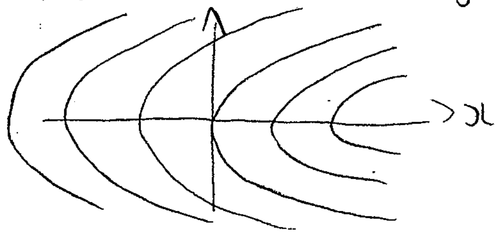
Math105: Solutions to Practice Questions

Chapter 7

(1) Distance $= \sqrt{(-1-2)^2 + (-2+1)^2 + (-6+2)^2} = \sqrt{26}$

(2) A level curve of $f(x,y) = x - y^2$, with height z_0, has the equation $x - y^2 = z_0$, or $x = z_0 + y^2 \Rightarrow$ parabola.



z_0 is point where it hits the x-axis.

(3) Apply chain rule:

$$\frac{\partial f}{\partial x} = ye^{x^2+y^2}\frac{\partial}{\partial x}(x^2+y^2) + 6x = 2xye^{x^2+y^2} + 6x$$

(4) First find the critical points of the revenue function, & so we set $\frac{\partial R}{\partial x} = 30 - x = 0$ & $\frac{\partial R}{\partial y} = 20 - y = 0$

The solution is $x = 30$ & $y = 20$ & so only critical pt is (30,2

Computing second derivatives:

$$\frac{\partial^2 R}{\partial x^2} = -1, \quad \frac{\partial^2 R}{\partial y^2} = -1, \quad \frac{\partial^2 R}{\partial x \partial y} = 0$$

& so $D(x,y) = \left(\frac{\partial^2 R}{\partial x^2}\right)\left(\frac{\partial^2 R}{\partial y^2}\right) - \left(\frac{\partial^2 R}{\partial x \partial y}\right)^2 = 1 > 0$

Hence since $D > 0$ & $\frac{\partial^2 R}{\partial x^2} < 0$, $(30,20)$ is a local max.

Also since R is quadratic, it must be a global max.

(5) $f(x,y) = x^3 + y^3 - 3xy$. Now

$\frac{\partial f}{\partial x} = 3x^2 - 3y$, $\frac{\partial f}{\partial y} = 3y^2 - 3x$ & $\frac{\partial^2 f}{\partial x^2} = 6x$, $\frac{\partial^2 f}{\partial y^2} = 6y$, $\frac{\partial^2 f}{\partial x \partial y} = -3$.

Therefore $D = (6x)\cdot(6y) - (-3)^2 = 36xy - 9$.

Since $D(1,1) = 36 - 9 = 27 > 0$ & $\frac{\partial^2 f}{\partial x^2}(1,1) = 6 > 0$

we conclude f has a local min at $(1,1)$.

(6) Need to maximize $Q(K,L) = 50K^{0.3}L^{0.7}$ subject to the constraint $P(K,L) = 60K + 20L - 800{,}000 = 0$.

Thus we set $\frac{\partial Q}{\partial K} = \lambda \frac{\partial P}{\partial K}$ or $15K^{-0.7}L^{0.7} = 60\lambda$

$\frac{\partial Q}{\partial L} = \lambda \frac{\partial P}{\partial L}$ or $35K^{0.3}L^{-0.3} = 20\lambda$

Divide second by 1st, i.e. $\frac{35K^{0.3}L^{-0.3}}{15K^{-0.7}L^{0.7}} = \frac{1}{3}$

$\Rightarrow \frac{7}{3}KL^{-1} = \frac{1}{3} \Rightarrow L = 7K.$

subst into $60K + 20L = 800,000 \Rightarrow 60K + 140K = 800,000$

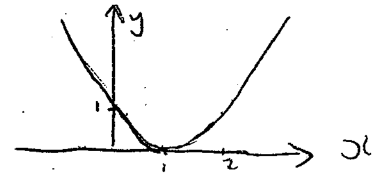
$\Rightarrow K = 4000.$

& so $L = 28,000.$

(7) setting $f(x,y) = -x^2 + 2x + y = 1$ we have $y = x^2 - 2x + 1$, or

$y = (x-1)^2$

i.e. a parabola with minimum at $x=0$



(8) Partial Derivatives are $\frac{\partial f}{\partial x} = \ln y + ye^x$ & $\frac{\partial f}{\partial y} = \frac{x}{y} + e^x.$

Then $\frac{\partial f}{\partial x}(0,1) = \ln 1 + e^0 = 1$ & $\frac{\partial f}{\partial y}(0,1) = 1$ & $f(0,1) = 1$

$\therefore f(x,y) \approx f(0,1) + (x-0)\frac{\partial f}{\partial x}(0,1) + (y-1)\frac{\partial f}{\partial y}(0,1)$

$= \cancel{1} + x + (y - \cancel{1})$

$\Rightarrow f(x,y) \approx x + y$ is (x,y) is close to $(0,1)$.

Thus $f(0.05, 0.9) \approx 0.05 + 0.9 = 0.95.$

(9) $f(x,y) = \ln(x^2+y^2+3)$. Thus $\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2+3} = 2y(x^2+y^2+3)^{-1}$

$\therefore \frac{\partial^2 f}{\partial x \partial y} = -2y(x^2+y^2+3)^{-2}.2x = \frac{-4xy}{(x^2+y^2+3)^2}.$

(10) set $\frac{\partial f}{\partial x} = -2x + 6y = 0$ & $\frac{\partial f}{\partial y} = 6x - 2y + 16 = 0$

Hence $x = 3y \Rightarrow 18y - 2y = 16 \Rightarrow y = 1$ & so $x = 3$.

$\therefore$ the only critical point is $(3,1)$.

(11) $f(x,y) = x^2 + y^2$. Write $g(x,y) = 2x - y^2 - 2$.
Setting $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$ & $\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$ gives

$$\begin{cases} 2x = 2\lambda \\ 2y = -2\lambda y \\ 2x - y^2 = 2 \end{cases}$$

From 2nd eqn, either $y = 0$ or $\lambda = -1$.

- If $\lambda = -1$, then from 1st eqn $x = -1$. But 3rd eqn is then $-2 - y^2 = 2 \Rightarrow y^2 = -4$. impossible.
- If $y = 0$, then from 3rd eqn $x = 1$

Thus the global minimium occurs at $(1,0)$, with $f(1,0) = 1$.

(2) The marginal productivity function of labour is

$$\frac{\partial Q}{\partial L} = 2K^{0.5}L^{-0.5}$$

At the pt $K = 25, L = 16$ thus equals $\frac{2\sqrt{25}}{\sqrt{16}} = 5/2$.

Chapter 5

(1) use integration by parts: $u = t$ & $v' = e^{t/2}$
$\therefore u' = 1$ & $v = 2e^{t/2}$

$$\therefore \int_0^2 te^{t/2}\,dt = [2te^{t/2}]_0^2 - 2\int_0^2 e^{t/2}\,dt$$
$$= 4e^1 - 4[e^{t/2}]_0^2 = 4e^1 - 4e^1 + 4 = 4.$$

(2) Integrate $\frac{dy}{dt} = \frac{1}{t} - \frac{1}{t-1} \Rightarrow y = \ln|t| - \ln|t-1| + c$

$y(2) = \ln 2 + 1$ & so $\ln 2 + 1 = \ln 2 - \ln 1 + c = \ln 2 + c$

$\therefore c = 1$

$\therefore y = \ln|t| - \ln|t-1| + 1$ & so $y(3) = \ln 3 - \ln 2 + 1$.

(3) Since $n = 5$ $\Delta x = \frac{1-0}{5} = \frac{1}{5}$. Then the values of x_i are

$x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8, x_5 = 1$

Thus the trapezoidal rule gives

$$\int_0^1 \frac{x}{1+x^4}\,dx = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right]$$

$$= \frac{1}{10}\left[0 + 2 \cdot \frac{0.2}{1+0.2^4} + 2 \cdot \frac{0.4}{1+0.4^4} + 2 \cdot \frac{0.6}{1+0.6^4} + 2 \cdot \frac{0.8}{1+0.8^4} + \frac{1}{2}\right]$$

≈ 0.3877 (Actual answer is $\frac{\pi}{8} \simeq 0.3927$).

(4) Let $u = x^2 - 5x + 6$. Then $du = (2x-5)\,dx$ & when $x = 2$, $u = 2^2 - 5(2) + 6 = 0$ & when $x = 4$, $u = 4^2 - 5(4) + 6 = 2$.

$$\therefore \int_2^4 (2x-5)e^{x^2-5x+6}\,dx = \int_0^2 e^u\,du = [e^u]_0^2 = e^2 - 1.$$

(5) Integrate $R'(t) = e^{t/2} + 20t - 0.8$:

$R(t) = 2e^{t/2} + 10t^2 - 0.8t + c$

If $R(0) = 60$ then $60 = 2e^0 + 0 - 0 + c \implies c = 58$.

$\therefore R(t) = 2e^{t/2} + 10t^2 - 0.8t + 58$.

(6) Let $u = \ln x$ & $v' = x^2 + 2x + 3$ (use integration by parts).

$\therefore u' = \frac{1}{x}$ & $v = \frac{1}{3}x^3 + x^2 + 3x$

So
$$\int (x^2+2x+3)\ln x\,dx = \left(\tfrac{1}{3}x^3 + x^2 + 3x\right)\ln x - \int \tfrac{1}{x}\left(\tfrac{1}{3}x^3 + x^2 + 3x\right)dx$$
$$= \left(\tfrac{1}{3}x^3 + x^2 + 3x\right)\ln x - \int\left(\tfrac{1}{3}x^2 + x + 3\right)dx$$
$$= \left(\tfrac{1}{3}x^3 + x^2 + 3x\right)\ln x - \left(\tfrac{1}{9}x^3 + \tfrac{1}{2}x^2 + 3x\right) + c$$

(7) Use substitution: $u = x^3 + 3x^2 + 3x + 3$

$\therefore du = (3x^2 + 6x + 3)\,dx$

$\implies \frac{1}{3}du = (x^2 + 2x + 1)\,dx$

$$\therefore \int (x^2+2x+1)e^{x^3+3x^2+3x+3}\,dx = \int \tfrac{1}{3}e^u\,du = \tfrac{1}{3}e^u + c = \tfrac{1}{3}e^{x^3+3x^2+3x+3} + c$$

(8) Use partial fractions:

Let $\frac{x}{x^2+x-12} = \frac{x}{(x-3)(x+4)} = \frac{A}{x-3} + \frac{B}{x+4}$

$\therefore x = A(x+4) + B(x-3)$

- If $x = 3$: $3 = 7A \implies A = 3/7$
- If $x = -4$: $-4 = -7B \implies B = 4/7$.

Thus $\int \frac{x}{x^2+x-12}\,dx = \int \frac{3/7}{x-3}\,dx + \int \frac{4/7}{x+4}\,dx$
$= \frac{3}{7}\ln|x-3| + \frac{4}{7}\ln|x+4| + C.$

(9) Average value of $f(x)$ over $0 \le x \le 3$ is

$$Av = \frac{1}{3-0}\int_0^3 x(x^2+1)^4\,dx.$$

Let $u = x^2+1$ $\therefore du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

Also, $x = 0 \Rightarrow u = 1$, $x = 3 \Rightarrow u = 10$

$$\therefore Av = \frac{1}{3}\int_1^{10} \frac{1}{2}u^4\,du = \left[\frac{1}{30}u^5\right]_1^{10} = \frac{1}{30}(100{,}000-1) = \frac{33{,}333}{10}.$$

(10) Since $n = 4$, stepsize is $\Delta x = \frac{2-0}{4} = \frac{1}{2}$. Thus

$x_0 = 0,\ x_1 = 0.5,\ x_2 = 1,\ x_3 = 1.5,\ x_4 = 2$

Simpson's rule:

$$\int_0^2 f(x)\,dx \approx [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]\frac{\Delta x}{3}$$
$$= [-1 + 4(-0.5) + 2(1.5) + 4(2) + 1]\frac{1}{6}$$
$$= [\cancel{-1} - 2 + 3 + 8 \cancel{+1}]\frac{1}{6} = 1.5.$$

(11) The intersection points of the curves satisfy

$$3x = x^2 - x$$
$$\Rightarrow x(x-4) = 0$$

$\therefore x = 0$ or $x = 4$. $\Rightarrow$ No intersection points in $0 \le x \le 4$.

For $0 < x < 4$, $3x > x^2 - x$ & so

$$\text{Area} = \int_0^4 [3x - (x^2 - x)]\,dx = \int_0^4 (4x - x^2)\,dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \frac{32}{3}.$$

(12) This is an ivp: $P'(x) = MP(x) = -0.5x + 62$
$P(100) = 10{,}000$

$\therefore P(x) = \int(-0.5x + 62)\,dx = -0.25x^2 + 62x + C$

From i.c. $10{,}000 = -0.25(100)^2 + 62(100) + C \Rightarrow C = 6{,}300$

$\therefore P(x) = -0.25x^2 + 62x + 6{,}300$

Thus the profit of the company at $x = 110$ units per day is

$$P(110) = -0.25(110)^2 + 62(110) + 6{,}300 = \$10{,}095$$

Chapter 6

(1) Let S be the annual amount. From formula for future value of a continuous income stream, we have

$$1{,}800{,}000 = \int_0^{35} Se^{0.09(35-t)}\,dt = Se^{3.15}\int_0^{35} e^{-0.09t}\,dt$$
$$= \frac{S}{0.09}(e^{3.15}-1)$$

$$\therefore S = \frac{0.09(1800000)}{e^{3.15}-1} \approx \$7{,}253.$$

(2) If $y = at^2 + \frac{5}{t^2}$ then $y' = 2at - \frac{10}{t^3}$ & so

$$ty' + 2y = t\left[2at - \frac{10}{t^3}\right] + 2\left[at^2 + \frac{5}{t^2}\right] = 4at^2$$

$\therefore ty' + 2y = 8t^2$ if $a = 2$.

(3) Use integration by parts; let $u = t$ & $v' = e^{-t}$

$\therefore u' = 1$ & $v = -e^{-t}$

$$\therefore \int te^{-t}\,dt = -te^{-t} + \int e^{-t}\,dt = -te^{-t} - e^{-t} + C$$

$$\therefore \int_1^{\infty} te^{-t}\,dt = \lim_{h\to\infty}\left[-te^{-t} - e^{-t}\right]_1^h = \lim_{h\to\infty}\left[-he^{-h} - e^{-h} + 2e^{-1}\right] = 2e^{-1}$$

Hence the integral converges.

(4) Substitution: let $u = x^2 - 1$ & so $du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

Also when $x = 1$, $u = 0$ & when $x = 2$, $u = 3$

$$\therefore \int_1^2 \frac{x}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\int_0^3 u^{-1/2}\,du = \frac{1}{2}\lim_{h\to 0^+}\int_h^3 u^{-1/2}\,du = \frac{1}{2}\lim_{h\to 0^+}\left[2\sqrt{u}\right]_h^3$$
$$= \sqrt{3}.$$

Hence the integral converges.

(5) $$PV = \int_0^{\infty} 17{,}000e^{-0.085t}\,dt = \lim_{h\to\infty}\int_0^h 17{,}000e^{-0.085t}\,dt$$
$$= \lim_{h\to\infty}\left[\left(\frac{-1}{0.085}\right)17{,}000e^{-0.085t}\right]_0^h = 200{,}000$$

(6) with $S(t) = 12 - t$, $r = 0.04$ & $T = 10$, the present value is

$$PV = \int_0^{10}(12-t)e^{-0.04t}\,dt = 12\int_0^{10} e^{-0.04t}\,dt - \int_0^{10} te^{-0.04t}\,dt$$

1st integral: $\int_0^{10} e^{-0.04t}\,dt = \left[-25e^{-0.04t}\right]_0^{10} = 25(1 - e^{-0.4})$

2nd integral: $u = t$, $v' = e^{-0.04t} \Rightarrow u' = 1$ & $v = -25e^{-0.04t}$

$$\therefore \int_0^{10} te^{-0.04t}\,dt = \left[-25te^{-0.04t}\right]_0^{10} + \int_0^{10} 25e^{-0.04t}\,dt = 625 - 875e^{-0.4}$$

$$\therefore PV = 12\cdot 25(1 - e^{-0.4}) - 625 + 875e^{-0.4} = (575e^{-0.4} - 325)$$
$$\approx 60.434$$

(7) Separate variables: $\frac{dy}{y} = \left(2x + \frac{1}{x}\right)dx$

integrate both sides: $\ln|y| = x^2 + \ln x + C$

$\therefore |y| = e^{x^2 + \ln x + C} = e^C e^{\ln x} e^{x^2} = e^C x e^{x^2}$

$\Rightarrow y = Axe^{x^2}$ where $A = \pm e^C$.

when $x=1$, $y=1$ & so $1 = Ae^1$. Thus $y = e^{-1} x e^{x^2} = xe^{x^2-1}$.

(8) write $\frac{1}{(x-1)(x-2)} = \frac{A}{x-2} + \frac{B}{x-1}$ $\Rightarrow 1 = A(x-1) + B(x-2)$

$x=2: \quad 1 = A(2-1) \Rightarrow A = 1$

$x=1: \quad 1 = -B \Rightarrow B = -1$

$$\therefore \int \frac{dx}{(x-1)(x-2)} = \int \left(\frac{dx}{x-2} - \frac{dx}{x-1} \right) = \ln|x-2| - \ln|x-1| + C$$
$$= \ln\left|\frac{x-2}{x-1}\right| + C$$

Thus
$$\int_3^\infty \frac{dx}{(x-1)(x-2)} = \lim_{h\to\infty} \left[\ln\left|\frac{x-2}{x-1}\right| \right]_3^h = \lim_{h\to\infty} \left(\ln\left|\frac{h-2}{h-1}\right| - \ln\left|\frac{3-2}{3-1}\right| \right)$$
$$= \ln 1 - \ln(\tfrac{1}{2})$$
$$= \ln 2.$$

(9) Use substitution $u = 1-x$ $\therefore du = -dx$. when $x=0$, $u=1$; $x=1$, $u=0$

$$\therefore \int_0^1 \frac{x}{\sqrt{1-x}}\,dx = -\int_1^0 \frac{1-u}{\sqrt{u}}\,du = \int_0^1 (u^{-1/2} - u^{1/2})\,du$$
$$= \lim_{h\to 0^+} \int_h^1 (u^{-1/2} - u^{1/2})\,du$$
$$= \lim_{h\to 0^+} \left[2u^{1/2} - \tfrac{2}{3}u^{3/2} \right]_h^1$$
$$= \lim_{h\to 0^+} \left[(2 - \tfrac{2}{3}) - (2h^{1/2} - \tfrac{2}{3}h^{3/2}) \right]$$
$$= 2 - 2/3 = 4/3.$$

(10) Use substitution: $u = -x^2$ $du = -2x\,dx$ $\Rightarrow -\frac{1}{2}du = x\,dx$

$$\therefore \int xe^{-x^2}dx = -\tfrac{1}{2}\int e^u\,du = -\tfrac{1}{2}e^u + C = -\tfrac{1}{2}e^{-x^2} + C$$

So
$$\int_{-\infty}^\infty xe^{-x^2}dx = \int_0^\infty xe^{-x^2}dx + \int_{-\infty}^0 xe^{-x^2}dx$$
$$= \lim_{h\to\infty} \left[-\tfrac{1}{2}e^{-x^2} \right]_0^h + \lim_{h\to-\infty} \left[-\tfrac{1}{2}e^{-x^2} \right]_h^0$$
$$= \lim_{h\to\infty} (1 - \tfrac{1}{2}e^{-h^2}) + \lim_{h\to-\infty} (\tfrac{1}{2}e^{-a^2} - 1)$$
$$= (1-0) + (0-1) = 0$$

(11) For $x \geq 1$ we have $0 < \frac{1}{\sqrt{x^3+x}} \leq \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$

Since $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges, so does $\int_1^\infty \frac{1}{\sqrt{x^3+x}}\,dx$

(12) Let (q_e, p_e) be the equil point. Then

$$\frac{20}{q_e+1} = q_e + 2 \quad \Rightarrow \quad 20 = (q_e+2)(q_e+1)$$
$$\Rightarrow q_e^2 + 3q_e - 18 = 0$$
$$\Rightarrow (q_e+6)(q_e-3) = 0$$
$$\Rightarrow q_e = 3 \quad \text{is the positive solution}$$

Thus $p_e = D(q_e) = \frac{20}{3+1} = 5$.

consumer surplus is

$$\int_0^3 \frac{20}{q+1}\,dq - 3(5) = 20\left[\ln|q+1|\right]_0^3 - 15 = 40\ln 2 - 15.$$

Section 8.4

(1) Separating variables: $\frac{dy}{y^2} = \sec^2 x\,dx$

Integrating both sides: $-\frac{1}{y} = \tan x + c$

$\therefore y = \frac{-1}{\tan x + c}$

When $x = \frac{\pi}{4}$, $y = 1$ & so $1 = \frac{-1}{1+c} \Rightarrow 1 + c = -1 \Rightarrow c = -2$

$\therefore y = \frac{1}{2 - \tan x}$.

(2) $$\int \frac{1-\tan^2 x}{\sin^2 x}\,dx = \int\left(\frac{1}{\sin^2 x} - \frac{\tan^2 x}{\sin^2 x}\right)dx = \int\left(\operatorname{cosec}^2 x - \frac{1}{\cos^2 x}\right)dx$$
$$= -\cot x - \int \sec^2 x\,dx$$
$$= -\cot x - \tan x + C.$$

(3) Let $u = e^x$ & $v' = \cos x$. Then $u' = e^x$ & $v = \sin x$

$$\therefore \int_0^{\frac{\pi}{2}} e^x \cos x\,dx = \left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \sin x\,dx$$
$$= e^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \sin x\,dx \qquad (*)$$

Now use integration by parts again: $u = e^x$ & $v' = \sin x$

$\therefore u' = e^x$ & $v = -\cos x$

So $$\int_0^{\frac{\pi}{2}} e^x \sin x\,dx = \left[-e^x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^x \cos x\,dx = 1 + \int_0^{\frac{\pi}{2}} e^x \cos x\,dx$$

Substitute back into $(*)$:

$$\int_0^{\frac{\pi}{2}} e^x \cos x\,dx = e^{\frac{\pi}{2}} - 1 - \int_0^{\frac{\pi}{2}} e^x \cos x\,dx$$
$$\Rightarrow 2\int_0^{\frac{\pi}{2}} e^x \cos x\,dx = e^{\frac{\pi}{2}} - 1 \quad \Rightarrow \quad \int_0^{\frac{\pi}{2}} e^x \cos x\,dx = \tfrac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right).$$

(4) $$\text{Av value} = \frac{1}{\pi/2}\int_0^{\frac{\pi}{2}} \sin x \cos x\,dx$$
$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \sin 2x\,dx = \frac{1}{\pi}\left[-\tfrac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}} = \frac{1}{2\pi}\left[1-(-1)\right] = \frac{1}{\pi}.$$

(5) Area of region $= \int_0^{\pi/6} \frac{\cos x}{1+\sin x}\,dx$

Let $u = 1 + \sin x$. Then $du = \cos x\,dx$. When $x = 0$, $u = 1$, $x = \frac{\pi}{6}$, $u = \frac{3}{2}$.

& so $\int_0^{\pi/6} \frac{\cos x}{1+\sin x}\,dx = \int_1^{3/2} \frac{1}{u}\,du = [\ln|u|]_1^{3/2} = \ln(\frac{3}{2}) - \ln 1 = \ln(\frac{3}{2})$.
$= \ln 3 - \ln 2$.

Section 11.5

① $f(x)$ must satisfy
$\int_0^1 f(x)\,dx = \int_0^1 c(1-x^2)\,dx = c\underbrace{\int_0^1 (1-x^2)\,dx}_{=2/3} = 1$
$\Rightarrow \frac{2}{3}c = 1$
$\Rightarrow c = 3/2$.
Note that $f(x) \geq 0$ on $(0,1)$.
Now $E(X) = \int_0^1 x f(x)\,dx = \int_0^1 \frac{3}{2}x(1-x^2)\,dx = 3/8 \Rightarrow \mu = 3/8$.
$Var(X) = \int_0^1 x^2 f(x)\,dx - \mu^2 = \frac{1}{5} - \frac{9}{64} = \frac{19}{320}$.

② First, $\mu = E(X) = \int_0^1 x f(x)\,dx = \int_0^1 4x^4\,dx = [\frac{4}{5}x^5]_0^1 = \frac{4}{5}$.
Next, $var(X) = \int_0^1 x^2 f(x)\,dx - \mu^2 = \int_0^1 4x^5\,dx - \frac{16}{25} = \frac{2}{3} - \frac{16}{25} = \frac{2}{75}$.

③ If $f(x)$ is a pdf, it must satisfy $\int_0^1 c(1-x)^{-1/2}\,dx = 1$
Since $\int_0^1 (1-x)^{-1/2}\,dx = [-2(1-x)^{1/2}]_0^1 = 2$, we must have
$2c = 1 \Rightarrow c = 1/2$.

④(a) $P(X \geq 1) = P(1 \leq X \leq 3) = \int_1^3 \frac{1}{18}(9-x^2)\,dx = [\frac{1}{18}(9x - \frac{1}{3}x^3)]_1^3$
$= \frac{14}{27}$.

(b) $P(X \leq 2) = P(0 \leq X \leq 2) = \int_0^2 \frac{1}{18}(9-x^2)\,dx = [\frac{1}{18}(9x - \frac{1}{3}x^3)]_0^2$
$= \frac{23}{27}$.

(c) $P(1 \leq X \leq 2) = \int_1^2 \frac{1}{18}(9-x^2)\,dx = [\frac{1}{18}(9x - \frac{1}{3}x^3)]_1^2 = \frac{10}{27}$.

⑤ $E(X) = \int_0^\infty x f(x)\,dx = \int_0^\infty \frac{2x}{(x+1)^3}\,dx$
Let $u = x+1$ $\therefore du = dx$. When $x=0$, $u=1$ & when $x=\infty$, $u=\infty$.
$\therefore E(X) = \int_1^\infty \frac{2(u-1)}{u^3}\,du = \int_1^\infty 2(u^{-2} - u^{-3})\,du$
$= 2\lim_{h\to\infty} [-u^{-1} + \frac{1}{2}u^{-2}]_1^h$
$= 2\lim_{h\to\infty} [-h^{-1} + \frac{1}{2}h^{-2} + 1 - \frac{1}{2}] = 2(\frac{1}{2}) = 1$.
If m is the median of X, then $\frac{1}{2} = \int_0^m 2(x+1)^{-3}\,dx = 1 - (m+1)^{-2}$
Thus $(m+1)^2 = 2$ & so $m = -1 \pm \sqrt{2}$. For $m > 0$ choose $m = \sqrt{2} - 1$.