[10] **1.** Find the area between the curves $y = x^2$ and $y = \sqrt{x}$ over the interval $[0, 4]$.

Solution

The curves intersect when $x^2 = \sqrt{x}$

i.e. $x^4 = x$

$\Rightarrow x(x^3 - 1) = 0$

$\Rightarrow x = 0$ or $x = 1$.

(1) In the interval $0 \le x \le 1$, $x^2 < \sqrt{x}$

(2) " " " $1 \le x \le 4$, $x^2 > \sqrt{x}$

To show (1), $x \le 1 \Rightarrow x^2 \le x$.
but $x \le \sqrt{x}$ for $x \le 1$ & so $x^2 \le x \le \sqrt{x}$

Hence the area between the curves is:

$$\text{Area} = \int_0^1 [\sqrt{x} - x^2]\,dx + \int_1^4 [x^2 - \sqrt{x}]\,dx$$

$$= \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{3}x^3\right]_0^1 + \left[\tfrac{1}{3}x^3 - \tfrac{2}{3}x^{3/2}\right]_1^4$$

$$= \left(\tfrac{2}{3} - \tfrac{1}{3}\right) + \left(\tfrac{64}{3} - \tfrac{16}{3}\right) - \left(\tfrac{1}{3} - \tfrac{2}{3}\right)$$

$$= \tfrac{2}{3} + 16$$

$$= 16\,{}^{2}/_{3}.$$

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[12] **2.**

(a) If $g(t) = \ln t + \int_0^t \frac{x}{1+x^4}\, dx$, then find $g'(1)$.

(b) Evaluate the following definite integral: $\int_e^{e^2} \frac{\ln x}{x}\, dx$.

[Note: these two parts are **not** related.]

Solution

2(a) write $g(t) = g_1(t) + g_2(t)$ where $g_1(t) = \ln t$

$g_2(t) = \int_0^t \frac{x}{1+x^4}\, dx$

Now, $g_1'(t) = \frac{1}{t}$

& $g_2'(t) = \frac{t}{1+t^4}$, by Fundamental Theorem of Calculus

Thus $g'(t) = \frac{1}{t} + \frac{t}{1+t^4}$

$\Rightarrow g'(1) = 1 + \frac{1}{2} = 3/2$.

(b) Use substitution: Let $u = \ln x$

$\therefore du = \frac{1}{x}\, dx$

when $x = e$, $u = \ln e = 1$

when $x = e^2$, $u = \ln e^2 = 2\ln e = 2$

$$\therefore \int_e^{e^2} \frac{\ln x}{x}\, dx = \int_1^2 u\, du = \left[\frac{1}{2}u^2\right]_1^2 = 3/2.$$

or Let $u = \ln x$, $du = \frac{1}{x}\, dx$.

$$\int \frac{\ln x}{x}\, dx = \int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$$

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$$\therefore \int_e^{e^2} \frac{\ln x}{x}\, dx = \left[\frac{1}{2}(\ln x)^2\right]_e^{e^2} = \frac{1}{2}(\ln e^2)^2 - \frac{1}{2}(\ln e)^2 = \frac{1}{2}(2)^2 - \frac{1}{2}(1)^2 = 3/2$$

[8] **3.** The amount of medicine being continuously injected into a patient's bloodstream is given by the formula $q(t) = 3t^2 + \frac{4}{2t+1} + e^{-0.5t}$, where t is the time in hours and $q(t)$ is the dose in milligrams. What is the average number of milligrams in the patient's blood stream over the first 2 hours?

<u>Solution</u>

Average value $= \frac{1}{2}\int_0^2 q(t)\,dt$

$$= \frac{1}{2}\int_0^2 \left(3t^2 + \frac{4}{2t+1} + e^{-0.5t}\right) dt$$

$$= \frac{1}{2}\left[t^3 + 2\ln|2t+1| - 2e^{-0.5t} \right]_0^2$$

$$= \frac{1}{2}\left[8 + 2\ln 5 - 2e^{-1}\right] - \frac{1}{2}\left[-2\right]$$

$$= 5 + \ln 5 - e^{-1}.$$

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[10] 4. Approximate the definite integral $\int_{-1}^{2} \frac{1}{2x+3}\,dx$ using the midpoint rule with $n = 3$.

Solution

$-1 \leq x \leq 2$ and so $\Delta x = \frac{2-(-1)}{3} = 1$.

Since $n = 3$, the midpoint rule uses three points:

$$x_1 = -\tfrac{1}{2}, \quad x_2 = \tfrac{1}{2}, \quad x_3 = 3/2.$$

Hence

$$\int_{-1}^{2} \frac{1}{2x+3}\,dx \approx \Delta x\left[f(x_1) + f(x_2) + f(x_3)\right]$$

where $f(x) = \frac{1}{2x+3}$.

Thus

$$f(x_1) = \frac{1}{-1+3} = \frac{1}{2}$$

$$f(x_2) = \frac{1}{1+3} = \frac{1}{4}$$

$$f(x_3) = \frac{1}{3+3} = \frac{1}{6}.$$

$$\therefore \int_{-1}^{2} \frac{1}{2x+3}\,dx \approx \frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{11}{12}.$$

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[10] **5.** Claudia's bakery estimates that the supply curve for its Mediterraneo bread is $p = 25 + q^2$, while the demand curve is $p = 75 - 5q$, where the quantity q is measured in hundreds of loaves and the price p is in dollars. Find its producer surplus.

Solution

The supply curve is $S(q) = 25 + q^2$
& the demand curve is $D(q) = 75 - 5q$

First find the equilibrium value q_e which satisfies

$$25 + q^2 = 75 - 5q$$

$$\Rightarrow q^2 + 5q - 50 = 0$$

$$\Rightarrow (q-5)(q+10) = 0$$

$\therefore q_e = 5$ is the unique non-negative equilibrium value

& so $p_e = S(q_e)$ (or $D(q_e)$)

$$= 25 + 5^2$$

$$= 50.$$

The producer surplus satisfies

$$PS = p_e q_e - \int_0^{q_e} S(q)\,dq$$

$$= 250 - \int_0^5 (25 + q^2)\,dq$$

$$= 250 - \left[25q + \tfrac{1}{3}q^3\right]_0^5$$

$$= 250 - \left(125 + \tfrac{125}{3}\right)$$

$$= 83\tfrac{1}{3}.$$

The End