Chapter 5

Piecewise Linear Continuous Approximation

In this chapter, we will look at the continuum model with a piecewise linear, continuous approximation of the cubic. This function will be qualitatively, and even quantitatively, similar to the cubic. This piecewise linear function has roots at $v = 0$ and $v = 1$ and will attain the same local maximum and local minimum at the same points as the cubic.

The primary motivation for exploring this approximation is that there may be some internal functional structure in these model equations which is important in the behavior of the solution that is not captured in the piecewise linear discontinuous approximation. We will show that for certain parameter ranges, there are three steady states, two of which are stable in the space-clamped case, while the third is a saddle point. Furthermore, we show that there is, in fact, an inner region of the solution, which does not exist in the piecewise linear discontinuous approximation.

We now begin the study of the piecewise linear continuous (PWLC) model, which is given by

\[
\frac{\partial v}{\partial t} = \dot{f}(v) + \gamma (w - v), \tag{5.1}
\]

\[
\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} - \frac{w}{\tau} + \frac{\kappa}{\tau} (v - w), \tag{5.2}
\]

where $\gamma > 0$, $\kappa > 0$, $\tau > 0$, and $0 < a < 1$. 

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In order to write a formula for $\hat{f}$ explicitly, we first need to determine the local maximum/minimum of $f$ and the points where they occur. Differentiating $f$ with respect to $v$, we find the roots of

$$f'(v_0) = -3v_0^2 + 2(a + 1)v_0 - a = 0$$

(5.3)

to be

$$v_{\text{min}} = \frac{a + 1 - \sqrt{a^2 - a + 1}}{3},$$

(5.4)

$$v_{\text{max}} = \frac{a + 1 + \sqrt{a^2 - a + 1}}{3}.$$  

(5.5)

Given the restrictions on $a$, it is obvious that $0 < v_{\text{min}} < v_{\text{max}} < 1$. The local minimum and maximum of $f$ are given by:

$$f_{\text{min}} = f(v_{\text{min}}),$$

(5.6)

$$f_{\text{max}} = f(v_{\text{max}}),$$

(5.7)

respectively. If we choose $\hat{f}(0) = 0 = \hat{f}(1)$, then we may write

$$\hat{f}(v) = \begin{cases} m_1 v, & -\infty < v < v_{\text{min}}, \\ m_2 v + b_2, & v_{\text{min}} < v < v_{\text{max}}, \\ m_3 v + b_3, & v_{\text{max}} < v, \end{cases}$$

(5.8)

where

$$m_1 = \frac{f_{\text{min}}}{v_{\text{min}}},$$

(5.9)

$$m_2 = \frac{f_{\text{max}} - f_{\text{min}}}{v_{\text{max}} - v_{\text{min}}},$$

(5.10)

$$m_3 = \frac{f_{\text{max}}}{1 - v_{\text{max}}},$$

(5.11)

$$b_2 = \frac{f_{\text{min}} v_{\text{max}} - f_{\text{max}} v_{\text{min}}}{v_{\text{max}} - v_{\text{min}}},$$

(5.12)

$$b_3 = \frac{f_{\text{max}}}{1 - v_{\text{max}}}. $$

(5.13)
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Since \( f_{\text{min}} < 0 \) and \( f_{\text{max}} > 0 \), we conclude that \( m_1 < 0, m_2 > 0, b_2 < 0, b_3 > 0 \). This can be determined from the \( x \)- and \( y \)-intercepts of the different linear pieces of \( \hat{f} \).

Now that we have an explicit formula for \( \hat{f} \), we can examine the spatially homogeneous steady states.

### 5.1 Spatially homogeneous steady states and stability

We look for spatially homogeneous steady states, so we are looking for solutions to

\[
0 = \frac{\hat{f}(v)}{\gamma} + \gamma(w - v), \tag{5.14}
\]

\[
0 = \kappa v - (1 + \kappa)w. \tag{5.15}
\]

One obvious solution is the origin, \( v = 0 = w \).

The equation of the \( v \) nullcline is

\[
w = v - \frac{\hat{f}(v)}{\gamma},
\]

\[
= p(v), \tag{5.16}
\]

while the equation for the \( w \) nullcline is

\[
w = h(v) = \frac{\kappa}{\kappa + 1}v. \tag{5.17}
\]

Since \( \hat{f}'(v) < 0 \) for \( \forall \epsilon(-\infty, v_{\text{min}}) \cup (v_{\text{max}}, \infty) \), then \( p'(v) > 1 \) for \( \forall \epsilon(-\infty, v_{\text{min}}) \cup (v_{\text{max}}, \infty) \).

On the other hand, \( h'(v) = \frac{\kappa}{\kappa + 1} < 1 \). Thus, there will be three steady states if and only if

\[
h(v_{\text{max}}) > p(v_{\text{max}}). \tag{5.18}
\]

This condition is equivalent to

\[
\frac{\gamma}{\kappa + 1} < \frac{f_{\text{max}}}{v_{\text{max}}}. \tag{5.19}
\]
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To determine the middle steady state, \((v_2, w_2)\), we look for solutions of

\[
0 = (m_2 v_2 + b_2) + \gamma (w_2 - v_2),
\]

\[
0 = \kappa v_2 - (1 + \kappa) w_2,
\]

which yields

\[
v_2 = -\frac{b_2 (\kappa + 1)}{(\kappa + 1)m_2 - \gamma},
\]

\[
w_2 = -\frac{b_2 \kappa}{(\kappa + 1)}.
\]

Since \(m_2 > \frac{\ell_{\text{max}}}{t_{\text{max}}} > \frac{\gamma}{\kappa + 1}\) when condition (5.19) holds, we know that \(v_2 > 0\). If we substitute \(m_3\) for \(m_2\), and \(b_3\) for \(b_2\) in the above equations and use the fact that \(m_3 = -b_3\), then the third steady state, \((v_3, w_3)\), is given by

\[
v_3 = \frac{b_3 (\kappa + 1)}{b_3 (\kappa + 1) + \gamma},
\]

\[
w_3 = \frac{b_3 \kappa}{b_3 (\kappa + 1) + \gamma}.
\]

The stability of the steady states (let us denote the steady state at the origin as \((v_1, w_1)\)) can be determined by looking at the stability matrix

\[
A_i = \begin{bmatrix}
  m_i - \gamma & \gamma \\
  \kappa & -(1 + \kappa)
\end{bmatrix}, \quad i = 1, 2, 3.
\]

Its determinant, \(\text{det}(A_i)\), and trace, \(\text{tr}(A_i)\) are

\[
\text{det}(A_i) = -m_i (1 + \kappa) + \gamma, \quad i = 1, 2, 3,
\]

\[
\text{tr}(A_i) = m_i (1 + \kappa) - 1 - \gamma - \kappa, \quad i = 1, 2, 3.
\]

Since \(m_1, m_3 < 0\), we know that \(\text{det}(A_1), \text{det}(A_3) > 0\) and \(\text{tr}(A_1), \text{tr}(A_3) < 0\), so \((v_1, w_1)\) and \((v_3, w_3)\) represent stable steady states. Conversely, since \(m_2 > \frac{\ell_{\text{max}}}{t_{\text{max}}} > \frac{\gamma}{\kappa + 1}\), we know that \(\text{det}(A_2) < 0\), and this steady state is a saddle point.

Since there are two stable steady states separated by a saddle point in the space-clamped case when condition (5.19) holds, we now inquire into the existence of traveling front solutions between the two steady states.
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5.2 Traveling front solutions

We now seek out traveling fronts with constant speed $c$ which traverse a path from the steady state at $(v_1, w_1)$ to the one at $(v_3, w_3)$ as solutions to our PWLC model. As in the previous chapter, we look only for solutions with constant shape and make the reduction to the characteristic

$$z = x - ct.$$  \hspace{1cm} (5.29)

Substituting into (5.1) and (5.2), we get

$$-cv' = f'(v) + \gamma(w - v),$$  \hspace{1cm} (5.30)

$$-cw' = w'' - \frac{w}{\tau} + \frac{\kappa}{\tau}(v - w).$$  \hspace{1cm} (5.31)

where $'$ denotes differentiation with respect to $z$. The relevant boundary conditions are

$$\lim_{z \to -\infty} w(z) = w_3,$$ \hspace{1cm} (5.32)

$$\lim_{z \to -\infty} v(z) = v_3,$$ \hspace{1cm} (5.33)

$$\lim_{z \to \infty} w(z) = 0,$$ \hspace{1cm} (5.34)

$$\lim_{z \to \infty} v(z) = 0.$$ \hspace{1cm} (5.35)

As in the PWLD model, there is no spatial dependence in the right-hand side of our nonlinear ODE, and the associated boundary conditions are at $\pm \infty$. Thus, without loss of generality, we enforce the condition that

$$v(0) = v_{\min}.$$ \hspace{1cm} (5.36)

Furthermore, since we expect our solution to be monotonic, we shall require

$$v(z_1) = v_{\max}$$ \hspace{1cm} (5.37)

where $z_1 < 0$. 

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If we solve (5.31) for $v$, we get

$$v = -\frac{\tau}{\kappa} \left[ w'' + cw' - \frac{1 + \kappa}{\tau} \right]. \quad (5.38)$$

Differentiating the above equation with respect to $z$, we get

$$v' = -\frac{\tau}{\kappa} \left[ w''' + cw'' - \frac{1 + \kappa}{\tau} w' \right]. \quad (5.39)$$

We see that the boundary conditions on $v$ become redundant since if $w$ satisfies the boundary conditions (5.32) and (5.34), then the boundary conditions for $v$ are automatically satisfied. Eliminating $v$ and $v'$ in (5.30) using (5.38) and (5.39), respectively, we get

$$cw''' + [c^2 - (\gamma - m_i)]w'' - c \left[ \gamma - m_i + \frac{1 + \kappa}{\tau} \right] w' + \frac{\gamma - (1 + \kappa)m_i}{\tau} = \frac{\kappa b_i}{\tau} \quad (5.40)$$

where $i = 1$ for $\varepsilon(\infty, v_{min})$, $i = 2$ for $\varepsilon(v_{min}, v_{max})$, and $i = 3$ for $\varepsilon(v_{max}, \infty)$. We denote the characteristic polynomial of the above ODE by $cp_i$ where

$$cp_i(y) = cy^3 + [c^2 - (\gamma - m_i)]y^2 - c \left[ \gamma - m_i + \frac{1 + \kappa}{\tau} \right] y + \frac{\gamma - (1 + \kappa)m_i}{\tau}. \quad (5.41)$$

Notice that $cp_1(0), cp_3(0) > 0$ and $cp_1'(0), cp_3'(0) < 0$ (when $c > 0$), so both of these characteristic polynomials have one negative real root and two with positive real part which may be complex. However, $cp_2(0) < 0$, so it has one positive real root, but may have either two roots with negative real part or two roots with positive real part.

The next step is to search for stationary solutions, that is, solutions for $c = 0$. The reasons for doing this is that it reduces the order of the ODE, so it is easier to solve. Also, it allows us to determine the boundary in parameter space that corresponds to positive speed traveling front solutions.

### 5.2.1 Stationary front solutions

Here we set $c = 0$ in (5.40) to obtain

$$(m_i - \gamma)w'' + \frac{\gamma - m_i(1 + \kappa)}{\tau} w = \frac{b_i \kappa}{\tau}. \quad (5.42)$$
Thus, we define
\[
\lambda_i = \sqrt{\frac{\gamma - m_i(1 + \kappa)}{\tau(\gamma - m_i)}}, \quad i = 1, 3, \tag{5.43}
\]
\[
\omega = \sqrt{\frac{m_2(1 + \kappa) - \gamma}{\tau(\gamma - m_2)}}, \tag{5.44}
\]
and write down a solution for \( w \)
\[
w(z) = \begin{cases} 
  d_{1,1}e^{\lambda_1 z} + d_{1,2}e^{-\lambda_1 z}, & 0 < z < \infty, \\
  d_{2,1}\cos(\omega z) + d_{2,2}\sin(\omega z) + w_2, & z_1 < z < 0, \\
  d_{3,1}e^{\lambda_3(z-z_1)} + d_{3,2}e^{-\lambda_3(z-z_1)} + w_3, & -\infty < z < z_1.
\end{cases} \tag{5.45}
\]
In order to match the boundary condition (5.32), we require \( d_{3,2} = 0 \), and in order to match the boundary condition (5.34), we require \( d_{1,1} = 0 \). We now apply \( C^1 \) continuity conditions on \( w \), and the matching conditions (5.36) and (5.37).

If we apply continuity and differentiability across \( z = 0 \), we get
\[
\begin{align*}
  d_{1,2} - d_{2,1} & = w_2, \tag{5.46} \\
  -\lambda_1 d_{1,2} - \omega d_{2,2} & = 0, \tag{5.47}
\end{align*}
\]
and if we do the same at \( z = z_1 \), we obtain
\[
\begin{align*}
  d_{3,1} - d_{2,1}\cos(\omega z_1) - d_{2,2}\sin(\omega z_1) & = w_2 - w_3, \tag{5.48} \\
  \lambda_3 d_{3,1} + d_{2,1}\omega[\sin(\omega z_1) - d_{2,2}\sin(\omega z_1)] & = 0. \tag{5.49}
\end{align*}
\]

Now let us write down the solution for \( v \) using (5.38):
\[
v(z) = \begin{cases} 
  \frac{d_{1,2}}{\kappa} \left[ -\lambda_1^2 + \frac{1+\kappa}{\tau} \right] e^{\lambda_1 z}, & 0 < z < \infty, \\
  \frac{\omega}{\kappa} \left[ \omega^2 + \frac{1+\kappa}{\tau} \right] [d_{2,1}\cos(\omega z) + d_{2,2}\sin(\omega z)] + v_2, & z_1 < z < 0, \\
  \frac{d_{3,1}}{\kappa} \left[ -\lambda_3^2 + \frac{1+\kappa}{\tau} \right] e^{\lambda_3(z-z_1)} + v_3, & -\infty < z < z_1.
\end{cases} \tag{5.50}
\]
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If we apply our matching conditions (5.36) and (5.37), we get

\[
\frac{d_{1,2}}{\kappa} \left[ \frac{1}{\tau} + \kappa - \lambda_1^2 \right] = v_{\min}, \tag{5.51}
\]

\[
\frac{d_{3,1}}{\kappa} \left[ \frac{1}{\tau} - \lambda_3^2 \right] + v_3 = v_{\max}. \tag{5.52}
\]

If we substitute in for \( \lambda_1 \) and \( \lambda_3 \) in the above equations using (5.43), we get,

\[
d_{1,2} = \frac{v_{\min} (\gamma - m_1)}{\gamma}, \tag{5.53}
\]

\[
d_{3,1} = \frac{v_{\max} - v_3}{\gamma - m_3}. \tag{5.54}
\]

We may substitute in for \( d_{1,2} \) in (5.46) using (5.53) and solve for \( d_{2,1} \)

\[
d_{2,1} = \frac{v_{\min} (\gamma - m_1) - \gamma w_2}{\gamma}. \tag{5.55}
\]

Similarly, using (5.53) and (5.47), we may solve for \( d_{2,2} \)

\[
d_{2,2} = \frac{v_{\min}}{\gamma} \sqrt{\frac{[\gamma - m_2] \gamma - m_1 [\gamma - m_1 (1 + \kappa)]}{m_2 (1 + \kappa) - \gamma}}. \tag{5.56}
\]

We only have to solve for \( d_{3,1} \) and \( z_1 \) using (5.48) and (5.49). If we let

\[
y_1 = \cos(\omega z_1), \tag{5.57}
\]

\[
y_2 = \sin(\omega z_1), \tag{5.58}
\]

\[
\alpha_1 = d_{3,1} + w_3 - w_2, \tag{5.59}
\]

\[
\alpha_2 = -\frac{\lambda_3 d_{3,1}}{\omega}, \tag{5.60}
\]

then we may rewrite the problem as

\[
\begin{bmatrix}
  d_{2,1} & d_{2,2} \\
  -d_{2,2} & d_{2,1}
\end{bmatrix}
\begin{bmatrix}
  y_1 \\
  y_2
\end{bmatrix}
= \begin{bmatrix}
  \alpha_1 \\
  \alpha_2
\end{bmatrix},
\tag{5.61}
\]

with solution

\[
\begin{bmatrix}
  y_1 \\
  y_2
\end{bmatrix}
= \frac{1}{d_{2,1}^2 + d_{2,2}^2}
\begin{bmatrix}
  d_{2,1} (d_{3,1} + w_3 - w_2) + \frac{d_{2,2} d_{3,1} \lambda_3}{\omega} \\
  d_{2,2} (d_{3,1} + w_3 - w_2) - \frac{d_{2,1} d_{3,1} \lambda_3}{\omega}
\end{bmatrix}. \tag{5.62}
\]

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Enforcing the condition $y_1^2 + y_2^2 = 1$ gives us a solvability condition on the parameters. However, it turns out that there is an easier way of determining this restriction in a certain case, and we derive the restriction this way. Once this condition is known, we can use the above derivation to obtain the full solution to the problem. The method we use is to integrate (5.30) and (5.31).

First, multiply both sides of (5.31) by $w'$ and integrate from $z = -\infty$ to $z = \infty$ to obtain

$$
\int_{-\infty}^{\infty} \left[ w''w' - \frac{1 + \kappa}{\tau} w'w + \frac{\kappa}{\tau} vw' \right] dz = 0
$$

(5.63)

Using integration by parts and the fundamental theorem of calculus, the above expression can be simplified to:

$$
\frac{\tau (w'(z))^2 - (1 + \kappa) (w(z))^2 + 2\kappa v(z)w(z)}{2\tau} \bigg|_{z=-\infty}^{z=\infty} - \kappa \tau \int_{-\infty}^{\infty} v'w dz = 0,
$$

(5.64)

or if we use the boundary conditions $w'(\pm \infty) = 0$, $w(-\infty) = w_3$, $w(\infty) = 0$, $v(-\infty) = v_3$, $v(\infty) = 0$, and $w_3 = \frac{\kappa v_3}{\kappa + 1}$, we obtain

$$
\int_{-\infty}^{\infty} v'w dz = -\frac{\kappa v_3^2}{2(1 + \kappa)}.
$$

(5.65)

Now multiply (5.30) by $v'$ and integrate both sides from $z = -\infty$ to $z = \infty$ to obtain

$$
0 = \int_{-\infty}^{\infty} \left( \left[ \dot{f}(v(z)) - \gamma v(z) \right] v'(z) + \gamma w(z)v'(z) \right) dz.
$$

(5.66)

Using (5.65) and the fundamental theorem of calculus, we get

$$
\int_{0}^{v_3} \dot{f}(v) dv = \frac{\gamma v_3^2}{2(1 + \kappa)}.
$$

(5.67)
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We can do the integral on the left in three parts as follows:

\[
\int_0^v f(v) \, dv = \int_0^{v_{\min}} f(v) \, dv + \int_{v_{\min}}^{v_{\max}} f(v) \, dv + \int_{v_{\max}}^v f(v) \, dv \\
= \int_0^{v_{\min}} m_1 v \, dv + \int_{v_{\min}}^{v_{\max}} (m_2 v + b_2) \, dv + \int_{v_{\max}}^v b_3 (1 - v) \, dv \\
= \frac{m_1 v_{\min}^2}{2} + \frac{v_{\max} - v_{\min}}{2} [m_2 (v_{\max} - v_{\min}) + 2b_2] \\
+ b_3 [v_{\max} - v_{\max}] \left[1 - \frac{v_{\max} + v_{\min}}{2}\right] \\
= \frac{1}{2} \left[ f_{\max} (v_{\max} - v_{\min}) + f_{\min} v_{\max} \right] + b_3 [v_{\max} - v_{\max}] \left[1 - \frac{v_{3} + v_{\max}}{2}\right].
\]

(5.68)

Thus, going back to (5.67), we require

\[
\frac{1}{2} \left[ f_{\max} (v_{\max} - v_{\min}) + f_{\min} v_{\max} \right] + b_3 [v_{\max} - v_{\max}] \left[1 - \frac{v_{3} + v_{\max}}{2}\right] - \frac{\gamma v_{3}^2}{2(1 + \kappa)} = 0.
\]

(5.69)

If we let \( u = \frac{\gamma}{\kappa+1} \) and substitute in \( v_{3} = \frac{b_3}{b_3 + u} \), we get

\[
\frac{1}{2} \left[ f_{\max} (v_{\max} - v_{\min}) + f_{\min} v_{\max} \right] + b_3 \left[ \frac{b_3}{b_3 + u} - v_{\max} \right] \left[\frac{2 - v_{\max}}{2} - \frac{b_3}{b_3 + u} \right] - \frac{b_3 u}{2(b_3 + u)^2} = 0.
\]

(5.70)

If we multiply the above equation by \((b_3 + u)^2\) and collect powers of \( u \), we get

\[
a_2 u^2 + a_1 u + a_0 = 0,
\]

(5.71)

where

\[
a_0 = \frac{b_3^2}{2} \left[ f_{\max} (v_{\max} - v_{\min}) + f_{\min} v_{\max} + (1 - v_{\max})^2 \right],
\]

(5.72)

\[
a_1 = b_3 \left[ f_{\max} (v_{\max} - v_{\min}) + f_{\min} v_{\max} \right] + b_3 \left[ \frac{1}{2} - 2v_{\max} + v_{\max}^2 \right],
\]

(5.73)

\[
a_2 = \frac{1}{2} \left[ f_{\max} (v_{\max} - v_{\min}) + f_{\min} v_{\max} - b_3 v_{\max} (2 - v_{\max}) \right].
\]

(5.74)

Now \( a_0, a_1, \) and \( a_2 \) depend only upon \( a \), and it can be shown that for \( 0 < a < \frac{1}{2}, a_0 > 0, a_1 < 0, \) and \( a_2 < 0 \), so there is only one positive root to this equation. Also, for \( a = \frac{1}{2}, \)
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\( a_0 = 0 \) and thus there is one negative root and one zero root. What this shows, in essence, is that there can be no positive speed traveling wave solutions for \( a \geq \frac{1}{2} \).

It can be shown that one part of the boundary corresponding to traveling wave solutions is determined by

\[
\frac{\gamma}{\kappa + 1} < \frac{-a_1 - \sqrt{a_1^2 - 4a_0a_2}}{2a_2}. \tag{5.75}
\]

If we recall, the result for the PWLD model is that there were usually two values of \( \gamma \) which corresponded to stationary front solutions. It turns out that there is another boundary for zero speed waves in the PWLC model, but it has to be attained in a different manner than the above result. The reason why it does not appear in the integral method we just used is that it occurs as \( z_1 \to 0 \), and thus \( v' \) attains a jump discontinuity which it did not previously have and the above analysis is incorrect. Also, when we tried to solve the problem exactly, we were operating under the assumption that there was an internal matching layer. Again, as before, this second boundary emerges as a consequence of the discontinuity. The same methods and techniques that were used to solve the problem for the stationary fronts in the PWLD model could be used here, in principle.

In order to find the other boundary, we first have to recognize that it occurs for small values of \( \gamma \). In fact, we assume \( \gamma < m_2 \) unlike what we had tacitly assumed previously. In this case we have lost our internal matching layer, so our solution for \( w \) becomes

\[
w(z) = \begin{cases} 
\hat{d}_{1,2}e^{-\lambda_1 z}, & z > 0, \\
\hat{d}_{3,1}e^{\lambda_3 z} + w_3, & z < 0.
\end{cases} \tag{5.76}
\]

We can apply our matching conditions, (5.46) and (5.47) to obtain

\[
\hat{d}_{1,2} = \hat{d}_{3,1} + w_3, \tag{5.77}
\]

\[
-\lambda_1 \hat{d}_{1,2} = \lambda_3 \hat{d}_{3,1}. \tag{5.78}
\]
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We solve the above system of equations to obtain

\[
\hat{d}_{1,2} = \frac{\lambda_3 w_3}{\lambda_3 + \lambda_1}, \\
\hat{d}_{3,1} = -\frac{\lambda_1 w_3}{\lambda_3 + \lambda_1}.
\]  

(5.79)  

(5.80)

Now, we use equation (5.38) to write down a solution for \(v\)

\[
v(z) = \begin{cases} 
\frac{d_{1,2}}{\kappa} \left[ 1 + \frac{\kappa}{\tau} - \lambda_1^2 \right] e^{-\lambda_1 z}, & z > 0, \\
\frac{d_{3,1}}{\kappa} \left[ 1 + \frac{\kappa}{\tau} - \lambda_3^2 \right] e^{-\lambda_3 z} + v_3, & z < 0.
\end{cases}
\]  

(5.81)

Here we cannot enforce matching condition (5.36), but we try to enforce the condition we did before in the PWLD model

\[
\lim_{z \to 0^+} v(z) = v_{\text{min}},
\]

(5.82)

which is equivalent to

\[
\frac{d_{1,2}}{\kappa} \left[ 1 + \frac{\kappa}{\tau} - \lambda_1^2 \right] = \frac{\kappa v_{\text{min}}}{\tau}.
\]  

(5.83)

If we substitute for \(\hat{d}_{1,2}\) and \(\lambda_1\) in the above equation using (5.79) and (5.43), we get

\[
\frac{\lambda_3 \gamma}{(\lambda_1 + \lambda_3)(\gamma - m_1)} = \frac{v_{\text{min}}}{w_3}.
\]  

(5.84)

Thus we see that there is a second boundary condition for zero speed waves. The above equation could be solved explicitly to yield a lower bound for \(\gamma\) in terms of \(a\) and \(\kappa\).

Our next goal will be to tackle the problem of positive speed traveling wave fronts since we have some idea of the parameter set we require.

### 5.2.2 Positive speed traveling fronts

We are looking for solutions to equation (5.40) subject to the boundary conditions \(w(-\infty) = w_3\) and \(w(\infty) = 0\) with \(v(0) = v_{\text{min}}\) and \(v(z_1) = v_{\text{max}}\) for some \(z_1 < 0\).
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Let us denote by $\lambda_{i,j}$ one of the roots of the characteristic polynomial $c_p_i$. We know that $c_p_1$ and $c_p_3$ each have one negative real root and two roots with positive real part. We initially assume that all three roots are real and distinct. We also note that in the limiting case of zero speed solutions, $c_p_2$ had two imaginary roots and one positive real root. Here we initially assume that $c_p_2$ has two complex conjugate roots with negative real part. Thus the solution for $w$ is given by

$$w(z) = \begin{cases} 
  d_{1,1} e^{\lambda_{1,1} z}, & 0 < z < \infty, \\
  e^{\mu z} \left[d_{2,1} \cos(\omega z) + d_{2,2} \sin(\omega z)\right] + d_{2,3} e^{\lambda_{2,3} z} + w_2, & \lambda_1 < z < 0, \\
  d_{3,2} e^{\lambda_{3,2}(z-z_1)} + d_{3,3} e^{\lambda_{3,3}(z-z_1)} + w_3, & -\infty < z < \lambda_1,
\end{cases}$$

(5.85)

where $\lambda_{1,1} < 0$, $\mu \pm \omega$ are roots of $c_p_2$ with $\mu < 0$, $\lambda_{2,3} > 0$, and $0 < \lambda_{3,2} < \lambda_{3,3}$.

Now we apply our boundary conditions and matching conditions to pin down some of these constants. Applying $C^0$, $C^1$, and $C^2$ conditions at $z = 0$, we get

$$d_{1,1} - d_{2,1} - d_{2,3} = w_2,$$

(5.86)

$$\lambda_{1,1} d_{1,1} - \mu d_{2,1} - \omega d_{2,2} - \lambda_{2,3} d_{2,3} = 0,$$

(5.87)

$$\lambda_{1,1}^2 d_{1,1} - \mu^2 d_{2,1} - 2\mu \omega d_{2,2} + \omega^2 d_{2,3} = 0.$$  

(5.88)

At $z = \lambda_1$, we obtain

$$d_{3,2} + d_{3,3} + w_3 = e^{\mu \lambda_1} \left[d_{2,1} \cos(\omega \lambda_1) + d_{2,2} \sin(\omega \lambda_1)\right] + d_{2,3} e^{\lambda_{2,3} \lambda_1} + w_2,$$

(5.89)

$$\lambda_{3,2} d_{3,2} + \lambda_{3,3} d_{3,3} = \mu e^{\mu \lambda_1} \left[d_{2,1} \cos(\omega \lambda_1) + d_{2,2} \sin(\omega \lambda_1)\right]$$

$$+ \omega e^{\mu \lambda_1} \left[-d_{2,1} \sin(\omega \lambda_1) + d_{2,2} \cos(\omega \lambda_1)\right]$$

$$+ \lambda_{2,3} d_{2,3} e^{\lambda_{2,3} \lambda_1},$$

(5.90)

$$\lambda_{3,2}^2 d_{3,2} + \lambda_{3,3}^2 d_{3,3} = d_{2,1} e^{\mu \lambda_1} \left[\mu^2 \cos(\omega \lambda_1) - 2\mu \omega \sin(\omega \lambda_1) - \omega^2 \cos(\omega \lambda_1)\right]$$

$$+ d_{2,2} e^{\mu \lambda_1} \left[\mu^2 \sin(\omega \lambda_1) + 2\mu \omega \cos(\omega \lambda_1) - \omega^2 \sin(\omega \lambda_1)\right]$$

$$+ \lambda_{2,3}^2 d_{2,3} e^{\lambda_{2,3} \lambda_1}.$$  

(5.91)
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Unfortunately, no easy formula exists for finding the solution of the \(d_{i,j}\) in terms of the \(\lambda_{i,j}\). In principle, it is a linear system, but one that does not appear to simplify neatly as in the PWLD model. Thus, we do not attempt to write down a solution for the \(d_{i,j}\).

We can write down a solution for \(v\) easily using (5.38)

\[
v(z) = \begin{cases} 
  -\frac{d_{1,1}}{\kappa} g(\lambda_{1,1}) e^{\lambda_{1,1} z}, & 0 < z < \infty, \\
  -\frac{\tau}{\kappa} e^{\mu z} d_{2,1} \left[ (\mu^2 - \omega^2 + c\mu - \frac{1+i\kappa}{\tau}) \cos(\omega z) - \omega (2\mu + c) \sin(\omega z) \right] \\
  -\frac{\tau}{\kappa} e^{\mu z} d_{2,2} \left[ (\mu^2 - \omega^2 + c\mu - \frac{1+i\kappa}{\tau}) \sin(\omega z) + \omega (2\mu + c) \cos(\omega z) \right] \\
  -\frac{\tau}{\kappa} d_{3,3} g(\lambda_{3,3}) e^{-\lambda_{3,3} z} + v_2, & z_1 < z < 0, \\
  -\frac{\tau}{\kappa} \left[ d_{3,2} g(\lambda_{3,2}) e^{\lambda_{3,2} (z-z_1)} + d_{3,3} g(\lambda_{3,3}) e^{\lambda_{3,3} (z-z_1)} \right] + v_3, & -\infty < z < z_1,
\end{cases}
\]

where \(g\) is as given previously in (4.63). We can explicitly write down equations for our matching conditions (5.36) and (5.37)

\[
d_{1,1} g(\lambda_{1,1}) = -\frac{v_{\min} \kappa}{\tau}, \tag{5.93}
\]

\[
d_{3,2} g(\lambda_{3,2}) + d_{3,3} g(\lambda_{3,3}) = \frac{\kappa (v_3 - v_{\max})}{\tau}. \tag{5.94}
\]

Thus we are left with the problem of solving 14 transcendental equations (the 8 listed above and the six for the eigenvalues). In general, it is difficult to give a sufficiently accurate initial guess for the Newton method nonlinear equation solver that we are using to determine a solution to such a system. However, we showed previously how one can get two different zero speed solutions, one corresponding to a small value of \(\gamma\), and one corresponding to a larger value of \(\gamma\). The small \(\gamma\) problem is a very difficult one to perturb off of, so we do not attempt to do that here. One can perturb off of the large \(\gamma\) solution, but it requires a bit more work than in the PWLD model as we don't have explicit expressions for the \(d_{i,j}\) in terms of the eigenvalues. Here we need to use a bit of asymptotics to guess the appropriate scalings for the non–order 1 coefficients.
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In order to begin the process, we proceed as in the PWLD model and let \( c = \epsilon \ll 1 \) and assume that \( \kappa, a, \) and \( \tau \) are fixed. Let \( \gamma \) be expanded in a power series in \( \epsilon \) as

\[
\gamma = \gamma_0 + \epsilon \gamma_1 + O(\epsilon^2). \tag{5.95}
\]

We also need asymptotic expansions of our steady states

\[
\begin{align*}
    w_2 &= w_{2,1} + \epsilon w_{2,2} + O(\epsilon^2), \tag{5.96} \\
    w_3 &= w_{3,1} + \epsilon w_{3,2} + O(\epsilon^2), \tag{5.97} \\
    w_{2,1} &= - \frac{b_2(\kappa + 1)}{m_2(\kappa + 1) - \gamma_0}, \tag{5.98} \\
    w_{2,2} &= \frac{\gamma_1 b_2(\kappa + 1)}{[m_2(\kappa + 1) - \gamma_0]^2}, \tag{5.99} \\
    w_{3,1} &= \frac{b_3(\kappa + 1)}{b_3(\kappa + 1) + \gamma_0}, \tag{5.100} \\
    w_{3,2} &= - \frac{\gamma_1 b_3(\kappa + 1)}{[b_3(\kappa + 1) + \gamma_0]^2}. \tag{5.101}
\end{align*}
\]

Expansions of \( v_2 \) and \( v_3 \) can be attained by multiplying the equation for \( w_2 \) and \( w_3 \), respectively, by \( \frac{\kappa + 1}{\kappa} \).

In the next step, we seek to find expansions for the eigenvalues. The ones which were \( O(1) \) in the limit \( \epsilon \to 0 \) are easy to handle, and we deal with them first. Thus, we expand \( \lambda_{1,1}, \omega, \) and \( \lambda_{3,2} \) as follows

\[
\begin{align*}
    \lambda_{1,1} &= \lambda_{1,1}^{(1)} + \epsilon \lambda_{1,1}^{(2)} + O(\epsilon^2), \tag{5.102} \\
    \omega &= \omega^{(1)} + \epsilon \omega^{(2)} + O(\epsilon^2), \tag{5.103} \\
    \lambda_{3,2} &= \lambda_{3,2}^{(1)} + \epsilon \lambda_{3,2}^{(2)} + O(\epsilon^2). \tag{5.104}
\end{align*}
\]

Following the same techniques as before, we try to find expansions for the "missing eigenvalues" as in the PWLD model. Here, we can expand \( \lambda_{2,3} \) and \( \lambda_{3,3} \) as we did in the
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PWLD model for $\lambda_3$. Thus we expect

$$
\lambda_{2,3} = \frac{\lambda_{2,3}^{(1)}}{\epsilon} + \lambda_{2,3}^{(2)} + O(\epsilon),
$$

(5.105)

$$
\lambda_{3,3} = \frac{\lambda_{3,3}^{(1)}}{\epsilon} + \lambda_{2,3}^{(2)} + O(\epsilon).
$$

(5.106)

We also notice that $\mu = 0$ in the limiting zero speed wave case, so we expect to be able to expand $\mu$ as:

$$
\mu = \epsilon \mu^{(1)} + O(\epsilon^2).
$$

(5.107)

5.3 Numerical Results

5.3.1 Varying $a$ over its range

In Figure 5.1, we see the fairly classical result that the speed is a monotonic decreasing function of $a$ with the zero-speed wave occurring before $a = 0.5$.

In Figure 5.2, we see that the absolute value of $z_1$ is a monotonic increasing function of $a$. This is somewhat counterintuitive as one would expect that when $a$ is decreased, the dendrite should be becoming more excitable, and this should translate into faster moving, steeper solutions. One should note that the absolute value of $z_1$ corresponds to distance traveled within the wave for the value of $v$ to increase from $v_{\text{min}}$ to $v_{\text{max}}$. However, the difference between $v_{\text{max}}$ and $v_{\text{min}}$,

$$
v_{\text{max}} - v_{\text{min}} = \frac{2\sqrt{a^2 - a + 1}}{3},
$$

(5.108)

is a decreasing function of $a$. Thus, although $z_1$ is increasing as $a$ decreases, a greater portion of the total height achieved by the traveling front is being traversed, and the solution is not necessarily any less steep.

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![Graph](image)

*Figure 5.1: Speed of the wave as a function of $a$ for $\gamma = 1.5$, $\kappa = 5$, $\tau = 10$.*

### 5.3.2 Varying $\kappa$ over its range of values

In Figure 5.3, we see roughly the same relationship between $c$ and $\kappa$ that we saw in the PWLD model, that is, there is something of a skewed hump which intuitively should asymptote at some positive value of $c$.

In Figure 5.4, we have plotted $z_1$ against $\kappa$ over the range corresponding to traveling front solutions.

Here we see the relationship between $z_1$ and $\kappa$ is directly opposite to that between $c$ and $\kappa$. One can immediately reason out that this is the case because when the dendrite is highly excitable, this should correspond to high speed, steep solutions. That is, solutions in which $c$ is large and the absolute value of $z_1$ is small. Conversely, when the dendrite is not highly excitable, this should correspond to low speed, gently sloped solutions. That
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Figure 5.2: $z_1$ as a function of $\gamma = 1.5$, $\kappa = 5$, $\tau = 10$.

is, solutions in which $c$ is small and the absolute value of $z_1$ is large.

5.3.3 Varying $\gamma$ over its range of values

In Figure 5.5, we see the same relationship that we noted in the PWLD model, that is, there appears to be a hump shaped relationship between $\gamma$ and $c$. This is due to the balancing factors of the strength of local depolarization, voltage saturation, and current attenuation which are intricately linked to $\gamma$.

The next two graphs, Figures 5.6 and 5.7, show where the the assumed functional form of the wave breaks down. When $\gamma$ becomes small, the roots are no longer complex conjugates in the inner matching region. Instead, they become negative real roots. In Figure 5.6, we see the complex portion of the root going to zero. In Figure 5.7, we see
Figure 5.3: Speed of the wave as a function for $\kappa$ for $a = 0.1$, $\gamma = 1.5$, $\tau = 10$.

the emergence and splitting of the two real roots. This bifurcation was difficult to get numerically. The problem was that non-simple roots are typically very ill-conditioned. Thus, while only an $O(\epsilon)$ change is made in the coefficients of the polynomial, there is an $O(1)$ change in the roots. The easiest way around the problem that I found was to jump comfortably across the bifurcation, and project all other variables across the jump according to a simple linear approximation. Then go back, and accurately calculate the two negative real roots using Maple. One can head back towards the bifurcation site, or move on in the other direction.

In Figure 5.8, we have plotted a graph of $z_1$ against $\gamma$. Here we see that as $\gamma$ goes to its lower critical value corresponding to zero-speed waves, the inner matching region disappears (i.e., $z_1 \to 0$). This can be understood quite well mechanistically. If $\gamma$ is small,
there is very strong local depolarization so the spine head voltage should quickly jump to its equilibrium value. This corresponds to a slow-moving wave because it effectively chokes off the current source for the dendrite due to voltage saturation. Conversely, as $\gamma$ approaches its higher critical value corresponding to zero-speed waves, we see that the absolute value of $z_1$ is increasing. This is due to the low excitability of the spines which occurs because too much current is being lost to the dendrite. This results in a requirement for a long period of time for the spines to depolarize and thus a large absolute value of $z_1$.

5.3.4 Varying $\tau$ over its range of values

In Figure 5.9, we have graphed $\sqrt{\tau c}$ against $\tau$ to compensate for the factor involved in the nondimensionalization and to make it easier to interpret graphically. We see that the
speed of the wave, $c$, is a monotonic increasing function of $\tau$, the nonlinear time scale, which appears to asymptote to some finite positive value just as in the PWLD model.

In Figure 5.10, we have plotted the relationship of $z_1$ with $\tau$. The graph shows that the magnitude of $z_1$ is a monotonic decreasing function of $\tau$. This result is intuitively obvious, as the wave should become steeper as the nonlinear dynamics are sped up.

### 5.3.5 Numerical simulation of the PDE model

The analytical method we developed for numerically finding solutions to the system of ODEs that we derived was very valuable in mapping out some of the relevant parameter ranges, and allowed us quickly to verify numerically some of our intuitive expectations from the PWLD and full Hodgkin–Huxley models. However, these traveling fronts are
Figure 5.6: $\omega$ over the range of $\gamma$ values for which there are complex conjugate roots for $a = 0.1, \kappa = 5, \tau = 10$.

not of much experimental importance unless they can be shown to correspond to stable solutions. Lacking a proof of the stability of the system of PDEs, we instead decided to examine the solution numerically using PDEcol. The program was run with Neumann boundary conditions, and the initial guess was the computed solution from our analysis (slight perturbations to the initial guess were made and still gave the same result). The result is shown in Figure 5.11. The traveling front appears to propagate to the right with the theoretically computed speed without changing shape and thus acts as an important check of our analysis.
Figure 5.7: The real eigenvalues which exist on the other side of the bifurcation for $\alpha = 0.1, \kappa = 5, \tau = 10$. 
Figure 5.8: Graph of $z_1$ as a function of $\gamma$ for $a = 0.1, \kappa = 5, \tau = 10$. 
Figure 5.9: Speed of the wave as a function of $\tau$ for $a = 0.1, \gamma = 1, \kappa = 5$. 
Figure 5.10: $z_1$ as a function of $\tau$ for $\gamma = 1, \kappa = 5, \tau = 10$. 
Figure 5.11: This is a simulation of the traveling front for $a = 0.1$, $\gamma = 1$, $\kappa = 5$, $\tau = 10$. 