supply. We also assume that the demand for electricity at time t, D_t, follows a mean-reverting Ornstein-Uhlenbeck process defined by the stochastic differential equation

$$dD_t = a_1 \left(b_1 - D_t \right) dt + \sigma_1 dW_t,$$

where a_1, b_1 and σ_1 are constants and W is a standard one-dimensional Brownian motion.

We note that the electricity demand is inelastic, that is, it is independent of the current spot price. This is because most consumers receive electricity at a fixed price and will not reduce their consumption if spot prices on the electricity exchange rise. On the other hand, the supply function is increasing in price because if the income per unit of electricity rises, more electricity producers will be willing to start generating and supplying electricity.

The spot price process at time t is defined by the equilibrium of supply and demand at time t. So, we have

$$u_t(P_t) = D_t. \tag{2.1}$$

This is an implicit equation for P_t and a solution might not always exist. In particular, since the demand follows an Ornstein-Uhlenbeck process, there is a positive probability that D_t will exceed the maximum supply a_0 and if this happens P_t will not be well-defined. To handle this problem, Barlow (2002) suggests capping P_t at some maximum price whenever demand exceeds the maximum supply.

To simplify computations, we next assume that the supply function is nonrandom and independent of t and follows the functional form

$$u_t(p) = g(p) = a_0 - b_0 p^{\alpha}, \tag{2.2}$$

for some constant α. Then, for $D_t < a_0$, we have

$$P_t = g^{-1}(D_t) = \left(\frac{a_0 - D_t}{b_0}\right)^{\frac{1}{\alpha}}.$$

So, we obtain the model

$$P_t = \begin{cases} \left(\frac{a_0 - D_t}{b_0}\right)^{\frac{1}{\alpha}}, & D_t < a_0 - \epsilon_0 b_0, \\ \epsilon_0^{\frac{1}{\alpha}}, & D_t \geq a_0 - \epsilon_0 b_0, \end{cases} \tag{2.3}$$

where $\epsilon_0^{\frac{1}{\alpha}}$ is the maximum price if demand exceeds the maximum supply.

Now, we observe that for $D_t < a_0 - \epsilon_0 b_0$, P_t can be written as

$$P_t = \left(\frac{a_0 - D_t}{b_0}\right)^{\frac{1}{\alpha}} = (1 + \alpha X_t)^{\frac{1}{\alpha}},$$

where

$$X_t = \frac{a_0 - b_0}{\alpha b_0} - \frac{1}{\alpha b_0} D_t$$

is also an Ornstein-Uhlenbeck process defined by $dX_t = a(b - X_t)dt + \sigma dW_t$, with $a = a_1$, $b = (a_0 - b_0 - b_1)/(\alpha b_0)$ and $\sigma = -\sigma_1/\alpha b_0$.

Next, we define

$$f_\alpha(x) = \begin{cases} (1 + \alpha x)^{\frac{1}{\alpha}}, & \alpha \neq 0, \\ e^x, & \alpha = 0, \end{cases} \tag{2.4}$$

where the inverse function of $f_\alpha(x)$ is the Box-Cox transformation:

$$g_\alpha(x) = f_\alpha^{-1}(x) = \begin{cases} \frac{x^\alpha - 1}{\alpha}, & \alpha \neq 0, \quad x > 0, \\ \log x, & \alpha = 0, \quad x > 0. \end{cases} \tag{2.5}$$

Then, we can write the price process as

$$\begin{aligned} P_t &= \begin{cases} f_\alpha(X_t), & 1 + \alpha X_t > \epsilon_0, \\ \epsilon_0^{\frac{1}{\alpha}}, & 1 + \alpha X_t \leq \epsilon_0, \end{cases} \\ dX_t &= a(b - X_t)dt + \sigma dW_t. \end{aligned} \tag{2.6}$$

This is called a non-linear Ornstein-Uhlenbeck process with parameters $(a, b, \sigma, \alpha, \epsilon_0)$.

2.3 Numerical Simulation

In this section, we perform a simulation of the price process P_t to illustrate the non-linear Ornstein-Uhlenbeck process.

The simulation procedure

From equation (2.6), we know that the price process P_t is a function of the normalized demand X_t which follows an Ornstein-Uhlenbeck process. To simulate X_t, we will use the exact solution to the Ornstein-Uhlenbeck stochastic differential equation and the fact that, given X_t, for any $0 < t < T$, X_T is normally distributed with mean $e^{-a(T-t)}X_t + b\left(1 - e^{-a(T-t)}\right)$ and variance $\frac{\sigma^2}{2a}\left(1 - e^{-2a(T-t)}\right)$. (Refer to Appendix A).

According to Glasserman (2004), we can use the following time-stepping equation to simulate X at time $0 = t_0 < t_1 < \cdots < t_n$:

$$X_{t_{i+1}} = e^{-a(t_{i+1}-t_i)}X_{t_i} + b\left(1 - e^{-a(t_{i+1}-t_i)}\right) + \sigma\sqrt{\frac{1}{2a}\left(1 - e^{-2a(t_{i+1}-t_i)}\right)}Z_{i+1} \tag{2.7}$$

where $Z_1, \ldots, Z_n$ are independent draws from a standard normal distribution.

Once we have simulated X_t, we can compute the prices P_t for $t = t_0, t_1, \ldots, t_n$ using the relation (2.6)(Refer to Appendix B.1 for the MATLAB code).

The following figure shows a simulation of the price process P_t. We have taken $X_{t_0} = 1$. The parameter values have been taken from estimated values given in Barlow (2002).

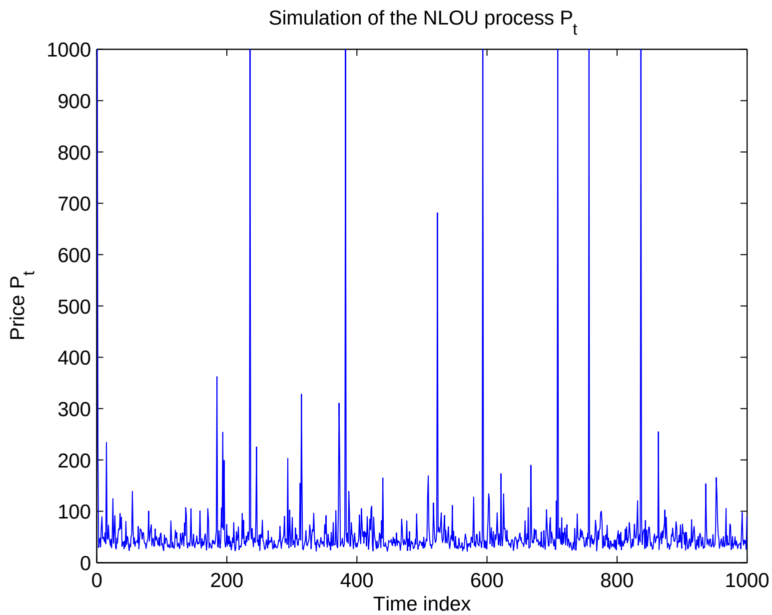

Figure 2.1: Simulation of price process for parameter values $a = 172, b = 0.91, \sigma = 0.12, \alpha = -1.08, \epsilon_0 = (1000)^{\alpha}$

Chapter 3

A structural risk-neutral model for electricity prices

3.1 Introduction

In this chapter, we describe the model for electricity spot prices and forward contracts on electricity spot prices proposed by Aid et al. (2009). The basic idea behind the proposed model is to express electricity spot prices in terms of prices of fuels used to produce electricity, so that we can adopt the classical no-arbitrage approach when pricing the corresponding forward contracts and thus, avoid the problem of non-storability of electricity. Aid et al. (2009) first explain how electricity demand, prices of fuels and capacities of electricity production technologies influence electricity prices. They present a structural relationship between electricity spots and fuel prices and then use this structural relationship in conjuction with classical no-arbitrage arguments to derive explicit formulas for forward contracts on electricity spot prices.

3.2 The model for spot prices

3.2.1 The market setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We assume that $0 < T^\star < \infty$ is a fixed time horizon which is long enough to contain all delivery times of financial contracts of interest. Let $(W^0, W^1, \ldots, W^n)$ be an $(n+1)$-dimensional standard Brownian motion where $n \geq 1$. We denote the filtration generated by W^0 by $\mathcal{F}^0 = (\mathcal{F}^0_t)$ and the filtration generated by the n-dimensional Brownian motion $W = (W^1, \ldots, W^n)$ by $\mathcal{F}^W = (\mathcal{F}^W_t)$.

The market setting is as follows:

- The market consists of a riskless asset with price process $S^0_t = e^{\int_0^t r_u du}, t \geq 0$, where $(r_t)_{t\geq 0}$ is the instantaneous interest rate process which is non-negative and $\mathcal{F}^W$-adapted.

- The market contains $n \geq 1$ tradeable commodities (fuels) from which electricity can be produced. For convenience,we identify each production technology with its corresponding fuel.

- S^i_t is defined to be the price of the amount of commodity i necessary to produce 1 KWh of electricity and is assumed to satisfy the stochastic differential equation given by
$$dS^i_t = S^i_t \left(\mu^i_t dt + \sum_{j=1}^n \sigma^{ij}_t dW^j_t \right), \quad t \geq 0, \quad i = 1, \ldots, n,$$
where μ^i_t and σ^{ij}_t are stochastic processes adapted to the filtration $\mathcal{F}^W$.

We also make the following technical assumption.

Assumption 3.1 *The volatility matrix $\sigma_t = (\sigma^{ij}_t)_{1\leq i,j\leq n}$ is invertible and the matrices σ and σ^{-1} are bounded uniformly on $[0, T^\star] \times \Omega$. The market price of risk θ satisfies the*

Novikov condition

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^{T^\star}\|\theta_t\|^2 dt\right\}\right] < \infty \quad a.s$$

where $\theta_t = \sigma_t^{-1}\left[\mu_t - r_t \boldsymbol{1}_n\right], t \geq 0$ *and* $\boldsymbol{1}_n$ *is an* n*-dimensional vector of ones.*

3.2.2 Market demand for electricity

We model the electricity market demand by a real-valued continuous process $D = (D_t)_{t\geq 0}$ which is $\mathcal{F}^0$-adapted. Under the probability $\mathbb{P}$, the processes S^i are independent of the process D since the Brownian motions W^0 and W are independent. In the analysis that follows, we will use the residual demand instead of the demand. The residual demand is defined to be the total demand minus the production of some generation assets (for example,nuclear power, hydroelectric plants, wind farms). The residual demand can be negative.

3.2.3 Capacity process of the production technology

Let $\Delta_t^i > 0$ be the capacity of the i^{th} technology. (Δ_t^i) is a stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that (Δ_t^i) is independent of (W^0, W) and we denote the filtration generated by Δ_t^i by $\mathcal{F}^\Delta = (\mathcal{F}_t^\Delta)$. We also assume that $\Delta_t^i \in [m_i, M_i]$ where $0 \leq m_i \leq M_i$ and m_i and M_i are the minimal and maximal capacities of the i^{th} technology and are known to the producer.

3.2.4 Electricity spot prices

Let P_t be the electricity spot price at time t. At any time t, the electricity producer chooses the most convenient (cheap) technology among the n technologies to manufacture electricity, and the spot price P_t is determined by the producer's choice of commodity.

We will now derive an expression for the spot price of electricity P_t in terms of the fuel prices S^i, the residual demand of electricity D_t and the capacity processes Δ_t^i. For every

(t, ω), we have an ordering of commodities from the cheapest to the most expensive:

$$S_t^{(1)}(\omega) \leq S_t^{(2)}(\omega) \leq \cdots S_t^{(n)}(\omega).$$

We denote the ordering at time t by the permutation $\pi_t = \{\pi_t(1), \pi_t(2), \cdots \pi_t(n)\}$. Next, we set up the capacity intervals:

$$\begin{aligned}
I_1^{\pi_t}(t) &= \left[0, \Delta_t^{\pi_t(1)}\right), \\
I_2^{\pi_t}(t) &= \left[\Delta_t^{\pi_t(1)}, \Delta_t^{\pi_t(1)} + \Delta_t^{\pi_t(2)}\right), \\
&\vdots \\
I_k^{\pi_t}(t) &= \left[\sum_{i=1}^{k-1} \Delta_t^{\pi_t(i)}, \sum_{i=1}^{k} \Delta_t^{\pi_t(i)}\right), \quad 2 \leq k \leq n.
\end{aligned}$$

Now, we observe that if, at time t,the electricity residual demand D_t falls in the interval $I_k^{\pi_t}(t)$, then the last unit of electricity will be produced using commodity $\pi_t(k)$ and therefore, the electricity spot price will be equal to $S_t^{(k)}$. Thus, we obtain the relation:

$$P_t = \sum_{i=1}^{n} S_t^{(i)} \mathbf{1}_{\{D_t \in I_i^{\pi_t}(t)\}}, \quad t \geq 0. \tag{3.1}$$

This is the structural relationship between electricity spot prices and fuel prices which forms the crux of the model proposed by Aid et al. (2009). This relation allows us to transfer the risk-neutral probabilities of the market of fuels to the electricity market.

3.3 Pricing of electricity forwards

3.3.1 Pricing using the no-arbitrage principle

We consider an asset with price S_t in a *financial* market and a forward contract on the asset with expiry T, with $t \leq T$. We assume that the riskless interest rate r is constant. Classical no-arbitrage theory (Bjork 2004) tells us that the relationship between the spot price S_t

and the forward price $F(t,T)$ is given by:

$$F(t,T) = e^{r(T-t)}S_t. \tag{3.2}$$

For example, if $F(t,T) > e^{r(T-t)}S_t$, then we have the following arbitrage strategy:

- At time t
  - Borrow S_t at interest r.
  - Buy the asset for S_t.
  - Enter into a forward contract to sell the asset for $F(t,T)$ at time T.
- At time T
  - Sell the asset for $F(t,T)$.
  - Repay the loan at $e^{r(T-t)}S_t$.

This results in a riskless profit of $F(t,T) - e^{r(T-t)}S_t > 0$ which violates the no-arbitrage principle. Similarly, if $F(t,T) < e^{r(T-t)}S_t$, an arbitrage opportunity exists. At time t, we can short sell the asset at S_t, invest the proceeds at interest rate r until time T, and enter a forward contract to buy the asset at $t = T$. This will produce, at time T, a riskless profit of $e^{r(T-t)}S_t - F(t,T) > 0$.

Clearly, the no-arbitrage arguments discussed above relies very much on the fact that the asset is storable. As mentioned previously in the introduction, such arguments no longer hold in electricity markets because of the fact that electricity cannot be stored. Once purchased, it has to be consumed.

However, because of relation (3.1), we can look at all electricity derivatives as basket options on fuels. This allows us to overcome the non-storability restriction of electricity and apply the usual arbitrage arguments directly to the market of fuels, instead of the market of electricity.

3.3.2 The choice of a risk-neutral measure

Let $\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{F}_t^W \vee \mathcal{F}_t^\Delta$, $t \in [0, T^*]$, be the market filtration. We recall that equation (3.1) gives us an expression for the electricity spot in terms of the fuel prices $S_t^i, i = 1, 2 \ldots, n$. Therefore, in order to price derivatives on the spot P_t, it will be enough to identify an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$ defined on $\mathcal{F}_{T^*}$, such that the discounted fuel prices $\widetilde{S}_t^i := S_t^i / S_t^0, i = 1, \ldots, n$ are $\mathbb{Q}$-martingales with respect to the filtration $(\mathcal{F}_t)$. As mentioned in (Bjork 2004), the absence of arbitrage assumption in the market of fuels guarantees the existence of such an equivalent martingale measure $\mathbb{Q}$.

However, the market of fuels and electricity is an incomplete market. This is, firstly, because of the non-hedgeable risk W^0 influencing the electricity demand,and secondly, because of the fact that we cannot use the underlying process P_t to hedge derivatives due to inefficiencies in storing electricity. Incompleteness of the market implies that the equivalent measure $\mathbb{Q}$ is not uniquely determined. In such a case, it is common to fix one equivalent martingale measure $\mathbb{Q}$ and then pricing all derivatives under the chosen measure $\mathbb{Q}$. Aid et al. (2009) suggests using $\mathbb{Q} = \mathbb{Q}^{\min}$, a minimal martingale measure defined by the equation

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{ -\int_0^{T^*} \theta_u . dW_u - \frac{1}{2}\int_0^{T^*} \|\theta_u\|^2 du \right\}, \tag{3.3}$$

where $\theta_t = \sigma_t^{-1}(\mu_t - r_t \mathbf{1}_n)$ is the market price of risk for the commodities market $(S^1, \ldots, S^n)$. Due to assumption (3.1), such a measure is well-defined.

We note the following:

- Under $\mathbb{Q}$, the laws of the processes W^0 and $\Delta^i, i = 1, \ldots, n$ are unchanged. The independence between the filtrations $\mathcal{F}^0, \mathcal{F}^\Delta$ and $\mathcal{F}^W$ is also preserved under $\mathbb{Q}$.

- We recall, from section (3.2.1), that under the objective probability $\mathbb{P}$, the commodities prices S^i satisfy the stochastic differential equations given by

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^{n} \sigma_t^{ij} dW_t^j \right), \quad t \geq 0, \quad i = 1, \ldots, n. \tag{3.4}$$

  Therefore, using equation (3.3), we can deduce the $\mathbb{Q}$-dynamics of the prices of fuels. This is given by the stochastic differential equations

$$dS_t^i = S_t^i \left(r_t dt + \sum_{j=1}^{n} \sigma_t^{ij} dW_t^{\mathbb{Q},j} \right), \quad S_0^i > 0, \quad i = 1, \ldots, n, \tag{3.5}$$

  with solutions

$$S_t^i = S_0^i \exp \left\{ \int_0^t \left(r_u - \frac{1}{2} \|\sigma_u^i\|^2 \right) du + \int_0^t \sigma_u^i . dW_u^{\mathbb{Q}} \right\}, \quad t \geq 0, \quad i = 1, \ldots, n, \tag{3.6}$$

  where $W^{\mathbb{Q}} = \left(W^{\mathbb{Q},1}, \ldots W^{\mathbb{Q},n} \right)$ is an n-dimensional Brownian motion under $\mathbb{Q}$ and $\sigma^i = (\sigma^{i,1}, \ldots, \sigma^{i,n})$.

Equipped with our chosen risk-neutral measure $\mathbb{Q}$ and the $\mathbb{Q}$-dynamics of the prices of the fuels S_t^i, we can now proceed to the pricing of forward contracts.

3.3.3 Calculation of forward prices

We recall, from Chapter (1), section (1.2), that for an electricity forward with maturity $T_1 > 0$, delivery period $[T_1, T_2]$ with $T_1 < T_2 \leq T^\star$ and forward time-t price $F_t(T_1, T_2)$, the payoff is given by

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} P_T dT - F_t(T_1, T_2) \tag{3.7}$$

where P_T is the electricity spot price at time T.

The arbitrage-free forward price $F_t(T_1, T_2)$ is chosen so that the forward contract has price zero at time t and is obtained by solving the equation

$$\frac{1}{D(t)}\mathbb{E}^{\mathbb{Q}}\left[D(T)\left(\frac{1}{T_2-T_1}\int_{T_1}^{T_2} P_T dT - F_t(T_1,T_2)\right)|\mathcal{F}_t\right] = 0, \tag{3.8}$$

where $D(t)$ defines a discount process given by $D(t) = e^{-\int_0^t r_u du}$, and r_t is the interest rate process (Shreve 2004).

The solution to equation (3.8) is given by

$$F_t(T_1,T_2) = \frac{1}{T_2-T_1}\int_{T_1}^{T_2} \frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u du} P_T|\mathcal{F}_t\right]}{B(t,T)} dT, \tag{3.9}$$

where $B(t,T)$ is the price at time t of a zero-coupon bond paying 1 at time T and is given by

$$\begin{aligned} B(t,T) &= \frac{1}{D(t)}\mathbb{E}^{\mathbb{Q}}\left[D(T)|\mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}|\mathcal{F}_t\right]. \end{aligned}$$

We now introduce the forward measure $\mathbb{Q}_T$ defined by

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}} := \frac{e^{-\int_t^T r_u du}}{B(t,T)} \quad \text{on} \quad \mathcal{F}_T^W. \tag{3.10}$$

Then

$$\mathbb{E}^{\mathbb{Q}_T}\left[X\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{e^{-\int_t^T r_u du}}{B(t,T)} X\right].$$

Therefore, equation (3.9) becomes

$$
\begin{aligned}
F_t(T_1, T_2) &= \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \mathbb{E}^{\mathbb{Q}_T} \left[P_T | \mathcal{F}_t \right] dT, \\
&= \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \mathbb{E}^{\mathbb{Q}_T} \left[\sum_{i=1}^{n} S_T^{(i)} \mathbf{1}_{\{D_T \in I_i^{\pi_T}(T)\}} | \mathcal{F}_t \right] dT, \\
&= \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \underbrace{\sum_{i=1}^{n} \mathbb{E}^{\mathbb{Q}_T} \left[S_T^{(i)} \mathbf{1}_{\{D_T \in I_i^{\pi_T}(T)\}} | \mathcal{F}_t \right]}_{F_t(T)} dT, \qquad (3.11)
\end{aligned}
$$

where in the second step we have used the structural relationship described by equation (3.1).

Let Π_n be the set of all permutations over the index set $\{1, \ldots, n\}$ and let $\pi \in \Pi_n$ be a given permutation. Let $F_t(T)$ denote the forward time-t price with instantaneous delivery at maturity T. Then

$$
\begin{aligned}
F_t(T) &= \mathbb{E}^{\mathbb{Q}_T} \left[P_T | \mathcal{F}_t \right], \\
&= \sum_{i=1}^{n} \mathbb{E}^{\mathbb{Q}_T} \left[S_T^{(i)} \mathbf{1}_{\{D_T \in I_i^{\pi_T}(T)\}} | \mathcal{F}_t \right], \\
&= \sum_{i=1}^{n} \sum_{\pi \in \Pi_n} \mathbb{E}^{\mathbb{Q}_T} \left[S_T^{\pi(i)} \mathbf{1}_{\{D_T \in I_i^{\pi}(T)\}} \mathbf{1}_{\{\pi_T = \pi\}} | \mathcal{F}_t \right]. \qquad (3.12)
\end{aligned}
$$

Because of the mutual conditional independence between W, W_0 and Δ, equation (3.12) becomes

$$
F_t(T) = \sum_{i=1}^{n} \sum_{\pi \in \Pi_n} \mathbb{E}^{\mathbb{Q}_T} \left[S_T^{\pi(i)} \mathbf{1}_{\{\pi_T = \pi\}} | \mathcal{F}_t^W \right] \mathbb{Q}_T \left[D_T \in I_i^{\pi}(T) | \mathcal{F}_t^{0,\Delta} \right], \qquad (3.13)
$$

where $\mathcal{F}_t^{0,\Delta}$ denotes the natural filtration generated by both W^0 and Δ.

We now introduce the change of probability

$$\frac{d\mathbb{Q}_T^i}{d\mathbb{Q}_T} = \frac{S_T^i}{\mathbb{E}^{\mathbb{Q}_T}\left[S_T^i\right]} \quad \text{on} \quad \mathcal{F}_T^W, \quad 1 \leq i \leq n, \quad T \leq T^\star. \tag{3.14}$$

Then

$$\mathbb{E}^{\mathbb{Q}_T^i}\left[X\right] = \mathbb{E}^{\mathbb{Q}_T}\left[\frac{S_T^i}{\mathbb{E}^{\mathbb{Q}_T}\left[S_T^i\right]}X\right].$$

Therefore, applying the change of probability, we get

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_T}\left[S_T^{\pi(i)}\mathbf{1}_{\{\pi_T=\pi\}}|\mathcal{F}_t^W\right] &= \mathbb{E}^{\mathbb{Q}_T}\left[\mathbb{E}^{\mathbb{Q}_T}\left[S_T^{\pi(i)}\right]\frac{S_T^{\pi(i)}}{\mathbb{E}^{\mathbb{Q}_T}\left[S_T^{\pi(i)}\right]}\mathbf{1}_{\{\pi_T=\pi\}}|\mathcal{F}_t^W\right], \\
&= \mathbb{E}^{\mathbb{Q}_T}\left[\mathbb{E}^{\mathbb{Q}_T}\left[S_T^{\pi(i)}\right]|\mathcal{F}_t^W\right]\mathbb{E}^{\mathbb{Q}_T^{\pi(i)}}\left[\mathbf{1}_{\{\pi_T=\pi\}}|\mathcal{F}_t^W\right], \\
&= F_t^{\pi(i)}(T)\mathbb{Q}_T^{\pi(i)}\left[\pi_T=\pi|\mathcal{F}_t^W\right].
\end{aligned} \tag{3.15}$$

We now combine equations (3.11), (3.13) and (3.15) to get the following expression for the forward price $F_t(T_1, T_2)$:

$$F_t(T_1,T_2) = \frac{1}{T_2-T_1}\sum_{i=1}^{n}\sum_{\pi\in\Pi_n}\int_{T_1}^{T_2} F_t^{\pi(i)}(T)\mathbb{Q}_T^{\pi(i)}\left[\pi_T=\pi|\mathcal{F}_t^W\right]\mathbb{Q}_T\left[D_T\in I_i^\pi(T)|\mathcal{F}_t^{0,\Delta}\right]dT, \tag{3.16}$$

where $t \in [0, T_1]$ and $F_t^i(T)$ denotes the time-t price of a forward contract on commodity i having maturity T.

In the following two sections, we will illustrate how we can compute the probabilities $\mathbb{Q}_T^{\pi(i)}\left[\pi_T=\pi|\mathcal{F}_t^W\right]$ and $\mathbb{Q}_T\left[D_T\in I_i^\pi(T)|\mathcal{F}_t^{0,\Delta}\right]$ explicitly when we make certain specific assumptions on the demand process D_t and the capacity processes Δ^i.

3.3.4 Case 1:A model with two technologies,constant coefficients and constant capacity

In this section, we make the following assumptions:

- the market has only two technologies $i = 1, 2$.
- the volatilities of the fuels are constant, that is, $\sigma_t^{i,j} = \sigma^{i,j} > 0$, for $1 \leq i, j \leq 2$.
- the interest rate is constant, that is, $r_t = r > 0$.
- the capacity of each technology is constant, that is, $\Delta_t^i = m_i$ for $i = 1, 2$.
- the electricity (residual) demand follows a mean reverting Ornstein-Uhlenbeck process with constant coefficients.

We first note that under the constant interest rate assumption,the forward measure $\mathbb{Q}_T$ equals the minimal martingale measure $\mathbb{Q}$.

We start by computing the expressions $\mathbb{Q}_T\left[D_T \in I_i^\pi(T)|\mathcal{F}_t^{0,\Delta}\right]$ for $i = 1, 2$. We have

$$\begin{aligned}\mathbb{Q}_T\left[D_T \in I_1^\pi(T)|\mathcal{F}_t^{0,\Delta}\right] &= \mathbb{Q}_T\left[D_T \leq \Delta_T^{\pi(1)}|\mathcal{F}_t^{0,\Delta}\right], \\ &= \mathbb{Q}_T\left[D_T \leq m_{\pi(1)}|\mathcal{F}_t^{0}\right],\end{aligned}$$

and

$$\begin{aligned}\mathbb{Q}_T\left[D_T \in I_2^\pi(T)|\mathcal{F}_t^{0,\Delta}\right] &= \mathbb{Q}_T\left[\Delta_T^{\pi(1)} < D_T \leq \Delta_T^{\pi(1)} + \Delta_T^{\pi(2)}|\mathcal{F}_t^{0,\Delta}\right], \\ &= \mathbb{Q}_T\left[m_{\pi(1)} < D_T \leq m_{\pi(1)} + m_{\pi(2)}|\mathcal{F}_t^{0}\right].\end{aligned}$$

In order to compute the above expressions explicitly, we need to consider the dynamics of the electricity demand.

Dynamics of the electricity demand D

We assume that the residual demand follows a mean reverting Ornstein-Uhlenbeck process given by

$$dD_t = a(b - D_t)dt + \delta dW_t^0, \quad D_0 > 0, \tag{3.17}$$

where $a, \delta > 0$ and b is a constant long-term mean.

We know, from Appendix A, that given the knowledge of the state at any time $t < T$, the process D_T is normally distributed with the following mean and variance

$$D_T|\mathcal{F}_t^0 \sim \mathcal{N}\left(b + (D_t - b)e^{-a(T-t)}, \frac{\delta^2}{2a}(1 - e^{-2a(T-t)})\right). \tag{3.18}$$

Therefore, for $x_1, x_2 \in \mathbb{R}$,

$$\mathbb{Q}_T\left[D_T \leq x_1|\mathcal{F}_t^0\right] = \Phi\left(\frac{x_1 - b - (D_t - b)e^{-a(T-t)}}{\delta\sqrt{\frac{1}{2a}(1 - e^{-2a(T-t)})}}\right), \tag{3.19}$$

and

$$\begin{aligned}\mathbb{Q}_T\left[x_1 < D_T \leq x_1 + x_2|\mathcal{F}_t^0\right] &= \Phi\left(\frac{(x_1 + x_2) - b - (D_t - b)e^{-a(T-t)}}{\delta\sqrt{\frac{1}{2a}(1 - e^{-2a(T-t)})}}\right) \\ &\quad -\Phi\left(\frac{x_1 - b - (D_t - b)e^{-a(T-t)}}{\delta\sqrt{\frac{1}{2a}(1 - e^{-2a(T-t)})}}\right),\end{aligned} \tag{3.20}$$

where Φ denotes the distribution function of a standard normal variable.

Next, we compute explicitly the conditional probability $\mathbb{Q}_T^{\pi(i)}[\pi_T = \pi|\mathcal{F}_t^W]$ for any $\pi \in \Pi_2$ and any $i = 1, 2$. To do that, we first need to determine the law of the couple (S_T^1, S_T^2) under each probability $\mathbb{Q}_T^{\pi(i)}$. Let

$$Z_t^i := \frac{d\mathbb{Q}_T^i}{d\mathbb{Q}}|_{\mathcal{F}_t^W}.$$

Then, from equation (3.14), we have

$$\begin{aligned} Z_t^i &= \frac{S_t^i}{\mathbb{E}^{\mathbb{Q}}\left[S_t^i\right]} \\ &= \frac{S_0^i \exp\{\left(r - \frac{1}{2}\|\sigma^i\|^2\right) t + \sigma^i \cdot W_t^{\mathbb{Q}}\}}{S_0^i \exp\{rt\}} \\ &= \exp\left\{\sigma^i \cdot W_t^{\mathbb{Q}} - \frac{1}{2}\|\sigma^i\|^2 t\right\}, \quad t \in [0,T], \end{aligned} \tag{3.21}$$

where σ^i denotes the 2-dimensional vector $(\sigma^{i,1}, \sigma^{i,2})$.

Now, we recall (from equations (3.5) and (3.6)) that under the probability $\mathbb{Q}$, the prices of commodities S_j follow an Ito process given by

$$dS_t^j = S_t^j \left(rdt + \sigma^j \cdot dW_t^{\mathbb{Q}}\right), \quad S_0^i > 0,$$

with solutions given by

$$S_t^i = S_0^i \exp\left\{ \left(r - \frac{1}{2}\|\sigma^j\|^2\right) t + \sigma^j \cdot W_t^{\mathbb{Q}} \right\}.$$

From equation (3.21), we have

$$\frac{dZ_t^i}{Z_t^i} = \sigma^i \cdot dW_t^{\mathbb{Q}}.$$

By Girsanov's Theorem, there exists a 2-dimensional Brownian motion $\widehat{W} = (\widehat{W}^1, \widehat{W}^2)$ under $\mathbb{Q}_T^i$ such that the price process S^j satisfies the following $\mathbb{Q}_T^i$-Ito process

$$dS_t^j = S_t^j \left(\left(r + \sigma^j \cdot \sigma^i\right) dt + \sigma^j \cdot d\widehat{W}_t^{\mathbb{Q}}\right), \quad S_0^i > 0,$$

whose solution is given by

$$S_T^j = S_t^j \exp\left\{ \left(r - \frac{1}{2}\|\sigma^j\|^2 + \sigma^j \cdot \sigma^i\right)(T-t) + \sum_{k=1}^{2} \sigma^{j,k}\left(\widehat{W}_T^k - \widehat{W}_t^k\right) \right\}, \tag{3.22}$$

where $t \in [0, T]$ and $j = 1, 2$.

Having obtained the $\mathbb{Q}_T^i$-dynamics of the price process, we can now compute the conditional probability $\mathbb{Q}_T^{\pi(i)}[\pi_T = \pi | \mathcal{F}_t^W]$.

$$\begin{aligned} \mathbb{Q}_T^{\pi(i)}\left[\pi_T = \pi | \mathcal{F}_t^W\right] &= \mathbb{Q}_T^{\pi(i)}\left[S_T^{\pi(1)} \leq S_T^{\pi(2)} | \mathcal{F}_t^W\right] \\ &= \mathbb{Q}_T^{\pi(i)}\left[\ln\left(\frac{S_T^{\pi(1)}}{S_T^{\pi(2)}}\right) \leq 0 | \mathcal{F}_t^W\right]. \end{aligned}$$

Let

$$\begin{aligned} X &:= \ln\left(\frac{S_T^{\pi(1)}}{S_T^{\pi(2)}}\right), \\ &= \ln\left(\frac{S_t^{\pi(1)}}{S_t^{\pi(2)}}\right) + \sum_{k=1}^{2}\left(\sigma^{\pi(1),k} - \sigma^{\pi(2),k}\right)\left(\widehat{W}_T^k - \widehat{W}_t^k\right) \\ &\quad - \sum_{k=1}^{2}\left(\frac{1}{2}\left((\sigma^{\pi(1),k})^2 - (\sigma^{\pi(2),k})^2\right) - \left(\sigma^{\pi(1),k} - \sigma^{\pi(2),k}\right)\sigma^{\pi(i),k}\right)(T-t). \end{aligned}$$

Therefore, the random variable X, conditional upon $\mathcal{F}_t^W$, is normally distributed with mean $m(t)$ and variance $\gamma^2(t)$ given by

$$\begin{aligned} m(t) &= \ln\left(\frac{S_t^{\pi(1)}}{S_t^{\pi(2)}}\right) - \sum_{k=1}^{2}\left(\frac{1}{2}\left((\sigma^{\pi(1),k})^2 - (\sigma^{\pi(2),k})^2\right) - \left(\sigma^{\pi(1),k} - \sigma^{\pi(2),k}\right)\sigma^{\pi(i),k}\right)(T-t), \\ \gamma^2(t) &= \sum_{k=1}^{2}\left(\sigma^{\pi(1),k} - \sigma^{\pi(2),k}\right)^2 (T-t). \end{aligned} \tag{3.23}$$

Hence, we have

$$\begin{aligned} \mathbb{Q}_T^{\pi(i)}\left[\pi_T = \pi | \mathcal{F}_t^W\right] &= \mathbb{Q}_T^{\pi(i)}\left[X \leq 0 | \mathcal{F}_t^W\right] \\ &= \mathbb{Q}_T^{\pi(i)}\left[\frac{X - m(t)}{\gamma(t)} \leq -\frac{m(t)}{\gamma(t)} | \mathcal{F}_t^W\right] \\ &= \Phi\left(-\frac{m(t)}{\gamma(t)}\right) = 1 - \Phi\left(\frac{m(t)}{\gamma(t)}\right). \end{aligned}$$

Putting everything together, we obtain the following price for a forward contract, at time t, with maturity T_1 and delivery period $[T_1, T_2]$:

$$F_t(T_1, T_2) = \frac{1}{T_2 - T_1} \sum_{\pi \in \Pi_2} \int_{T_1}^{T_2} (A_1(t,T) + A_2(t,T))\, dT, \tag{3.24}$$

where

$$\begin{aligned} A_1(t,T) &= F_t^{\pi(1)}(T)\mathbb{Q}_T^{\pi(1)}\left[\pi_t = \pi | \mathcal{F}_t^W\right] \mathbb{Q}_T\left[D_T \leq m_{\pi(1)} | \mathcal{F}_t^0\right], \\ A_2(t,T) &= F_t^{\pi(2)}(T)\mathbb{Q}_T^{\pi(2)}\left[\pi_t = \pi | \mathcal{F}_t^W\right] \mathbb{Q}_T\left[m_{\pi(1)} < D_T \leq m_{\pi(1)} + m_{\pi(2)} | \mathcal{F}_t^0\right], \end{aligned}$$

and where the probabilities $\mathbb{Q}_T\left[D_T \leq m_{\pi(1)} | \mathcal{F}_t^0\right]$ and $\mathbb{Q}_T\left[m_{\pi(1)} < D_T \leq m_{\pi(1)} + m_{\pi(2)} | \mathcal{F}_t^0\right]$ are given by equations (3.19) and (3.20) respectively and

$$\begin{aligned} \mathbb{Q}_T^{\pi(i)}\left[\pi_T = \pi | \mathcal{F}_t^W\right] &= 1 - \Phi\left(\frac{m(t)}{\gamma(t)}\right), \\ m(t) &= \ln\left(\frac{S_t^{\pi(1)}}{S_t^{\pi(2)}}\right) - \left(\frac{1}{2}\|\sigma^{\pi(1)} - \sigma^{\pi(2)}\|^2 - (\sigma^{\pi(1)} - \sigma^{\pi(2)}).\sigma^{\pi(i)}\right)(T-t), \\ \gamma^2(t) &= \|\sigma^{\pi(1)} - \sigma^{\pi(2)}\|^2 (T-t). \end{aligned} \tag{3.25}$$

3.3.5 Case 2:A model with two technologies,constant coefficients and non-constant capacity

In this section, we make the following assumptions:

- the market has only two technologies $i = 1, 2$.
- the volatilities of the fuels are constant, that is, $\sigma_t^{i,j} = \sigma^{i,j} > 0$, for $1 \leq i, j \leq 2$.
- the interest rate is constant, that is, $r_t = r > 0$.
- the electricity (residual) demand follows a mean reverting Ornstein-Uhlenbeck process.

Dynamics of capacity processes Δ^i

Further, we assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supports four independent standard poisson processes $N_t^{1,u}$, $N_t^{1,d}$, $N_t^{2,u}$ and $N_t^{2,d}$ with constant intensities λ_1^u, λ_1^d, λ_2^u, λ_2^d and that the dynamics of each capacity process Δ^i follows the stochastic differential equation:

$$d\Delta_t^i = (m_i - M_i)\mathbf{1}_{(\Delta_t^i = M_i)} dN_t^{i,d} + (M_i - m_i)\mathbf{1}_{(\Delta_t^i = m_i)} dN_t^{i,u}, \quad \Delta_0^i = M_i, \quad i = 1, 2, \tag{3.26}$$

where m_i and M_i are the constant minimal and maximal values of the capacity processes Δ^i respectively. This assumption on the capacities implies that each capacity Δ^i can only take one of the two values m_i or M_i, and that it switches from m_i to M_i when the poisson process $N^{i,u}$ jumps and switches from M_i to m_i when the process $N^{i,d}$ jumps. Each capacity evolves independently of each other.

We will proceed to compute the probabilities $\mathbb{Q}_T\left[D_T \in I_i^\pi(T) | \mathcal{F}_t^{0,\Delta}\right]$ for $i = 1, 2$ appearing in formula (3.16). Using the fact that Δ is independent of W^0 and that its law is the same under both the objective probability $\mathbb{P}$ and the risk-neutral probability $\mathbb{Q} = \mathbb{Q}_T$, we can decompose the probability $\mathbb{Q}_T\left[D_T \in I_1^\pi(T) | \mathcal{F}_t^{0,\Delta}\right]$ as follows:

$$\begin{aligned}
\mathbb{Q}_T\left[D_T \in I_1^\pi(T) | \mathcal{F}_t^{0,\Delta}\right] &= \mathbb{Q}_T\left[D_T \le \Delta_T^{\pi(1)} | \mathcal{F}_t^{0,\Delta}\right], \\
&= \mathbb{Q}_T\left[\Delta_T^{\pi(1)} = m_{\pi(1)} | \mathcal{F}_t^{\Delta}\right] \mathbb{Q}_T\left[D_T \le m_{\pi(1)} | \mathcal{F}_t^0\right] \\
&\quad + \mathbb{Q}_T\left[\Delta_T^{\pi(1)} = M_{\pi(1)} | \mathcal{F}_t^{\Delta}\right] \mathbb{Q}_T\left[D_T \le M_{\pi(1)} | \mathcal{F}_t^0\right], \\
&= \mathbb{P}\left[\Delta_T^{\pi(1)} = m_{\pi(1)} | \mathcal{F}_t^{\Delta}\right] \mathbb{Q}_T\left[D_T \le m_{\pi(1)} | \mathcal{F}_t^0\right] \\
&\quad + \mathbb{P}\left[\Delta_T^{\pi(1)} = M_{\pi(1)} | \mathcal{F}_t^{\Delta}\right] \mathbb{Q}_T\left[D_T \le M_{\pi(1)} | \mathcal{F}_t^0\right], \\
&= \sum_{x_1 = m_{\pi(1)}, M_{\pi(1)}} \mathbb{P}\left[\Delta_T^{\pi(1)} = x_1 | \mathcal{F}_t^{\Delta}\right] \mathbb{Q}_T\left[D_T \le x_1 | \mathcal{F}_t^0\right].
\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbb{Q}_T\left[D_T \in I_2^{\pi}(T)|\mathcal{F}_t^{0,\Delta}\right] &= \mathbb{Q}_T\left[\Delta_T^{\pi(1)} < D_T \leq \Delta_T^{\pi(1)} + \Delta_T^{\pi(2)}|\mathcal{F}_t^{0,\Delta}\right], \\ &= \sum_{\substack{\{x_1=m_{\pi(1)},M_{\pi(1)}; \\ x_2=m_{\pi(2)},M_{\pi(2)}\}}} \mathbb{P}\left[\Delta_T^{\pi(1)} = x_1, \Delta_T^{\pi(2)} = x_2|\mathcal{F}_t^{\Delta}\right] \\ &\quad \times \mathbb{Q}_T\left[x_1 < D_T \leq x_1 + x_2|\mathcal{F}_t^{0}\right].\end{aligned}$$

We observe that, under the assumption that the demand process follows a mean-reverting Ornstein-Uhlenbeck process, the probabilities $\mathbb{Q}_T\left[D_T \leq x_1|\mathcal{F}_t^0\right]$ and $\mathbb{Q}_T\left[x_1 < D_T \leq x_1 + x_2|\mathcal{F}_t^0\right]$ will still be given by equations (3.19) and (3.20) respectively, as calculated previously in section (3.3.4). However, we still need to compute the probabilities $\mathbb{P}\left[\Delta_T^k = x|\mathcal{F}_t^{\Delta}\right]$ for $x = m_k, M_k$.

For a start, we will compute $\mathbb{P}\left[\Delta_T^k = m_k|\Delta_0^k = M_k\right]$. Let τ^d be the last jump time of the process $N_t^{k,d}$ before T, that is, $\tau^d = \sup\{t \in [0,T] : \Delta N_t^{k,d} = 1\}$. We consider the events $\{\tau^d > 0\}$ and $\{\tau^d = 0\}$. If $\{\tau^d > 0\}$ occurs, then it means that Δ^k jumps downwards to m_k at τ^d and, that will be by definition, the last downward jump of Δ^k in $[0,T]$. So, if we want $\Delta_T^k = m_k$, then we cannot have any upward jump in $[\tau^d, T]$. Therefore, $\{\Delta_T^k = m_k\} = \{N_{\tau^d}^{k,u} = N_T^{k,u}\}$. Otherwise, if $\{\tau^d = 0\}$ occurs, then it implies that Δ^k has no downward jump in $[0,T]$, so that $\mathbb{P}\left[\Delta_T^k = m_k, \tau^d = 0|\Delta_0^k = M_k\right] = 0$. Hence, we have

$$\begin{aligned}&\mathbb{P}\left[\Delta_T^k = m_k|\Delta_0^k = M_k\right] \\ =\ &\mathbb{E}\left[\mathbb{P}\left(N_{\tau^d}^{k,u} = N_T^{k,u}|\tau^d\right)\mathbf{1}_{\tau^d>0}\right] \\ =\ &\mathbb{E}\left[\mathbb{P}\left(N_{T-\tau^d}^{k,u} = 0|T-\tau^d\right)\mathbf{1}_{T-\tau^d<T}\right] \quad \text{(by stationarity of } N^{k,u}\text{)} \\ =\ &\mathbb{E}\left[e^{-\lambda_k^u(T-\tau^d)}\mathbf{1}_{T-\tau^d<T}\right],\end{aligned}$$

where in the last step, we have used the independence of $N^{k,u}$ and $N^{k,d}$ and the fact that for the Poisson process $N^{k,u}$, the number of events in any interval t is Poisson distributed with mean $(\lambda_k^u t)$. Now, since the process $\left(N_T^{k,d} - N_{T-t}^{k,d}\right)_{t\geq 0}$ has the same law as the process $\left(N_t^{k,d}\right)_{t\geq 0}$, the random variable $T - \tau^d$ has the same law as $T_1^d \wedge T$, where T_1^d is the first jump time of $\left(N_t^{k,d}\right)_{t\geq 0}$. We also recall that the waiting time T_1^d has exponential law with parameter λ_k^d. Hence,

$$\begin{aligned}
\mathbb{P}\left[\Delta_T^k = m_k | \Delta_0^k = M_k\right] &= \mathbb{E}\left[e^{-\lambda_k^u T_1^d \wedge T}\mathbf{1}_{T_1^d < T}\right] \\
&= \mathbb{E}\left[e^{-\lambda_k^u T_1^d}\mathbf{1}_{T_1^d < T}\right] \\
&= \int_0^T \lambda_k^d e^{-\left(\lambda_k^u + \lambda_k^d\right)T_1^d} dT_1^d \\
&= \frac{\lambda_k^d}{\lambda_k^d + \lambda_k^u}\left(1 - e^{-\left(\lambda_k^d + \lambda_k^u\right)T}\right).
\end{aligned}$$

By stationarity, we obtain the following expression:

$$\mathbb{P}\left[\Delta_T^k = m_k | \Delta_t^k = M_k\right] = \frac{\lambda_k^d}{\lambda_k^d + \lambda_k^u}\left(1 - e^{-\left(\lambda_k^d + \lambda_k^u\right)(T-t)}\right), \quad k = 1, 2. \tag{3.27}$$

Similarly, for $k = 1, 2$

$$\mathbb{P}\left[\Delta_T^k = M_k | \Delta_t^k = m_k\right] = \frac{\lambda_k^u}{\lambda_k^d + \lambda_k^u}\left(1 - e^{-\left(\lambda_k^d + \lambda_k^u\right)(T-t)}\right). \tag{3.28}$$

$$\mathbb{P}\left[\Delta_T^k = m_k | \Delta_t^k = m_k\right] = \frac{\lambda_k^d}{\lambda_k^d + \lambda_k^u}\left(1 - e^{-\left(\lambda_k^d + \lambda_k^u\right)(T-t)}\right) + e^{-\lambda_k^u (T-t)}. \tag{3.29}$$

$$\mathbb{P}\left[\Delta_T^k = M_k | \Delta_t^k = M_k\right] = \frac{\lambda_k^u}{\lambda_k^d + \lambda_k^u}\left(1 - e^{-\left(\lambda_k^d + \lambda_k^u\right)(T-t)}\right) + e^{-\lambda_k^d (T-t)}. \tag{3.30}$$

The computation for the conditional probability $\mathbb{Q}_T^{\pi(i)}[\pi_T = \pi | \mathcal{F}_t^W]$ for any $\pi \in \Pi_2$ and any $i = 1, 2$ is exactly identical to the computation discussed in section (3.3.4).

We can now put everything together to obtain the following price for a forward contract, at time t, with maturity T_1 and delivery period $[T_1, T_2]$:

$$F_t(T_1, T_2) = \frac{1}{T_2 - T_1} \sum_{\pi \in \Pi_2} \int_{T_1}^{T_2} \left(A_1(t,T) + A_2(t,T)\right) dT, \tag{3.31}$$

where

$$\begin{aligned}
A_1(t,T) &= \sum_{\{x_1 = m_{\pi(1)}, M_{\pi(1)}\}} F_t^{\pi(1)}(T)\mathbb{Q}_T^{\pi(1)}\left[\pi_t = \pi | \mathcal{F}_t^W\right] \mathbb{P}\left[\Delta_T^{\pi(1)} = x_1 | \Delta_t\right] \\
&\qquad \times \mathbb{Q}_T\left[D_T \leq x_1 | \mathcal{F}_t^0\right], \\
A_2(t,T) &= \sum_{\substack{\{x_1 = m_{\pi(1)}, M_{\pi(1)}; \\ x_2 = m_{\pi(2)}, M_{\pi(2)}\}}} F_t^{\pi(2)}(T)\mathbb{Q}_T^{\pi(2)}\left[\pi_t = \pi | \mathcal{F}_t^W\right] \mathbb{P}\left[\Delta_T^{\pi(1)} = x_1, \Delta_T^{\pi(2)} = x_2 | \Delta_t\right] \\
&\qquad \times \mathbb{Q}_T\left[x_1 < D_T \leq x_1 + x_2 | \mathcal{F}_t^0\right],
\end{aligned}$$

where the probabilities $\mathbb{Q}_T\left[D_T \leq x_1 | \mathcal{F}_t^0\right]$ and $\mathbb{Q}_T\left[x_1 < D_T \leq x_1 + x_2 | \mathcal{F}_t^0\right]$ are given by equations (3.19) and (3.20) respectively, the probabilities $\mathbb{P}\left[\Delta_T^k = x | \Delta_t^k\right]$ are given by equations (3.27)-(3.30) and

$$\begin{aligned}
\mathbb{Q}_T^{\pi(i)}\left[\pi_T = \pi | \mathcal{F}_t^W\right] &= 1 - \Phi\left(\frac{m(t)}{\gamma(t)}\right), \\
m(t) &= \ln\left(\frac{S_t^{\pi(1)}}{S_t^{\pi(2)}}\right) - \left(\frac{1}{2}\|\sigma^{\pi(1)} - \sigma^{\pi(2)}\|^2 - (\sigma^{\pi(1)} - \sigma^{\pi(2)}).\sigma^{\pi(i)}\right)(T-t), \\
\gamma^2(t) &= \|\sigma^{\pi(1)} - \sigma^{\pi(2)}\|^2 (T-t).
\end{aligned} \tag{3.32}$$

Chapter 4

A comparison between the two models

4.1 Introduction

In this chapter, we explore the connection between the two models discussed in chapters (2) and (3). We will use Barlow's model to price electricity forwards and derivatives. Because there is no analytic formula for expectations of the form $\mathbb{E}\left[P_T|\mathcal{F}_t\right]$ under Barlow's model, we will have to resort to numerical procedures to obtain actual prices.

4.2 Pricing forwards and derivatives using Barlow's model

Barlow's model can be viewed as a continuous version of the model proposed by Aid et al. (2009). Instead of considering only n technologies for the production of electricity, we will look a continuum of technologies s where $0 < \underline{s} \leq s \leq \bar{s}$. We denote by s, the price of technology/fuel s and by $\varphi(s)$, the density (or capacity as in chapter (3)) of technology s.

If the price of electricity at time t is P_t, it implies that all technologies with $s \leq P_t$ are being used to meet the demand for electricity. Thus, we have, at time t, a global supply of electricity given by

$$u_t(P_t) = g(P_t) = \int_{\underline{s}}^{P_t} \varphi(h)dh. \tag{4.1}$$

4.2 Pricing forwards and derivatives using Barlow's model

The only source of uncertainty in the model is the electricity demand D_t. As in previous chapters, we will again assume that this demand process follows a constant-coefficient mean-reverting Ornstein-Uhlenbeck process described by the stochastic differential equation

$$dD_t = a_1(b_1 - D_t) + \sigma_1 dW_t^{\mathbb{P}},$$

where $\mathbb{P}$ is the objective probability.

Now, if we assume the form $g(x) = a_0 - b_0 x^\alpha$ for the supply curve as in equation (2.2), the supply-demand equilibrium yields the following relation

$$\begin{aligned} P_t &= g^{-1}(D_t) \\ &= \left(\frac{a_0 - D_t}{b_0}\right)^{1/\alpha}. \end{aligned}$$

Then, following the transformation discussed by Barlow, (equations (2.4)-(2.6)), we get

$$\begin{aligned} P_t &= (1 + \alpha X_t)^{1/\alpha}, \quad \alpha \neq 0, \\ dX_t &= a(b - X_t)dt + \sigma dW_t^{\mathbb{P}}. \end{aligned} \tag{4.2}$$

We note that electricity cannot be stored but fuel can. This brings us back to the idea behind the model proposed by Aid et al. (2009), that is, we can still price electricity forwards and derivatives by no-arbitrage on the market of fuels, thus avoiding the electricity non-storability restriction on electricity. Thus, we can proceed to price electricity forwards and derivatives under the risk-neutral probability $\mathbb{Q}$ defined on the market of fuels. To do that, we first derive the $\mathbb{Q}$-equation for dP_t.

We set $f(X_t) = (1+\alpha X_t)^{1/\alpha}$ and we apply Ito's formula. This gives

$$\begin{aligned} dP_t = df &= \left[a(b-X_t)f_X + \frac{1}{2}\sigma^2 f_{XX}\right]dt + \sigma f_X dW_t^{\mathbb{P}} \\ &= \left[a(b-X_t)(1+\alpha X_t)^{\frac{1-\alpha}{\alpha}} + \frac{1}{2}\sigma^2(1-\alpha)(1+\alpha X_t)^{\frac{1-2\alpha}{\alpha}}\right]dt + \sigma(1+\alpha X_t)^{\frac{1-\alpha}{\alpha}} dW_t^{\mathbb{P}} \\ &= \left[a\left(b - \frac{P_t^\alpha - 1}{\alpha}\right)P_t^{1-\alpha} + \frac{1}{2}\sigma^2(1-\alpha)P_t^{1-2\alpha}\right]dt + \sigma P_t^{1-\alpha} dW_t^{\mathbb{P}}. \end{aligned}$$

The $\mathbb{Q}$-equation for dP_t is then obtained by replacing the drift term by $r_t P_t dt$ in the $\mathbb{P}$-equation. Thus,

$$dP_t = r_t P_t dt + \sigma P_t^{1-\alpha} dW_t^{\mathbb{Q}}. \tag{4.3}$$

Remark

From this continuous model, we can get closer to the model proposed by Aid et al. (2009) by making the following modifications:

- We take n points $s_1, \ldots, s_n$. At each of the s_i, we define the density $\varphi(s)$ such that it is close to the dirac masses, that is,

$$\begin{aligned} \varphi(s) &= 0 \text{ unless } |s - s_i \leq \epsilon \text{ for some } i|, \\ \int_{s_i-\epsilon}^{s_i+\epsilon} \varphi(h)dh &= 1. \end{aligned}$$

  This leads to model with a finite number of technologies.

- We introduce n factors X_i defined by the diffusion

$$dX_i = \mu_i(X,t)dt + \sum_{j=1}^{k} \sigma_{ij}(X,t)dW_j,$$

  and set the price of fuel to be $c(s,X)$ instead of s. Then, equation (4.1) becomes

$$g(P_t, X) = \int_{\underline{S}}^{P_t} \varphi(h)c(h,X)dh.$$

Thus, we have a second source of noise.

Forward Prices

To price forwards on electricity, we use the formula given by equation (3.9). Thus, for a forward contract on electricity with maturity $T_1 > 0$ and delivery period $[T_1, T_2]$ for $T_1 < T_2$, the time-t price is given by

$$F_t(T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u du} P_T | \mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u du} | \mathcal{F}_t\right]} dT.$$

If we assume the interest rate to be constant, this formula simplifies to

$$\begin{aligned} F_t(T_1, T_2) &= \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \mathbb{E}^{\mathbb{Q}}\left[P_T | \mathcal{F}_t\right] dT \\ dP_t &= rP_t dt + \sigma P_t^{1-\alpha} dW_t^{\mathbb{Q}}. \end{aligned} \tag{4.4}$$

Option prices

To price options, we use the following formulas:
For a call option on electricity with maturity T and strike price K, the option price at time $0 \leq t < T$ is given by:

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \max\left(P_T - K, 0\right) | \mathcal{F}_t\right]. \tag{4.5}$$

Similarly, for a put option on electricity with maturity T and strike price K, the option price at time $0 \leq t < T$ is given by:

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \max\left(K - P_T, 0\right) | \mathcal{F}_t\right]. \tag{4.6}$$

4.3 Numerical Simulation

We describe the numerical procedure for simulating prices for electricity forwards and derivatives and give some simulation results.

4.3.1 Forward prices

To simulate forward prices, we divide the delivery period $[T_1, T_2]$ into n intervals $T_1 = t_1 < t_2 < \ldots < t_{n+1} = T_2$. We calculate the expectation $\mathbb{E}^{\mathbb{Q}}\left[P_{t_i}|\mathcal{F}_t\right]$ for each $t_i, i = 1, \ldots, n+1$, using the following procedure:

1. Divide the interval (t, t_i) into N equal time steps $h = \frac{t_i - t}{N}$.

2. Simulate the paths for P_{t_i} using the following time-stepping equation.

$$P_k = P_{k-1}\left[1 + rh + \sigma P_{k-1}^{-\alpha}\sqrt{h}Z_k\right], \quad \text{for} \quad k = 2, \ldots, N+1, \tag{4.7}$$

   where Z_k are independent random draws from a standard normal distribution. This time-stepping scheme is an Euler discretization scheme and is essential in this case because we do not know the exact solution to the stochastic differential equation (4.3) (Glasserman 2004). We note that $P_1 = P_t$ is known at time t and that this recursion generates $P_{t_i} = P_{N+1}$ from $P_1 = P_t$.

3. Repeat step (2) for M runs to get $P_{N+1}^1, \ldots, P_{N+1}^M$.

4. $\mathbb{E}^{\mathbb{Q}}[P_{t_i}|\mathcal{F}_t]$ is then given by the statistical mean:

$$f(t_i) = \mathbb{E}^{\mathbb{Q}}[P_{t_i}|\mathcal{F}_t] = \frac{1}{M}\sum_{k=1}^{M} P_{N+1}^k$$

We then calculate the forward price using the trapezoidal rule.

$$F_t(T_1, T_2) = \frac{1}{2n}\left[f(t_1) + f(t_{n+1}) + 2\sum_{i=2}^{n} f(t_i)\right].$$

We implement this procedure in MATLAB (Refer to Appendix B.2 for code). Figure 4.1 shows the forward prices $F_t(T_1, T_2)$ obtained for some fictitious parameter values.

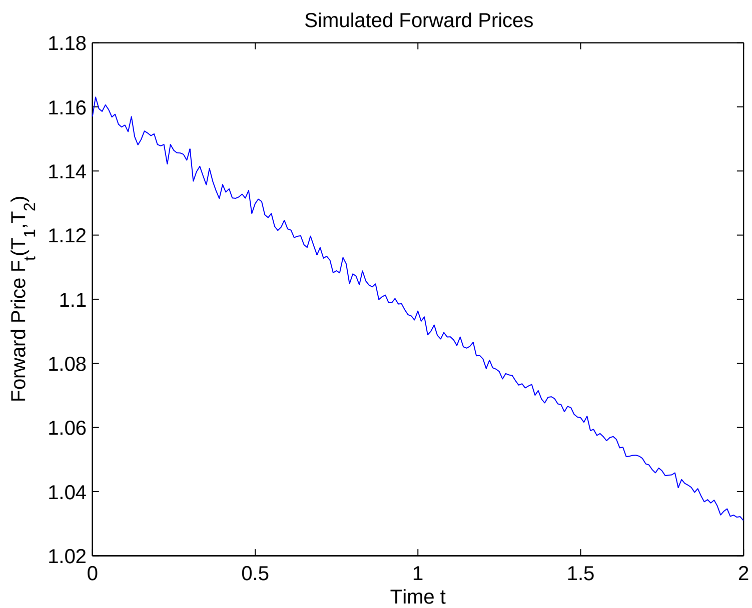

Figure 4.1: Simulation of forward prices for parameter values $\sigma = 0.12, \alpha = -1.08, T_1 = 2, T_2 = 3, r = 0.06$

4.3.2 Option prices

To calculate option prices, we use the following numerical procedure.

1. Divide the interval (t, T) into N equal time steps $h = \frac{T-t}{N}$.

2. Simulate the paths for P_T using the Euler discretization scheme.

$$P_k = P_{k-1}\left[1 + rh + \sigma P_{k-1}^{-\alpha}\sqrt{h}Z_k\right], \quad \text{for} \quad k = 2, \dots, N+1, \tag{4.8}$$

   where Z_k are independent random draws from a standard normal distribution. This generates $P_T = P_{N+1}$ from $P_1 = P_t$.

3. Repeat step (2) for M runs to get $P^1_{N+1}, \ldots, P^M_{N+1}$.

Then, the price for a call option with strike K is given by

$$V_t = e^{-r(T-t)} \left[\frac{1}{M} \sum_{i=1}^{M} \max(P^i_N - K, 0) \right], \tag{4.9}$$

and the price for a put option with strike K is given by

$$V_t = e^{-r(T-t)} \left[\frac{1}{M} \sum_{i=1}^{M} \max(K - P^i_N, 0) \right]. \tag{4.10}$$

We implement this procedure in MATLAB (Refer to Appendix B.3 for code). Figures 4.2 and 4.3 show the call and put prices obtained for some fictitious parameter values.

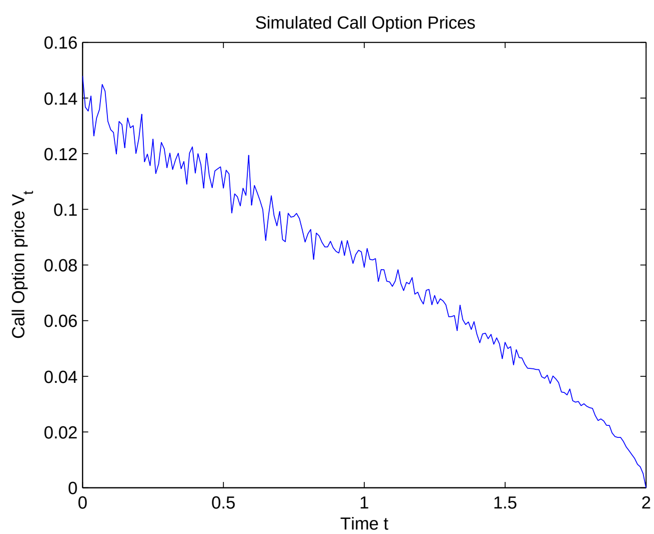

Figure 4.2: Simulation of call prices for parameter values $\sigma = 0.12, \alpha = -1.08, T = 1; r = 0.06, K = 1$

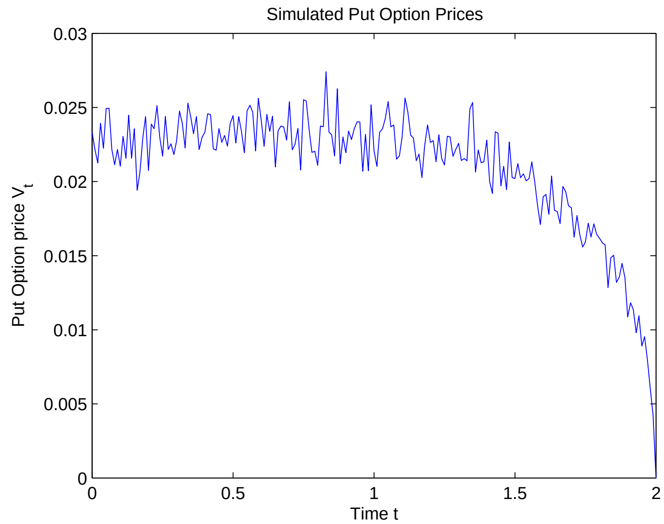

Figure 4.3: Simulation of put prices for parameter values $\sigma = 0.12, \alpha = -1.08, T = 1; r = 0.06, K = 1$

Chapter 5

Conclusion

We have presented two models for the pricing of electricity and we have shown how the model Barlow (2002) can be viewed as a continuous version of the model Aid et al. (2009).

We need to point out that throughout the essay, we have made simplifying assumptions which do not hold in the real world. In order to make the models more realistic and more suitable to be fitted to real data, we need to consider several extensions:

- The constant interest rate r must be replaced by a stochastic process r_t.
- Seasonal effects must be incorporated into the demand process. This can be done by replacing the long-term constant mean in the Ornstein-Uhlenbeck process characterizing the demand process by a sinusoidal time-varying function (Barlow 2002).
- The single factor model of Barlow must be extended to multifactor models by introducing more sources of noise along the lines discussed in section (4.2).

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Appendix A

The Ornstein-Uhlenbeck process

A stochastic process X_t which satisfies the stochastic differential equation

$$dX_t = a(b - X_t)dt + \sigma dW_t \tag{A.1}$$

is called an Ornstein-Uhlenbeck process. The parameters a, b and σ represent the speed of mean reversion, the long-term mean and the volatility respectively. These parameters can be time-varying.

A.1 Solution to the stochastic differential equation

We solve equation (A.1) in the case of constant coefficients a, b and σ. We set $f(t, X_t) = X_t e^{at}$ and apply Ito's formula to f. This gives

$$\begin{aligned} df(t, X_t) &= \left[aX_t e^{at} + a(b - X_t)e^{at}\right] dt + \sigma e^{at} dW_t \\ &= abe^{at}dt + \sigma e^{at} dW_t, \end{aligned}$$

which can be directly integrated. Then, the solution to equation (A.1), for any $0 \leq t \leq T$, is given by

$$X_T e^{aT} - X_t e^{at} = \int_t^T abe^{as} ds + \int_t^T \sigma e^{as} dW_s,$$

which simplifies to

$$X_T = b + (X_t - b)\, e^{-a(T-t)} + \sigma \int_t^T e^{-a(T-s)} dW_s. \tag{A.2}$$

For any fixed t and T, the random variable X_T, conditional upon X_t, is normally distributed with mean and variance given by

$$\begin{aligned}
\mathbb{E}\left[X_T | X_t = x\right] &= b + (x-b)e^{-a(T-t)}, && \text{(A.3)} \\
Var\left[X_T | X_t = x\right] &= \mathbb{E}\left[\sigma^2 \left(\int_t^T e^{-a(T-s)} dW_s\right)^2\right], \\
&= \sigma^2 \mathbb{E}\left[\int_t^T e^{-2a(T-s)} ds\right], \quad \text{(Ito's isometry)} \\
&= \frac{\sigma^2}{2a}\left(1 - e^{-2a(T-t)}\right). && \text{(A.4)}
\end{aligned}$$

Remark: Case of time-varying long-term mean b=b(t)

When $b = b(t)$, the solution to the stochastic differential equation is given by

$$X_T = e^{-a(T-t)} X_t + a \int_t^T e^{-a(T-s)} b(s) ds + \sigma \int_t^T e^{-a(T-s)} dW_s, \tag{A.5}$$

with conditional mean and variance given by

$$\begin{aligned}
\mathbb{E}\left[X_T | X_t = x\right] &= e^{-a(T-t)} x + a \int_t^T e^{-a(T-s)} b(s) ds, && \text{(A.6)} \\
Var\left[X_T | X_t = x\right] &= \frac{\sigma^2}{2a}\left(1 - e^{-2a(T-t)}\right). && \text{(A.7)}
\end{aligned}$$

Appendix B

MATLAB codes for the simulations

B.1 Code for simulating Barlow's NLOU process

```
function nlou(par)
%Simulation of a non-linear Ornstein-Uhlenbeck process
%Barlow's model
%dX=a*(b-X)*dt+sigma*dW
%P=(1+alpha*X)^(1/alpha) if 1+alpha*X>epsilon
%P=epsilon^(1/alpha) if 1+alpha*X<=epsilon
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
a=par(1);b=par(2); sigma=par(3); alpha=par(4); epsilon=par(5);
N=1000;%number of time steps
T=10;%time interval
dt=T/N;
tvec=[1:N];
X(1)=1;%X_0
%time stepping algorithm
for i=2:N
    X(i)=exp(-a*dt)*X(i-1)+b*(1-exp(-a*dt))+sigma*...
    ...sqrt((1-exp(-2*a*dt))/(2*a))*randn(1);
end
for i=1:N
    if 1+alpha*X(i)>epsilon
        P(i)=(1+alpha*X(i))^(1/alpha);
    else
        P(i)=epsilon^(1/alpha);
    end
end
%generate the plot of the simulated NLOU
plot(tvec,P) xlabel('Time index') ylabel('Price P_t')
title('Simulation of the NLOU process P_t')
```

B.2 Code for simulating forward prices under Barlow's model

```
function [forwd]=forwardprice(par)
%Forward prices simulation
%Barlow's model
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
sig=par(1); alpha=par(2);
t=par(3); %time at which forward price is being calculated
T1=par(4);%start of delivery period
T2=par(5);%end of delivery period
N=par(6);%number of time steps
M=par(7);%number of samples
r=par(8);%interest rate
S=par(9);%S at time t
%discretisation for numerical integration
nint=10; h=(T2-T1)/nint; tvec=[T1:h:T2];
%Monte Carlo simulations
for i=1:nint+1
    dt=(tvec(i)-t)/N;
    St=repmat(S,1,M);
    for n=1:N %time stepping
        e=1+r*dt+sig*sqrt(dt)*(St.^(-alpha)).*randn(1,M);
        St=St.*e;
    end
    meanst(i)=mean(St);
end
%numerical integration trapezoidal rule
forwd=(0.5*(meanst(1)+meanst(nint+1))+sum(meanst(2:nint)))/(nint);
```

B.3 Code for simulating option prices under Barlow's model

```
function [call]=optionprice(par)
%Call option prices simulation
%Barlow's model
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
sig=par(1); alpha=par(2);
t=par(3);%time at which option price is being calculated
N=par(4);%number of time steps
M=par(5);%number of samples
r=par(6);%interest rate
S=par(7);%S at time t
K=par(8);%strike price
T=par(9);%maturity
%Monte Carlo Simulations
dt=(T-t)/N;
St=repmat(S,1,M);
    for n=1:N %time stepping
        e=1+r*dt+sig*sqrt(dt)*(St.^(-alpha)).*randn(1,M);
        St=St.*e;
    end
Ct=max(zeros(1,M),St-repmat(K,1,M))*exp(-r*(T-t));
call=mean(Ct);
```

```
function [put]=optionprice2(par)
%Put option prices simulation
%Barlow's model
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
sig=par(1); alpha=par(2);
t=par(3);%time at which option price is being calculated
N=par(4);%number of time steps
M=par(5);%number of samples
r=par(6);%interest rate
S=par(7);%S at time t
K=par(8);%strike price
T=par(9);%maturity
%Monte Carlo Simulations
dt=(T-t)/N; St=repmat(S,1,M);
    for n=1:N %time stepping
        e=1+r*dt+sig*sqrt(dt)*(St.^(-alpha)).*randn(1,M);
        St=St.*e;
    end
Pt=max(zeros(1,M),repmat(K,1,M)-St)*exp(-r*(T-t));
put=mean(Pt);
```