Chapter 3

Two at a time

In the last chapter, I gave you winning strategies for tetris played with only one piece. The logical next step is to show you how to play tetris with two pieces.

3.1 Squares with bars or elbows

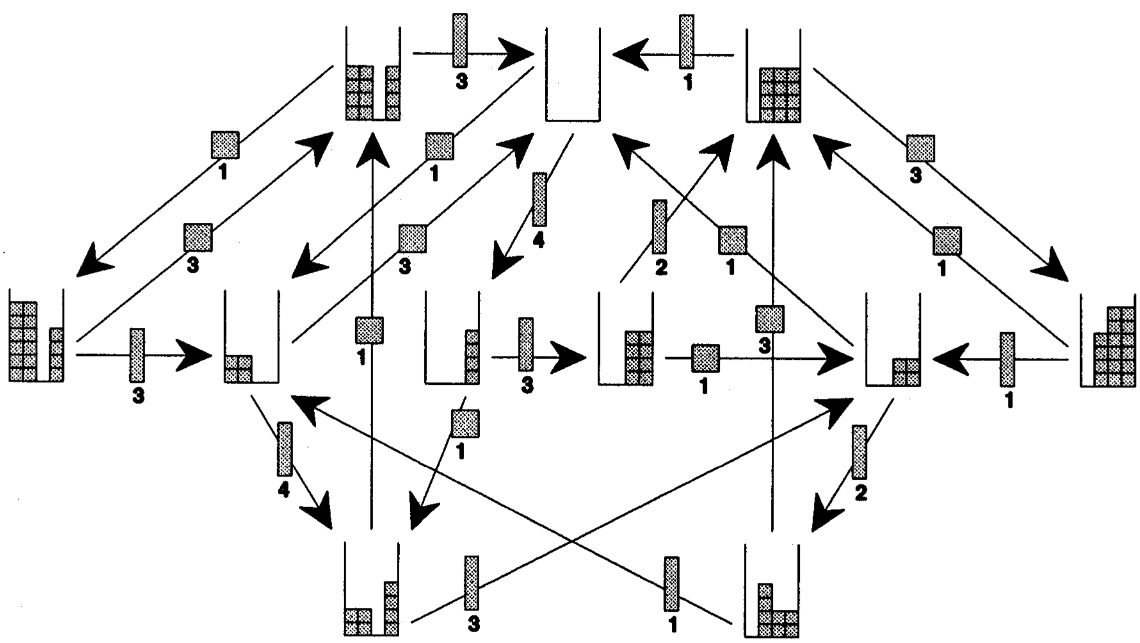

Figure 3.13: A winning strategy for bars and squares in a well of width 4.

In Figure 3.13, I've drawn a winning strategy for bars and squares in a well of width 4. A similar winning strategy for right elbows and squares appears in Figure 3.14 (for left

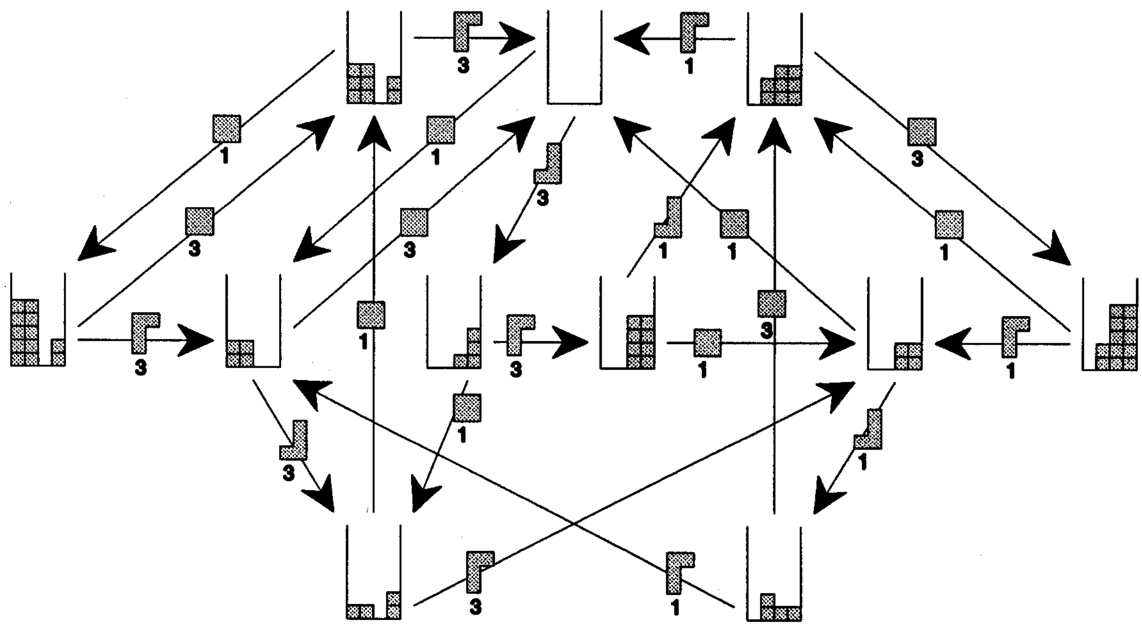

Figure 3.14: A winning strategy for right elbows and squares in a well of width 4.

elbows and squares, use a mirror). Neither strategy uses lookahead information. Similar strategies exist for wells of any even width:

Lemma 1 *If you play tetris using only the square and one of left elbow, right elbow, or bar, then for a well of any even width, you have a winning strategy whose height doesn't depend on the width of the well.*

Proof: You can probably discover the proof yourself, by staring at the diagrams. I'll call the elbow or bar the **non-square** piece. Just as for states and columns, the **height** of a lane is the height of the highest full cell in that lane. Notice that in every state shown in Figures 3.13 and 3.14, each lane, except possibly one, is **smooth**, consisting of a 2 column wide rectangle of full cells (of zero height, if the lane is empty). The possible exception is a **bumpy** lane which, in at least one row, has only a single full cell. You create a bumpy lane when you play the non-square piece in a smooth lane. You can restore that lane to smoothness by playing the same piece in it again — see Figure 3.15.

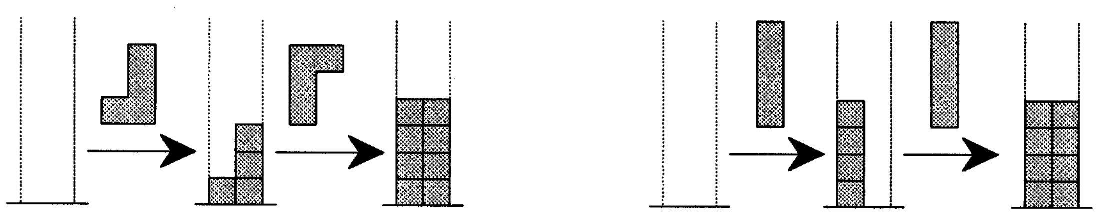

Figure 3.15: How to make a bumpy lane and how to smooth it. By playing a single elbow (left) or bar (right) in a smooth lane, you make it bumpy. By playing the same piece in a bumpy lane, you smooth it. (These plays don't necessarily occur one right after the other.)

For any even width of well, I describe the strategy by telling you how to play each piece:

Square: If there is a bumpy lane and if its height is at least two less than that of any other lane, then play the square in the bumpy lane. Otherwise, play the square in a lowest smooth lane.

Non-Square: If there is a bumpy lane, then play this piece to make it smooth. Otherwise, play this piece in a lowest smooth lane, making it bumpy.

Notice that if every lane in a state is smooth, then at least one lane is actually empty. Otherwise, the bottom row in the state would be full, which is impossible. Therefore, when you play the non-square piece in a lowest smooth lane, you are actually playing it in an empty lane. Also, notice that there is never more than one bumpy lane.

I claim that the strategy above will never result in a state whose height is more than 7. To see this, consider the following three statements:

Statement S1: There is at most one bumpy lane in the well.

Statement S2: If there is a bumpy lane, its height is at most 4.

Statement S3: The height of every smooth lane is at most 7.

Notice that if both S2 and S3 are true for some state of the well, then the height of that state is at most 7. Each of S1, S2, and S3 is certainly true of the empty inital well. I will show that the strategy has a height of at most 7 by showing that every type of play in the strategy preserves the truth of these three statements. That is, if S1, S2, and S3 are all true for a state, then playing either piece leads to another state for which they are also true. There are four cases:

- If you play a square in a lowest smooth lane, then either that lane is empty (if there is no bumpy lane in the well), or that lane is less than two rows higher than the bumpy lane. In the first case, the new height of the lane will be 2, and in the second case, the new height will be at most 3 plus the height of the bumpy lane, which is at most 7, since S2 was true of the state you played on. In both cases, the new height of the lane you played in is at most 7, and so S3 is still true. S2 is still true since the bumpy lane is unaffected. Also, S1 is true since playing a square never makes a lane bumpy.

- If you play a square in the bumpy lane, it disappears, clearing two rows (see Figure 3.16). The height of the smooth lanes decreases by 2, and the height of the bumpy lane doesn't change. Therefore, S2 and S3 are both still true. Again, S1 remains true since you're playing a square.

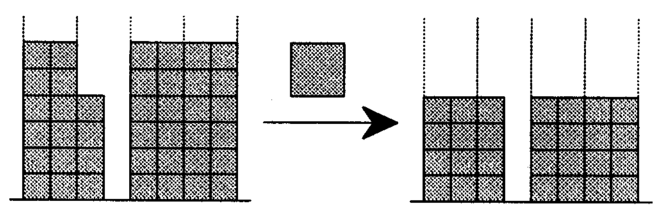

Figure 3.16: Playing a square in the bumpy lane clears two rows.

- If you play the non-square piece in a lowest smooth lane, then there is no bumpy lane and so any lowest smooth lane is actually empty. Thus, you create a bumpy lane whose height is at most 4, and you don't affect the heights of the other smooth lanes. This means S2 and S3 remain true. S1 is still true since you've just created the only bumpy lane in the well.

- Finally, if you play the non-square piece in the bumpy lane, then you make it smooth, but its height will still be at most 4, since a pair of bars or elbows form a 2 by 4 rectangle. The heights of the other smooth lanes only changes if this play decreases them by clearing rows. Again, S2 and S3 will still be true. Moreover, S1 remains true since you've just smoothed the only bumpy lane.

More careful accounting can actually show that the smooth lanes never reach a height of more than 6 when you play with the bar, and 5 when you play with an elbow, but this is a minor improvement. ■

In this lemma, the description of the strategy was incomplete: I told you to play pieces in *a* lowest smooth lane, giving you the freedom to choose which one. This really means there are several strategies which satisfy the description. Using any of them, you can win at two-piece tetris in an even well if the two pieces are the square and either the bar or one of the elbows.

3.2 Elbow couples, single elbows, and the bar

If you play tetris with the two elbows, or with one elbow and the bar, you have winning strategies which are very similar to those in the previous section. I won't give proofs, but will just describe how these strategies differ from the previous ones. The idea is again to have as few bumpy lanes as possible, but now you might need two of them, one for each type of piece. If there is a bumpy lane of each type in the well, then whatever the current piece is, you can use it to smooth one of those lanes. There is a new problem you might run into, which I'll illustrate using bars and left elbows. Suppose that at some point, the well has a bumpy lane, say l, which requires a left elbow to smooth it. I'll call l an **elbow-deficient** lane. What happens if the machine sends a long sequence of bars? As before, you can play these bars in lanes other than l, taking them to quite a height. Eventually, these lanes will all be smooth, and at least four rows higher than lane l. At this point, you must make a decision (see Figure 3.17). If the current piece is an elbow, you can just play it to smooth lane l, clearing three rows. However, if the current piece is a bar, playing it in lane l to clear rows would make lane l into an elbow-deficient lane, but of the opposite type. That is, lane l would now require a *right* elbow to smooth it. To avoid this problem, you examine the lookahead piece. If it's an elbow, you play the current bar on top of a smooth lane, and use the elbow to smooth lane l, thus clearing three rows. On the other hand, if the lookahead piece is a bar, then play the current and next pieces (both bars) in lane l, thus clearing four rows and leaving l left-elbow-deficient.

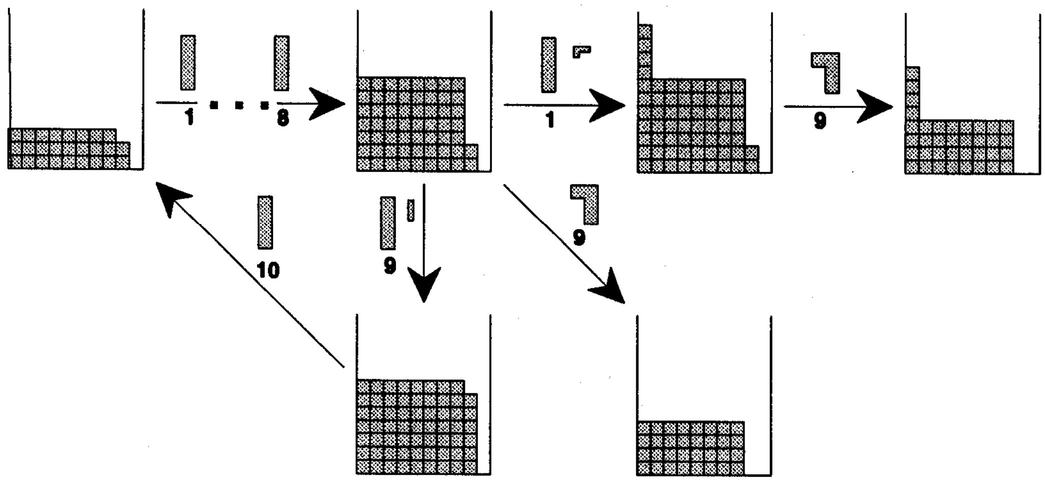

Figure 3.17: Where lookahead is used in playing bars and elbows. Above: if there is an elbow-deficient lane, and you get a long sequence of bars, play them as shown until you get to the second state. There, depending on the lookahead piece, you will either play the bar on a smooth lane, or play two bars in the elbow-deficient lane. Either way, you end up in a state with only one bumpy lane, and with lower smooth lanes.

The trick in the last paragraph also works when the well has a bar-deficient lane and the machine sends you a long sequence of elbows (see Figure 3.18), or when the well has an elbow-deficient lane of one type (left or right), and the machine sends you a long sequence of elbows of the opposite type. Notice that there are never more than two bumpy lanes, and the bumpy portion of those lanes (i.e. the rows in which those lanes are not completely full) each consist of at most four rows. Moreover, no pair of lanes will

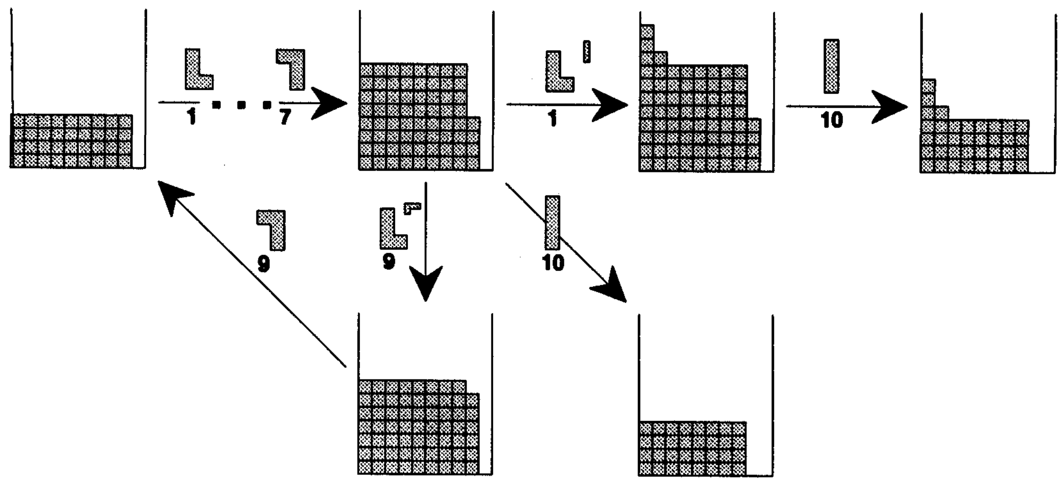

Figure 3.18: If there is a bar-deficient lane, and you get a long sequence of elbows, play them as shown until you get to the second state. There, you use lookahead to determine how to proceed. You end up in a state with only one bumpy lane, and with lower smooth lanes.

ever differ in height by more than a fixed amount. This is enough to show that these are winning strategies, at least for deep enough wells. In fact, careful counting can show that the heights of these strategies are 10 for the pair of elbows, and 11 for one elbow and the bar.

I've summarized the two-piece strategies in this chapter in Table 3.2. You might find it amusing to discover two-piece strategies for other pairs of pieces, say in a well of width 4. In that setting, I quickly and easily found strategies for the pairs of pieces listed in the table. However, for some other pairs of pieces, I didn't find strategies even though I spent a long time trying. There is a good reason for this, but I'll need the next two chapters to give it to you.

Table 3.2: A summary of winning two-piece strategies for wells of even width.

Pieces	Height of Strategy	Uses Lookahead?
Square, Bar	6	No
Square, Elbow	5	No
Elbow, Bar	11	Yes
Both Elbows	10	Yes