MATH 105 (Section 211): Practice Questions for the Final Examination

These are intended to be practice problems for the final examination, you should try them without using any notes. Please note that these questions do not cover everything you need to know: a topic not included here **could** *still be on the final exam (provided it is from the covered Sections studied in class.) Also, you might need a calculator for some of the questions, but you are* **not** *allowed one on the final exam: the calculations there will be easier.*

- **From Chapter 7**:

1. Compute the distance between the points $(2, -1, -2)$ and $(-1, -2, -6)$.
2. Sketch the level curves for the function $f(x, y) = x - y^2$.
3. Let $f(x, y) = ye^{x^2+y^2} + 3x^2$. Compute $\frac{\partial f}{\partial x}$.
4. A company produces two kinds of goods, X and Y. Suppose that x is the number of units (in thousands per month) of X being produced and that y is the number of units (in thousands per month) of Y being produced. The revenue function is given by $R(x, y) = 30x + 20y - \frac{1}{2}x^2 - \frac{1}{2}y^2$. Find the production levels that maximize revenue.
5. Let $f(x, y) = x^3 + y^3 - 3xy$. This function has a critical point at $(1, 1)$. What is the nature of this critical point?
6. The quantity Q of a product manufactured by a company is given by $Q(K, L) = 50K^{0.3}L^{0.7}$, where K is the quantity of capital and L is the quantity of labor used. Capital costs are 60 per unit, labor costs are 20 per unit, and the company wants to restrict the combined cost of capital and labor to $800,000$. Find the combination of capital and labor that maximize the quantity produced.
7. Let $f(x, y) = -x^2 + 2x + y$. Draw the level curve $f(x, y) = 1$ in the xy-plane.
8. Let $f(x, y) = x \ln y + ye^x$. Use the linear approximation of $f(x, y)$ at $(0, 1)$ to estimate the value of $f(0.05, 0.9)$.
9. Let $f(x, y) = \ln(x^2 + y^2 + 3)$. Find $\frac{\partial^2 f}{\partial x \partial y}$.
10. Find the critical points of the function $f(x, y) = -x^2 + 6xy - y^2 + 16y$.
11. The function $f(x, y) = x^2 + y^2$ has a global minimum value subject to the constraint $2x - y^2 = 2$. Use the method of Lagrange multipliers to find this global minimum point and the corresponding global minimum value.
12. Suppose that the production function of a machine-parts company is the following Cobb-Douglas function: $Q(K, L) = 4K^{0.5}L^{0.5}$, where K is the number of capital units and L is the number of labor units. Find the marginal productivity of labor when $K = 25$ and $L = 16$.

- **From Chapter 5**:

1. Evaluate $\int_0^2 te^{t/2}\,\mathrm{d}t$.
2. Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{t} - \frac{1}{t-1}$ with initial condition $y(2) = \ln 2 + 1$. Find $y(3)$.
3. Estimate $\int_0^1 \frac{x}{1+x^4}\,\mathrm{d}x$ using the trapezoidal rule with $n = 5$.

x	0	0.5	1	1.5	2
f(x)	-1	-0.5	1.5	2	1

Table 1: Data for question 10.

4. Evaluate $\int_2^4 (2x-5)e^{x^2-5x+6},\, udx$.
5. A rumor begins to circulate by email. Assume that the "infection" rate at which it spreads, measured by the number of new people receiving the rumor per day, is given by $R'(t) = e^{t/2} + 20t - 0.8$. If the rumor was originally ($t = 0$) received by 60 people, find $R(t)$, the number of people who have received the rumor after t days.
6. Evaluate $\int (x^2 + 2x + 3) \ln x \, dx$.
7. Evaluate $\int (x^2 + 2x + 1)e^{x^3+3x^2+3x+3} \, dx$.
8. Evaluate $\int \frac{x}{x^2+x-12} \, dx$.
9. Find the average value of the function $f(x) = x(x^2+1)^4$ over the interval $0 \le x \le 3$.
10. Use Simpson's rule to estimate the definite integral $\int_0^2 f(x) \, dx$ given the data for $f(x)$ in Table 1.
11. Find the area of the region between the curves $y = 3x$ and $y = x^2 - x$ over the interval $0 \le x \le 4$.
12. Assume that a company's marginal profit at a production level of x units per day is given by $MP(x) = -0.5x + 62$, $100 \le x \le 150$. The company is currently operating at a production level of 100 units per day and earning a profit of $10,000$. Find the profit of the company if its production level increases to 110 units per day.

- **From Chapter 6**:

1. Assume that you are 30 years old and you estimate that to retire comfortably at the age of 65 you will need $1,800,000$ in savings. If you can earn 9% compounded continuously, how much should you deposit each year in order to meet this goal? Treat the deposits as a continuous income stream and round your answer to the nearest dollar.
2. Find the constant a for which the function $y = at^2 + \frac{5}{t^2}$ is a solution of the differential equation $ty' + 2y = 8t^2$ [Note that $y' = \frac{dy}{dt}$.]
3. Determine whether the improper integral $\int_1^\infty te^{-t} \, dt$ converges, and if so, evaluate it. [Hint: $\lim_{b\to\infty} be^{-b} = 0$.]
4. Determine whether the improper integral $\int_1^2 \frac{x}{\sqrt{x^2-1}} \, dx$ converges, and if so, evaluate it.
5. Assume that you own a property which brings in a perpetual income stream flowing continuously at a rate of $17,000$ per year. What is its fair sale price assuming that money can be invested earning interest at the annual rate of 8.5% compounded continuously.
6. Suppose that a company manufactures a certain type of machine. Each machine is estimated to generated a continuous income stream at a rate of $(12-t)$ million dollars, where t is the life of the machine in years. The expected lifetime of such a machine is 10 years. Assume that the money can be invested at 4% interest, compounded continuously. Find the present value of one of these machines.

7. Solve the initial value problem

$$\begin{aligned}\frac{dy}{dx} &= \left(2x + \frac{1}{x}\right) y, \qquad \text{for } x > 0 \\ y(1) &= 1.\end{aligned}$$

8. Evaluate the improper integral $\int_3^{\infty} \frac{1}{(x-1)(x-2)}\, dx$.
9. Evaluate the improper integral $\int_0^1 \frac{x}{\sqrt{1-x}}\, dx$.
10. Evaluate the improper integral $\int_{-\infty}^{\infty} xe^{-x^2}\, dx$.
11. Use the comparison test for improper integrals to determine whether or not the following improper integral converges:

$$\int_1^{\infty} \frac{1}{\sqrt{x^2+x}}\, dx.$$

12. The demand curve of a certain item is $D(q) = \frac{20}{q+1}$ and its supply curve is $S(q) = q + 2$. Find the equilibrium price and equilibrium quantity. What is the consumer surplus?

- **From Section 8.4**:

1. Solve the initial value problem $\frac{dy}{dx} = y^2 \sec^2 x$, $y(\pi/4) = 1$.
2. Evaluate the indefinite integral $\int \frac{1-\tan^2 x}{\sin^2 x}\, dx$.
3. Evaluate the definite integral $\int_0^{\pi/2} e^x \cos x\, dx$.
4. Find the average value of the function $y = \sin x \cos x$ over the interval $[0, \pi/2]$.
5. Find the area of the region between the x-axis and the curve $y = \frac{\cos x}{1+\sin x}$, for $0 \leq x \leq \frac{\pi}{6}$.

- **From Section 11.5**:

1. Consider the function $f(x) = c(1 - x^2)$ on the interval $(0, 1)$, where c is some constant. Find c so that $f(x)$ is a probability density function. Then determine $E(X)$ and $\text{Var}(X)$.
2. A random variable X has probability density function $f(x) = 4x^3$ on the interval $(0, 1)$. Compute the variance $\text{Var}(X)$.
3. Find the value of the constant c so that the function $f(x) = c(1 - x)^{-1/2}$ for $0 < x < 1$ is the probability density function of a continuous random variable.
4. Assume that the daily demand for a certain product in thousands of units has probability density function $f(x) = \frac{1}{18}(9 - x^2)$, $0 \leq x \leq 3$.
   (a) Find the probability that the demand is at least $1,000$ units;
   (b) Find the probability that the demand is at most $2,000$ units;
   (c) Find the probability that the demand is between $1,000$ and $2,000$ units.
5. Compute the median and the expected value of the continuous random variable with probability density function $f(x) = 2(x + 1)^{-3}$, $x \geq 0$.