Chapter 5

Kinky Sets

Now that I've told you what cycle-bands and defeating algorithms are, I'm ready to show you why there is no winning strategy for standard tetris. In fact, I'll prove that as long as you are playing with a set of pieces that includes both kinks, you have no winning strategy in any well. Before giving a recipe for your defeat, I'll use the next two sections to establish facts which I need in order to prove that the recipe actually works.

5.1 Playing in the cycle-band

You might recall that the well can be divided into pairs of adjacent columns that I call lanes. In each lane, a flat can have zero, one, or two full cells, and I'll use **segment** to refer to the portion of a flat that occupies one lane. Moreover, I'll use the symbols □□, □■, ■□, and ■■ to represent the four possible configurations of empty and full cells in a segment.

If a flat has an □■ as its leftmost segment, then you can't kill that flat with a left kink (see Figure 5.24). I'll call such a flat **left-immune**. Similarly, a flat with a ■□ as its rightmost segment is **right-immune**. I now show that if the machine gives you a long enough sequence of left kinks, and you play them without ending your game, then the well will develop a certain kind of structure.

Lemma 2 *While following a strategy, if you pass through a cycle in which every piece is a left kink, then you must have played each one entirely within a single lane. Moreover,*

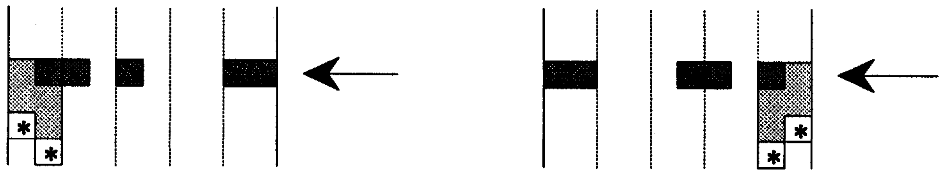

Figure 5.24: Arrows indicate left- and right-immune flats. The plays shown are impossible since they require the pieces to be moved upward. Even if I allow you upward moves, the [*] cells will have to be empty for you to slide the pieces into place, and gravity will then pull the kinks down into these empty cells. You still can't kill the immune flats.

for every lane in every state of the cycle, the portion of that lane within the cycle-band will consist of a stack of zero or more [■■] *segments, topped with a single* [■□].

Proof: To begin, I will define a sequence of statements which might describe your plays during the cycle. Each of these statements tells how your plays affected one lane, and can be either true or false. The statements are called $P(1)$, $P(2)$, and so on, where the numbers represent which lane the statement is about. The statements have the following meaning:

> For a lane x, $P(x)$ says that whenever you played a left kink so as to fill at least one cell to the left of lane x, you played that kink entirely within a single lane. This means, among other things, that during the cycle, you never played a left kink so that it filled cells in both lane x in the lane to the left of x (see Figure 5.25).

I repeat: each of $P(1)$, $P(2)$, and so on could be either a true or a false statement about how you played the kinks during the cycle. I want to show that, in fact, all of them are true.

What does $P(1)$ really mean? It means that during the cycle, any kink you played which filled some cells to the left of lane 1 was played entirely within a single lane. This is true, in a trivial way, simply because there are no cells to the left of lane 1, and so you couldn't have filled any. More importantly, this means that you never played a kink which stuck partly into lane 1 from the left (again, this just isn't possible).

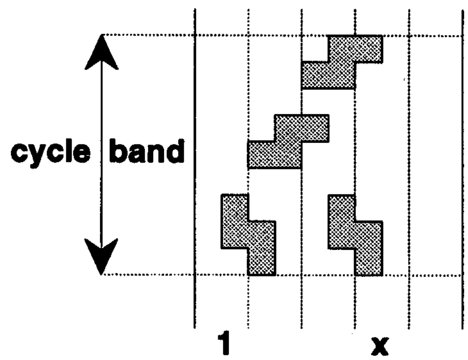

Figure 5.25: $P(x)$ is true if and only if plays of the type illustrated ***don't*** occur: lane boundaries to the left of lane x, including x's own left boundary, are not crossed by pieces played in the cycle. $P(1)$ is true simply because no piece can cross the left wall of the well.

Now that you know $P(1)$ is true, I will use that fact to prove that $P(2)$ is true. Even better, I will prove that if for some lane n, $P(n)$ is true, then $P(n+1)$ (i.e. the statement for the next lane to the right) is also true. This will mean $P(2)$ is true, and thus that $P(3)$ is true, and so on, all the way across the well.

Let n be a lane for which $P(n)$ is true. I first prove that no flat in the cycle-band can have an □■ segment in lane n. Suppose, to the contrary, that f is the oldest flat (in the cycle-band) that has an □■ in lane n. If you fill the empty cell of that segment during the cycle, it can only be by making the play shown in Figure 5.26, because $P(n)$ implies no left kink can be played to fill the cell from the left. However, this play creates another □■ in lane n, but in a flat, say g, which is lower (and hence older) than f. Moreover, since this play changes g, it is also in the cycle-band, contradicting my choice of f (see

Figure 5.26). Therefore, you can't fill the empty cell in that segment of f, and so even if the cycle is endlessly repeated, you won't kill f. But this contradicts the fact that f is in the infinite cycle-band, since every flat therein must eventually die. Therefore, neither f nor any other flat in the cycle-band can have a segment in lane n.

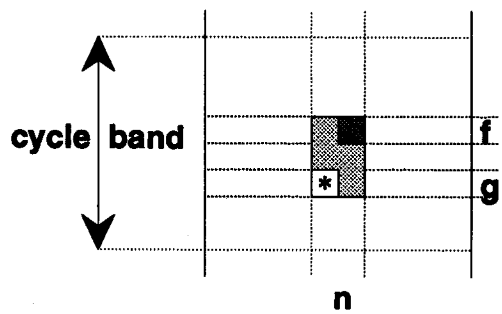

Figure 5.26: The only way you can fill the empty cell of an segment in lane n of flat f, if $P(n)$ is true. Doing so requires that the * cell be empty: think of where the kink could have been just before ending up as drawn. Thus, if you play the kink this way, you create a new segment in a flat g that's below f.

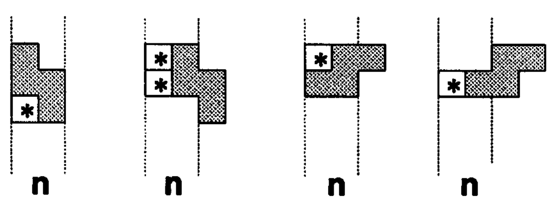

Figure 5.27: When $P(n)$ is true, there are at most four ways you can play a left kink so as to fill a cell in lane n. In each case, the * cells must be full or else you will create an in lane n.

In Figure 5.27 I've shown the four possible ways you could fill a cell in lane n (when $P(n)$ is true, as I've assumed). In order not to create the forbidden in lane n, the

[*] cells must already have been full when you played the kink. But that means the segment containing that cell must have been a [■][ ] (consisting of the full [*] cell and its empty right neighbour) before you played the kink. Therefore, if you filled a cell in lane n, there must already have been a [■][ ] there. But notice that the last three ways of playing a left kink change at least one [■][ ] into a [■][■] and so decrease the number of [■][ ] in lane n by at least one. In contrast, suppose you played the left kink in the first way I've shown (i.e. entirely within lane n). You would have destroyed one [■][ ] segment (the one with the [*]), but created another (the one at the top of the piece; its right cell is empty since otherwise, that segment would have been the forbidden [ ][■] before you played the piece). Thus the number of [■][ ] segments in lane n of the cycle-band could only have declined or remained constant each time you played a left kink. In fact, it must have remained constant since the initial and final states of the cycle are identical. Therefore, during the cycle, you could have used only the first method to play a left kink when filling a cell in lane n. Hence, during the cycle, you never played a left kink that filled cells in both lane n and lane $n+1$. Together with $P(n)$ being true, this is enough to imply that $P(n+1)$ is true. Therefore, all of the statements $P(1)$, $P(2)$, and so on are true: during the cycle, you played every left kink entirely within a single lane.

The second part of the lemma is a consequence of the first. Notice that during the cycle, you must have played each left kink at the *top* of a lane: playing it below the top would have required an [ ][ ] segment below the non-empty top segment of the lane. But how could this empty segment have arisen? During the cycle, gravity would prevent you from leaving any empty segments in a lane, and so in infinite repetition of the cycle, all such empty segments would eventually disappear. But then the play which required the empty segment could never be repeated. Therefore, such a play doesn't occur in the cycle.

Playing left kinks within and at the top of lanes can give rise to lane structures only of the type shown in Figure 5.28. ■

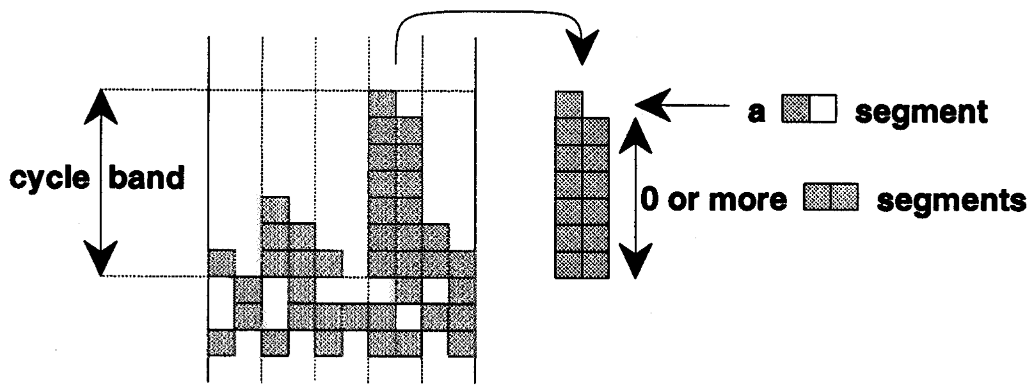

Figure 5.28: During a left kink cycle, the lanes in the cycle band all have a simple structure: a ■□ segment on top of a (possibly empty) stack of ■■.

5.2 The shield in the cycle-band

The importance of this kind of cycle-band structure is that it contains a collection of flats which together protect underlying flats from right kinks, and which no number of right kinks can kill. This is what I call a **shield** against right kinks.

Lemma 3 *In a cycle induced by a sequence of left kinks, the cycle-band contains a shield against right-kinks.*

Proof: Look at any state in the cycle. Let f be the highest flat in that state whose rightmost segment is not an □□. The previous lemma shows that this segment must be a ■□, and so f is right-immune. This is the highest flat in the shield, and you can find the other ones by this procedure:

1. f is a flat in the cycle-band which can't be deleted by right kinks, and whose rightmost segment is not an □□.

2. **If** none of f's segments is an □□ (for example, if f is the lowest flat in the cycle-band for this state) then each of f's segments is either a ▣□ or a ▣▣. But then f has no pair of consecutive empty cells at all, and so it prevents right kinks from reaching any rows below (kinks need a space of two empty cells to pass through a row). This flat is the lowest one in the shield, so you can stop.

3. **Otherwise** one of f's segments is an □□. Let l be the lane with the rightmost such segment, and notice that f has a full cell in the left column of lane $l+1$ — see Figure 5.29. (Notice that l is not the rightmost lane in the well, since f's rightmost segment isn't an □□.)

4. Put your hand on this full cell and descend lane l, dragging your hand down along the "wall" of full cells to your immediate right, until you reach a flat, say g, whose segment in lane l is not an □□. Flat g is still in the cycle-band since the bottom flat of the cycle band, having no □□ segments (by the previous lemma), would have stopped your descent. Thus, g must have a ▣□ segment in lane l (previous lemma again). Notice that the empty cell of this segment can't be filled by a right kink until all the full cells you touched are gone (Figure 5.29 again). That is, flat g can't be killed (using only right kinks) until you've killed all of the flats above g, up to and including f. Since f can't be killed by right kinks, neither can g.

5. Reuse the labels so that "f" now refers to this flat g.

6. Go to step 1.

Since you descend at least one row each time you perform step 4, following this procedure will eventually lead you to stop, successfully, in step 3. At that point, the set of all flats

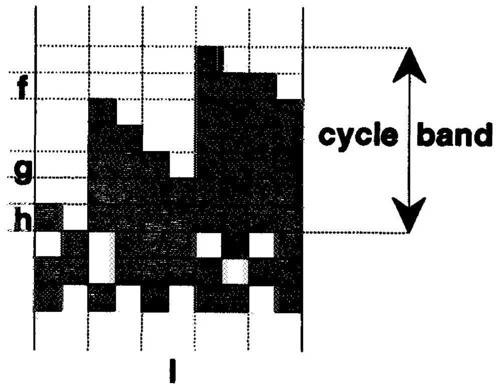

Figure 5.29: The flat f is right-immune, but has an □□ in lane l. The cells you touch while dropping down empty portions of lanes are shown notched. In this example, the shield you eventually find consists of flats f, g, and h.

you ever called f forms a shield against right kinks, since you can't kill any of these flats with right kinks, and since the lowest one actually prevents you from playing right kinks anywhere below. ■

5.3 Defeating you with kinks

When the machine performs the defeating algorithm for $\square_2$ and $\square_3$ which I gave in Chapter 4, it does not provide you with lookahead information. However, when it performs the following algorithm, it does. You can verify this by making sure that in every step, the piece being handed to you is the one displayed (as lookahead piece) in the previous step.

Algorithm 2 *The following procedure generates a sequence of kinks which defeats any strategy for tetris:*

1. Send left kink and display left kink until a cycle is detected.

2. Send one left kink and display right kink.

3. Send right kink and display right kink until a cycle is detected.

4. Send one right kink and display left kink.

5. Go to step 1.

Proof: For reasons I've already mentioned, if your game doesn't end in step 1 or 3, you must eventually pass through a cycle. The machine will detect this when you revisit a state within step 1 or 3. Lemmas 2 and 3 show that when a cycle is detected in step 1, the cycle-band will contain a shield against right kinks. Can you disable this shield when you play the left kink sent in step 2? No: the shield consists of flats having structures of the type illustrated in Figure 5.30. The only way you can fill the empty cell in one of those structures (and hence the only possible way you can disable the shield) with the single left kink of step 2 is to play that kink within a lane. But doing so preserves the structural form of the cycle-band: it will contain a (possibly new) shield against right kinks.

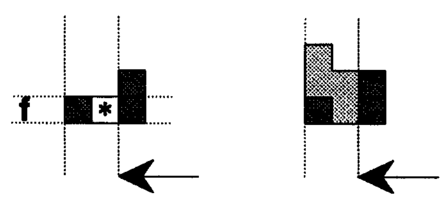

Figure 5.30: Every flat, say f, in the shield must have an empty cell, shown as a [*], with a full cell in the position one up and one to the right. The only way you can fill this empty cell using a single left kink is shown. (In both cases, the lane boundary indicated with an arrow might actually be the right wall of the well.)

So, even after step 2, the well still has a shield against right kinks. Therefore, nothing you do with the right kinks you get in steps 3 and 4 will kill any of the flats in that shield.

Also, notice that the cycle-band of the cycle detected in step 3 must lie entirely above the shield from steps 1 and 2. This is simply because the top flat of that shield can't be killed by right kinks, and so isn't in the infinite cycle-band corresponding to repetition of the right-kink cycle. (Remember, every flat in the infinite cycle-band must die.) But now the mirror images of Lemmas 2 and 3 show that the cycle-band induced by right kinks will contain a shield against left kinks. Since this new cycle-band is above the old shield, so is the new shield. Thus, the second time the machine follows step 1, you will be playing left kinks above this new shield, without killing any of the flats in it. As the machine repeats steps 1 to 4, you will be forced to create a stack of shields of alternating type until eventually, one of the flats in those shields is above row 20, and your game ends. Obviously, you could have chosen any other maximum well depth and the machine would still have (eventually) defeated you. ■

When following Algorithm 2, how can the machine detect a cycle? One way (popular with the American recreational vehicle crowd) is for it to create a list of states visited. Each time you enter step 1 or 2, the machine empties the list. Every time you reach a state within that step, it checks to see if it's already in the list. If so, you have just passed through a cycle. If not, it adds the state to its list. It's quite possible for this list to get arbitrarily large, even if a cycle is always (eventually) detected, so a better method is for the machine to wait for the cycle-band structure itself. This structure can arise even if you haven't yet completed a cycle. Moreover, the machine requires no list for this, but only needs to scan each state you visit to see if it has the required structure.

"Now that I know how the machine is defeating me, can't I choose to avoid creating the shields?" No. Remember that with my concept of "strategy", the way you play a piece can depend only on the current state of the well and on which piece is shown as lookahead. Therefore, if you revisit a situation, you have no choice but to play the current piece in the same way you did when you first encountered that situation. This is enough to guarantee that if you are given a long enough sequence of left kinks, you must eventually pass through a cycle (or end your game). Once you've done this, Lemma 3 guarantees that there will be a shield in the corresponding cycle-band.

I repeat: no matter what the width of the well, no matter how great you set its depth, no matter what state the well is initially in, and even if you are allowed to move pieces up before dropping them, you can't beat tetris. The machine can defeat you by using just the two kinks. You can't win at tetris, and so you can't win at TETRIS.