NON-SYMMETRIC HOLMBOE WAVES

By

Susan Patricia Haigh

B. Math. (Pure & Applied Mathematics) University of Waterloo, 1988

M. A. Sc. (Aerospace) University of Toronto, 1990

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Abstract

When two flows of different velocity and density meet, a shear layer with a density gradient is formed. Under certain conditions this flow can be unstable. A statically stable stratified shear flow in which the density interface is much thinner than the shear layer thickness can be linearly unstable to two modes with equal growth rates and equal and opposite phase speeds. The superposition of these two modes is called a Holmboe instability. This instability is only possible when the flow is symmetric about the center of the shear layer. We examine the effect of breaking this symmetry by allowing the center of the density interface to be displaced with respect to the center of the shear layer. There are three major components to this study: linear stability analysis, nonlinear numerical simulations, and comparison with laboratory experiments.

Linear stability analysis is used to examine the effect of the density interface offset on the overall stability of the flow. Both inviscid theory with piecewise linear background velocity and density profiles, and viscous theory with smooth background profiles are used. As in previous studies, it was found that the growth rate of one mode increases and that of the other mode decreases as the density interface displacement is increased. The precise behaviour depends on the relative thickness of the density interface with respect to the shear layer thickness. For inviscid theory with piecewise linear background profiles, we show that the initial perturbations must be two-dimensional. When the effects of viscosity and diffusion are included, however, it may be possible that the weaker mode is initially three-dimensional. Detailed analysis of the energy transfer in the linear regime indicates that when the background flow loses its symmetry, it is the mode with the larger growth rate that is primarily responsible for the extraction of energy from the mean flow.
Two-dimensional numerical simulations are used to examine the nonlinear development of “non-symmetric” Holmboe instabilities. We start by perturbing the flow with the stronger mode predicted by linear theory. We then examine the response of the flow to weaker mode. Finally, we impose both modes. By comparing the development of these three flows we are able to study the interaction between the two modes and the effect of initial conditions on the development of instabilities. Although the initial development of instabilities depends on the initial conditions, this dependence weakens as the density interface offset is increased. Also, preliminary results indicate that long term behaviour of the perturbations are independent of initial conditions.

Results of the numerical simulations are compared to both symmetric and non-symmetric Holmboe instabilities that have been observed in laboratory experiments. Since we are unable to compute the flow at the high Prandtl number of the salt stratified experimental flows, we run the simulation for a range of increasing Prandtl numbers to determine how instabilities of thermally stratified flows (with low Prandtl number) and salt stratified flows differ. Results of these simulations indicate that differences between the experimental and numerical flows can be attributed to the thicker density interface and lower Prandtl number used in the numerical simulations. When the density interface is thicker and the Prandtl number lower, the waves or billows formed by the instabilities are not as sharply defined as in the experiments.
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Chapter 1

Introduction

Mixing of stably stratified shear flows is an important aspect of many problems in oceanography, meteorology, and several branches of engineering. Properly quantifying this mixing, in order to accurately model turbulence, remains a problem of great importance (see, for example, Gregg [19] and Fernando [14]). Studying mixing in a stably stratified flow has an advantage over other types of transition; stable stratification appears to sufficiently suppress the onset of turbulence to enable distinct events in the transition to turbulence to be classified (Sherman et al. [50]). Wave-like instabilities are the first to occur during transition. Thus understanding the types of wave-like instabilities that can occur, and how they develop, is the first step in gaining complete understanding of mixing. The study of such instabilities remains one of the fundamental problems in the field of hydrodynamic stability.

The traditional method used to determine the stability of a given flow is the normal mode approach to linear stability analysis. This method adds a perturbation (usually, but not necessarily, two-dimensional) to a background flow. The governing equations are then linearized about this state. The normal mode approach allows one to determine the wave number, growth rate, and spatial structure of the most unstable modes. A complete description of this method is given in section 2.2. The underlying assumption in linear stability analysis is that, if the perturbation from the mean flow initially consists of random noise, then it is the most unstable modes that grow fastest and dominate.

All investigations into the stability of a flow start with determining its linear stability.
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On its own, however, linear stability analysis does not provide an adequate portrait of the evolution of the flow, since it is only valid when the perturbation amplitudes are very small. It is well known that nonlinear effects rapidly become important and should not be ignored. With today’s computers it is possible to use numerical simulations to examine these nonlinear effects. One drawback, however, is that only specific cases can be examined. The results of linear stability analysis can be used as a guide, indicating which flow parameters may produce instabilities of interest. Also, since linear stability analysis tell us which modes are most likely to grow, the nonlinear simulation can be given a head start by using the results from linear stability analysis to initialize the perturbations.

In this thesis, we restrict our attention to the stability of a statically stable stratified shear flow whose density interface is much thinner than the shear layer thickness, as depicted in figure 1.1. Such background flows occur in many physical situations, for example, salt wedge flows (Yonemitsu [70]) and exchange flows (Zhu [72]). Yonemitsu [70] and Zhu [72] have conducted laboratory experiments to model such flows and have observed wave-like instabilities at the density interface of the two fluids. Since the density gradient acts as a stabilizing force, the instabilities caused by the presence of the velocity shear do not always develop into fully turbulent flow. In this case, studying the onset and development of instabilities deals directly with many practical issues: for example, determining the interface location and the interfacial friction are of interest in many engineering problems. In addition, by determining the effect on the mean flow of small scale structures, a study of the development of interfacial instabilities can lead to a better understanding of large scale dynamics of two layer systems.

We assume that the only non-zero component of the background velocity vector is in the $x$-direction and that the background flow does not vary in this direction. In addition, we allow the background flow to develop in time only, since temporally developing flows
are easier to compute than spatially developing flows. These assumptions result in the background flow having parallel streamlines, and hence the term parallel flow is used.

A good first approximation to the background flow shown in figure 1.1 is the piecewise linear inviscid flow studied by Holmboe [23] (figure 1.2). This flow is characterized by a single parameter, the bulk Richardson number, \( J = 2\frac{(\rho_2 - \rho_1)gh}{(\rho_2 + \rho_1)(U_1 - U_2)^2} \), which is a measure of the relative strength of the stratification to the strength of the velocity shear. The flow depicted in figure 1.2 is linearly unstable to two types of instabilities. When \( J < 0.07 \), linear stability analysis predicts a Kelvin-Helmholtz instability. This instability is characterized by zero phase speed and is easily identified by the rolling up of the density interface. Since the original work of Helmholtz [22] and Kelvin [25], much has been accomplished towards understanding this instability, including the first nonlinear numerical simulations by Tanaka [61] and Patnaik et al. [44]. Recent research in this area includes the study of secondary instabilities (see, for example, Klaassen & Peltier [31] and Staquet [58]), and understanding how the Kelvin-Helmholtz instability develops into fully three-dimensional turbulence (Caulfield & Peltier [8]).

The second type of instability for the flow depicted in figure 1.2 is a Holmboe instability. This is defined as the superposition of two unstable modes of equal growth rate and equal, but opposite, phase speeds, and is named after Holmboe [23]. A Holmboe instability is characterized by a right-moving wave whose energy is concentrated above the center of the shear layer and a left-moving wave whose energy is concentrated below the center of the shear layer. Although Holmboe instabilities occur for \( J > 0 \), they are the only possible instabilities when \( J > 0.07 \). These instabilities are of interest as they are the only mechanism through which the flow may become unstable when stratification is strong compared to the strength of the velocity shear (\( J > 0.07 \)).

Holmboe [23] studied the case where the flow is symmetric about the center of the shear layer. In nature, the background flow is not necessarily symmetric, e.g. the salt
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wedge flows described by Yonemitsu [70]. There are two ways that the flow may lose its symmetry: either by having horizontal boundaries placed at different distances from the center of the shear layer (Yonemitsu [70]), or by displacing the density interface with respect to the center of the shear layer (Lawrence et al. [38]). A more general case would allow both of these to occur. When the symmetry of the background flow is broken, one of the modes of the Holmboe instability dominates. Non-symmetric background flows have been used to explain why “one-sided” instabilities often occur in laboratory experiments, instead of the Holmboe instabilities predicted by the symmetric model (see, for example, Maxworthy & Browand [40], and Koop & Browand [33]). Since relatively little research has been done to understand these “non-symmetric” Holmboe instabilities, the main purpose of the present study is to understand how the displacement of the density interface affects the development of instabilities in a flow which, for the symmetric case, supports pure Holmboe instabilities.

In chapter 2, we present the background needed for the present study, including an introduction to the governing equations, a description of linear stability analysis, and a review of the relevant literature. Chapter 3 focuses on how displacing the position of the density interface with respect to the center of the shear layer affects the linear stability of the flow. This is separated into two distinct parts: inviscid theory for piecewise linear background profiles, and viscous theory for smooth background profiles. The inviscid theory is an extension of the work of Lawrence et al. [38] who modified Holmboe’s original model by allowing a displacement of the density interface. In section 3.2.1 we examine the effect of imposing horizontal boundaries on this model. We also examine the possibility that the perturbations are initially three-dimensional (section 3.2.2).

Since inviscid flows with piecewise linear profiles do not take into account such physical parameters as the Reynolds number and the Prandtl number, section 3.3 is devoted to the linear stability of viscous flows with smooth background profiles. In particular, we
examine how varying the thickness of the density interface affects the stability of the flow when the displacement of the density interface is also varied. As with the inviscid case, we examine the effect of horizontal boundaries and the possibility of three-dimensional instabilities. We also examine, in detail, the shapes of the normal modes and how they govern the growth mechanism of the instability.

In chapter 4, numerical simulations are used to examine the nonlinear evolution of non-symmetric Holmboe instabilities. Since a Holmboe instability consists of two waves propagating in opposite directions, all previous studies of the nonlinear evolution of Holmboe instabilities have initialized the perturbations with both modes predicted from linear stability analysis. As a starting point in our nonlinear investigations, we initialize the flow with one mode only. We start by perturbing the flow with only the right moving instability, because linear theory predicts that this mode dominates the flow as the offset increases. We then examine the response of the flow to left moving perturbations. Finally, we impose both a left moving and a right moving perturbation. By comparing the results of these different cases, we are able to determine how initial conditions affect the development of an instability and the interaction between the two modes.

As a final test of the validity of the numerical simulations, chapter 5 concentrates on comparing the results of the nonlinear simulations with those of laboratory experiments. We start by comparing results of chapter 4 with the recent laboratory observations of Holmboe instabilities by Zhu [72]. Next we run simulations which match the flow conditions of Lawrence et al. [38] and Guez & Lawrence [20]. These latter flows are clear examples of “one-sided” instabilities. Comparing results of the numerical simulations with both symmetric and non-symmetric instabilities observed in the laboratory helps us identify strengths and weaknesses of the present model.
Figure 1.1: Schematic of a two layer flow with a thin density interface
Figure 1.2: Two-layer flow studied by Holmboe [23]
Chapter 2

Background

2.1 Governing Equations

A two-dimensional, unsteady, stratified flow can be described completely in terms of the following variables: \( u \) and \( w \) (velocity components in the horizontal, \( x \), and vertical, \( z \), directions, respectively), \( \rho \) (density), \( \theta \) (concentration of stratifying agent, e.g. heat or salinity), and \( p \) (pressure). Assuming that the fluid is Newtonian, incompressible, and the equation of state linear, the governing equations are given by

1. conservation of momentum:
   \[
   \rho u_t + \rho uu_x + \rho w u_z = -p_x + \mu \left( \Delta u + \frac{1}{3}(u_x + w_z)_x \right), \tag{2.1}
   \]

   \[
   \rho w_t + \rho uw_x + \rho w w_z = -p_z - \rho g + \mu \left( \Delta w + \frac{1}{3}(u_x + w_z)_z \right), \tag{2.2}
   \]

2. conservation of mass:
   \[
   \rho_t + \rho u_x + \rho w_z + u \rho_x + w \rho_z = 0, \tag{2.3}
   \]

3. diffusion equation for stratifying agent:
   \[
   \theta_t + u \theta_x + w \theta_z = \kappa \Delta \theta, \tag{2.4}
   \]

4. the equation of state:
   \[
   \rho = \rho_0 [1 - \gamma (\theta - \theta_0)], \tag{2.5}
   \]
In the above set of equations, \( g, \mu, \kappa, \) and \( \gamma \) are the gravitational acceleration, the molecular viscosity, the diffusivity of the stratifying agent, and the coefficient of expansion for the stratifying agent, respectively, and are all assumed to be constant. In addition, \( \Delta \) represents the two-dimensional Laplacian operator, i.e. \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \). The equation of state (2.5) and the heat equation (2.4) are often good approximations when the fluid is a pure liquid, where the effects of compressibility are minor (see Pedlosky [45]). In (2.5), \( \rho_s \) is the density at a standard state \( \theta_0 \). The validity of the two-dimensional assumption will be examined in chapter 3.

If \( \rho \) and \( p \) are expanded about a state of hydrostatic equilibrium \( \rho_s \) and \( p_s \), as discussed, for example, in Turner [65], the equations of conservation of momentum can be written as

\[
\rho u_t + \rho w x + \rho w z = -p'_x + \mu \left( \Delta u + \frac{1}{3}(u_x + w_z)_x \right), \quad (2.6)
\]

\[
\rho w_t + \rho w x + \rho w z = -p'_z - \rho' g + \mu \left( \Delta w + \frac{1}{3}(u_x + w_z)_z \right), \quad (2.7)
\]

where \( \rho = \rho_s(z) + \rho'(x, z, t), \rho = p_s(z) + p'(x, z, t), \) and \( \frac{dp_s}{dz} = -g\rho_s \).

### 2.1.1 Small Density Variations

For the flows we consider, the maximum density variation \( \Delta \rho \), is small compared to the standard density \( \rho_0 \), i.e. \( |\Delta \rho|/\rho_0 = O(10^{-2}) \), or less. Using this property, the above equations can be simplified considerably. For a fluid in which the effects of compressibility are minor, changes in density have little effect on the conservation of mass, and (2.3) can be approximated by the continuity equation for an incompressible and nondiffusive flow.

\[
u_x + w_z = 0. \quad (2.8)
\]

In order to derive the above approximation, we use a scaling argument. For the mean background flow depicted in figure 1.1, motions in the \( x \)-direction are much larger than
those in the $z$-direction. We can write $u = O(U)$ and $w = O(W)$, where $W \ll U$. Also, we let $x = O(L)$ and $z = O(D)$. We expect the time scale in the vertical direction to be of the same order of magnitude as the time scale in the horizontal direction. Hence $t = O(L/U) = O(D/W)$, which implies that $D \ll L$. Finally, since variations in density are small, the equation of state (2.5) suggests that we may write $\rho = \rho_0(1 + r(x, z, t))$, where $\rho_0$ is constant and $r = O(\epsilon) \leq O(10^{-2}) \ll 1$.

Using the equation of state, we can write (2.4) as

$$
O \left( \frac{\rho_0 \epsilon U}{L} \right) + O \left( \frac{\rho_0 \epsilon U}{L} \right) + O \left( \frac{\rho_0 \epsilon W}{D} \right) = \kappa \rho_{xx} + \kappa \rho_{zz}.
$$

We notice that all terms on the left hand side of (2.9) have the same order of magnitude. The dominant term on the right hand side is $\kappa \rho_{zz}$, since $D \ll L$. As diffusive terms on the right hand side cannot dominate, we have $\kappa \leq O \left( \frac{L^2}{U} \right)$.

Subtracting (2.9) from the conservation of mass (2.3), we obtain

$$
O \left( \frac{\rho_0 U}{L} \right) + O \left( \frac{\rho_0 W}{D} \right) = -\kappa \rho_{xx} - \kappa \rho_{zz}.
$$

Now,

$$
\kappa \left( \frac{\rho_0 \epsilon}{L^2} \right) \ll \kappa \left( \frac{\rho_0 \epsilon}{D^2} \right) \leq O \left( \frac{\rho_0 \epsilon U}{L} \right) \ll O \left( \frac{\rho_0 U}{L} \right).
$$

The relations in (2.11) indicate that we can neglect the terms on the right hand side of (2.10) and obtain (2.8).

We can now simplify the momentum equations (2.6) and (2.7) using the Boussinesq approximation. This approximation assumes that variations in density are only important in the gravitational term of (2.7). This assumption is valid providing the maximum density variation $\Delta \rho$ is small compared to $\rho_0$, as is the case here (see Turner [65] for a more detailed discussion of the Boussinesq approximation and the types of flows for which
it is valid). Using the Boussinesq approximation and the incompressibility condition (2.8) derived above, we can rewrite (2.6) and (2.7) as

\[ u_t + uu_x + wu_z = -\frac{\rho'_x}{\rho_0} + \nu \Delta u, \tag{2.12} \]

\[ w_t + uw_x + ww_z = -\frac{\rho'_z}{\rho_0} - \frac{\rho'g}{\rho_0} + \nu \Delta w, \tag{2.13} \]

where \( \nu = \mu / \rho_0 \) is the kinematic viscosity.

Equations (2.12) and (2.13) along with (2.8) and (2.9) form our new set of governing equations. These equations are valid when the fluid studied is a pure liquid for which density variations are small and the effects of compressibility are negligible.

### 2.1.2 Stream Function Representation

We can use the continuity equation for incompressible flow (2.8) to define a stream function \( \psi \) such that

\[ u = \frac{\partial \psi}{\partial z} ; \quad w = -\frac{\partial \psi}{\partial x}. \tag{2.14} \]

Taking \( \partial(2.12)/\partial z - \partial(2.13)/\partial x \) and using (2.14) one obtains the following equation for the stream function.

\[ (\Delta \psi)_t + u(\Delta \psi)_x + w(\Delta \psi)_z = \frac{\rho'_x g}{\rho_0} + \nu \Delta \Delta \psi. \tag{2.15} \]

We have reduced the original system of five equations, (2.1) to (2.5) with five unknowns, \( u, w, p, \rho, \) and \( \theta \), to a system of two equations (2.15) and (2.9), with two unknowns, \( \psi(x, z, t) \) and \( \rho(x, z, t) = \rho_0(z) + \rho'(x, z, t) \).

### 2.1.3 Nondimensional Equations

It is convenient to examine the nondimensional equations. This has the advantage of classifying various flows by the means of certain nondimensional parameters. A schematic
of the background flow that we will be examining is depicted in figure 1.1. Precise definitions of the background profiles are discussed in chapter 3. Defining the mean velocity as $\bar{U} = (U_1 + U_2)/2$ and using the length scale $L$, the velocity scale $\delta U$, and the density scale $\delta \rho$, where $L$, $\delta U$, and $\delta \rho$ will be defined as needed, the nondimensional variables, denoted by $^*$, are defined as

$$
\begin{align*}
  u &= \delta U u^* + \bar{U} \\
  \psi &= \delta U L \psi^* \\
  x &= L x^* \\
  w &= \delta U w^* \\
  \rho &= \delta \rho \rho^* + \rho_0 \\
  z &= L z^* \\
  t &= \frac{L}{\bar{U}} t^*.
\end{align*}
$$

(2.16)

Substituting the above definitions into the governing equations (2.15) and (2.9), we obtain

$$
(\Delta \psi)_t + u(\Delta \psi)_x + w(\Delta \psi)_z = J\rho_x' + \frac{1}{\text{Re}} \Delta \Delta \psi,
$$

(2.17)

and

$$
\rho_t + u \rho_x + w \rho_z = \frac{1}{\text{RePr}} \Delta \rho.
$$

(2.18)

In the above equation the $^*$ notation for nondimensional quantities has been dropped. The three nondimensional parameters, namely, the bulk Richardson number $J$, the Reynolds number $\text{Re}$, and the Prandtl number $\text{Pr}$, are defined as

$$
J = \frac{\delta \rho g L}{\rho_0 (\delta U)^2}, \quad \text{Re} = \frac{\delta U L}{\nu}, \quad \text{Pr} = \frac{\nu}{\kappa}.
$$

(2.19)

The Prandtl number, $\text{Pr} = \nu/\kappa$, is the ratio of kinematic viscosity to thermal diffusivity, whereas the Schmidt number, $\text{Sc} = \nu/\kappa$, is the ratio of kinematic viscosity to the coefficient of mass diffusivity (we are interested in the diffusion of salt). Throughout the remainder of this study, however, we shall refer to the ratio $\nu/\kappa$ as the Prandtl number regardless of the stratifying agent (heat or salt).
2.2 Linear Stability Analysis

The normal mode approach to predicting linear stability is the most common analytic tool used to investigate the stability of a flow. For a parallel shear flow, we start with a steady flow in the $x$-direction which satisfies the governing equations. Perturbations are then superimposed onto the steady flow and linearized equations for the perturbations are derived. The normal mode approach assumes that the perturbations are proportional to $\exp[i\alpha(x - ct)]$. For example, for the governing equations (2.15) and (2.9) we write

$$\psi(x, z, t) = \Psi(z) + \psi'(x, z, t),$$  \hspace{1cm} (2.20)

$$\rho(x, z, t) = \rho_a(z) + \rho'(x, z, t),$$  \hspace{1cm} (2.21)

where

$$\psi'(x, z, t) = \Re\{\phi(z)e^{i\alpha(x - ct)}\},$$  \hspace{1cm} (2.22)

$$\rho'(x, z, t) = \Re\{\hat{\rho}(z)e^{i\alpha(x - ct)}\}.\hspace{1cm} (2.23)$$

Here the nondimensional wave number, $\alpha$, is real and positive, and $c = c_r + ic_i$ is the complex nondimensional wave speed. It is the amplification factor $\alpha c_i$ which determines the linear stability of the problem. If $c_i > 0$, the flow is linearly unstable since the perturbations will grow exponentially. Conversely for $c_i < 0$ it is stable. If $c_i = 0$ then the flow is said to be neutrally stable. Choosing $\alpha$ real and $c$ complex means that the solution will grow in time (known as the temporal modes). This is in contrast to the physical case where solutions tend to grow in space (known as the spatial modes). Spatial modes can be found by choosing $c$ real and $\alpha$ complex (see Drazin & Reid [12] for a discussion on the use of spatial modes). We will restrict our study to that of temporal modes, however, since the nonlinear evolution of these is more easily computed. Gaster [17] provides the following approximate transformation between the parameters of the temporal problem
and those of the corresponding spatial problem. If the temporal solution has parameters \( \alpha = \alpha_r \) and \( \alpha c = \alpha_r c_r + i\alpha_r c_i \), then the corresponding spatial solution has parameters \( \alpha = \alpha_r - i\alpha_r c_i/c_g \) and \( \alpha c = \alpha_r c_r \), where \( c_g = \partial(\alpha_r c_r)/\partial\alpha_r \) is the group velocity. This approximation is valid providing amplification rates are small.

When linear stability analysis was first used, many investigators hoped that transition to turbulence would follow directly from linear instability. It was quickly shown, however, that this is not necessarily the case. Linear analysis is only valid for a very short time before nonlinear effects become important. Nevertheless, linear instability does correctly describe the onset and early evolution of infinitesimal perturbations. It also seems to give a qualitatively correct indication of the overall stability of the flow (Maslowe [39]). For this reason, much of the literature has been devoted to linear stability theory including books by Chandrasekhar [9] and more recently by Drazin & Reid [12], as well as review articles by Maslowe [39] for viscous shear flows, Pellacani [46] for stratified shear flows, and Gage [16] for viscous stratified shear flows.

The normal-mode approach of linear stability analysis reduces the system of nonlinear partial differential equations into a system of linear ordinary differential equations where the boundary conditions depend on the flow configuration. These new differential equations describe an eigen-problem where \( c \) is the eigenvalue. The coefficients of the resulting differential equations are, in general, not constant. For this reason these equations can only be solved exactly for a handful of special cases. These are mostly limited to steady flows approximated by either piecewise constant or piecewise linear velocity and density profiles. Although these flows are not physically realistic, they often admit simple solutions. For more complicated flow profiles it is usually necessary to use either asymptotic approximations, or to solve the equations numerically.

Having described the linear stability problem, we will now discuss how it is applied. For a given flow configuration, the wave speed \( c \) depends on the value of the wave number.
\( \alpha \), which can take on any real value. Depending on the problem, there is, for each value of \( \alpha \), either a continuous spectrum of wave speeds \( c \), a countable infinite number of discrete values of \( c \), a finite number of discrete values of \( c \), or some combination of these cases (see Drazin & Reid [12]). The complete solution for the stream function perturbation \( \psi' \) is of the form

\[
\psi'(x, z, t) = \int \sum_n A_n^\alpha \phi_n^\alpha(z) e^{i\alpha(x-c_n^\alpha t)} d\alpha
\]  

(2.24)

where \( c_n^\alpha \) is the \( n \)th wave speed associated with the wavenumber \( \alpha \) and \( A_n^\alpha \) are coefficients which are determined by the boundary conditions. If there is a continuous spectrum of wave speeds, then the summation in (2.24) is replaced by an integration. Obviously for the solution to be stable, we need \( \Im\{c_n^\alpha\} < 0 \) for all \( \alpha \) and \( n \) whereas for instability we only need \( \Im\{c_n^\alpha\} > 0 \) for one value of \( \alpha \) and \( n \).

In view of the above form of the general solution, it is easy to see that solving the complete linear problem is not a trivial task. Fortunately, one does not usually need to determine \( \psi' \) completely to gain some understanding of the stability mechanism. In fact, for the most part, the use of linear analysis is restricted to determining the dependence of the wave speed \( c \) on the wavenumber \( \alpha \). This dependence is usually described in the form of a dispersion relation. Once this relation has been obtained, either exactly, with asymptotics, or numerically, one of two problems is usually addressed. The first deals with transition from laminar to turbulent flow and is concerned with determining at what value of a given parameter, say the Reynolds number, does the flow first become linearly unstable. This value is often called a critical value. The second problem deals with a flow which is known to be unstable. In this case, given a fixed value of a parameter, for example, the Reynolds number, we wish to know at what wave number \( \alpha \) does the most unstable mode occur (i.e., the largest value of \( \alpha c_i \)). Both approaches are used in the present study.
2.3 Literature Review

In this section we review the relevant literature. We concentrate on important results that hold for stratified shear flows in general and those that are specific to Holmboe instabilities.

Many of the major results obtained in the study of the stability of stratified shear flows have been for inviscid flows. An advantage of performing linear stability analysis on inviscid flows is that exact solutions to the dispersion relation can often be found. The simplest model of a stably stratified shear flow is depicted in figure 2.1. Instability in such a flow was first remarked upon by Helmholtz [22] and calculated by Kelvin [25]. Details of the calculations are given in section 232 of Lamb [36] and section 4 of Drazin & Reid [12]. Results of linear stability analysis indicate that a mode is unstable if 
\[ g(\rho_1^2 - \rho_2^2) < k \rho_1 \rho_2 (U_1 - U_2)^2, \]
where \( k \) is the wave number of the normal mode. When \( U_1 \neq U_2 \) we can always find a value of \( k \) which satisfies this condition, and hence the flow is unstable. If, in addition, \( \rho_1 = \rho_2 \), then the flow is unstable at all wave numbers. This type of instability is called a Kelvin-Helmholtz instability and is characterized by a ‘rolling up’ of the shear layer into periodic train of symmetric vortices which travel with the mean flow (Smyth [51]).

In 1962, Holmboe [23] examined the stability of a discontinuous density interface contained within a piecewise linear velocity shear of finite thickness (figure 1.2). The stability of such a flow is governed by the bulk Richardson number \( J \). When \( J < 0.07 \), the velocity shear dominates and Kelvin-Helmholtz instabilities are predicted. He discovered, however, that when \( J > 0.07 \), the flow is unstable to two modes having equal growth rates and equal, but opposite, phase speeds. Using the method of symmetric waves he was able to determine the time dependence of the phase speeds for the two modes. The superposition of these two modes is usually referred to as a Holmboe instability, a term
first used by Browand & Wang [4].

Smyth [52] offers the following physical explanation for the difference between a Holmboe instability and a Kelvin-Helmholtz instability. If one thinks of the density interface as a flexible boundary between the two fluids, then the boundary is almost rigid when stratification is strong (i.e., large J). In this case, the two modes of the Holmboe instability, i.e., the right moving mode located in the upper layer and the left moving mode in the lower layer, propagate through the flow with very little interaction. When stratification is weak (small J), however, the interface is very flexible and disturbances in the upper layer interact with those in the lower layer as they pass through each other. Below some critical value of J (J < 0.07 for the case studied by Holmboe [23]), the phase speeds of the right and left moving modes become zero and there is a phase locking of the two modes. The two modes then wrap around each other, forming the Kelvin-Helmholtz instability.

Hazel [21] developed a numerical method for solving the linear stability problem for continuous profiles. He examined the stability of flows where the density interface thickness has a finite value and is characterized by $R$, the ratio of the shear layer thickness to the density interface thickness. When $R = 1$, the only possible instabilities are Kelvin-Helmholtz instabilities. When $R = 5$, unstable modes with non-zero phase speed, i.e., Holmboe instabilities, were found. This indicates that there is some critical value of $R$ below which Holmboe instabilities do not exist. By numerically computing stability diagrams for various values of $R$, Smyth [51] found the critical value of $R$ to be approximately 2.4. Smyth [51] and Smyth et al. [53] used numerical simulations to examine the nonlinear evolution of Holmboe instabilities and were able to confirm Holmboe’s prediction of varying phase speed as the right and left modes interact. In addition, Smyth and Peltier have conducted two studies on the transition from Kelvin-Helmholtz to Holmboe instabilities. The first concentrated on linear theory [54], while the second examined the results of nonlinear numerical simulations [56].
Observations of both Kelvin-Helmholtz and Holmboe instabilities have been documented. Perhaps the best known examples of Kelvin-Helmholtz instabilities are the billow clouds near Denver, Colorado (Colson [11]) and the tilting tube experiments by Thorpe [63]. Holmboe instabilities are, in general, less well-known than Kelvin-Helmholtz instabilities. There are numerous examples, however, of laboratory observations of Holmboe instabilities. These include the experimental results of Browand & Winant [5], Koop [32] (reproduced in part in Tritton & Davies [64] figure 8.8), and, more recently, Pouliquen et al. [48], and Zhu [72]. Browand & Wang [4] measured the position of the stability boundary experimentally and found good agreement with the predictions of Holmboe [23]. The calculated growth rates for their low Reynolds number flows ($Re = (U_1 - U_2)h/\nu = 40-300$), however, were an order of magnitude lower. There are two possible explanations for this discrepancy. The first lies in the piecewise linear profiles used by Holmboe [23]. Smyth [51] has computed the stability of an inviscid flow with smooth background profiles. He observed that as the thickness of the density interface increases, the maximum growth rate in the Holmboe regime decreases. Secondly, Nishida & Yoshida [43] have computed the stability boundaries for a two-layer stratified shear flow at various Reynolds numbers. They show that the region of instability decreases significantly with decreasing Reynolds number, indicating a decrease in the maximum growth rates. These results suggest that comparison with a model that includes the effects of viscosity and has smooth background profiles might yield better agreement between experiments and theory.

In addition to Kelvin-Helmholtz and Holmboe instabilities, a third type of instability has been observed in stratified shear flows with thin density interfaces. This instability occurs for larger Richardson numbers when Holmboe waves are expected. Instead, the instability exhibits a “one-sidedness” with the wave protruding into one layer only and is believed to be either, the result of a shift of the density interface with respect to the
center of the shear layer, or, the result of horizontal boundaries being non-symmetrically placed with respect to the centers of the shear layer and density interface. This has been observed by Keulegan [26], Yoshida [71], Koop & Browand [33] (see also Maxworthy & Browand [40]), Lawrence et al. [38], and Yonemitsu [70]. In order to better understand this “one-sideness”, Lawrence et al. [38] modified Holmboe’s model by allowing the density interface to be displaced with respect to the center of the shear layer. They found that, as the density interface is displaced, one of the two modes of a Holmboe instability dominates (we will refer to this as a non-symmetric Holmboe instability). For a viscous flow, Yonemitsu [70] examined the effect of non-symmetrically placed horizontal boundaries. Similar results were obtained as discussed below. Smyth [51] made some preliminary computations on the nonlinear evolution of non-symmetric Holmboe waves. He found that these waves evolve, after many oscillation periods, into weakly nonlinear interfacial waves which are qualitatively similar to waves that have been observed in the lower atmosphere by Gossard & Richter [18] and Emmanuel et al. [13].

Inviscid theory has been used to determine several general criteria for instability. In 1880, Rayleigh [49] proved his famous inflection-point theorem: a necessary condition for instability is that the background velocity profile have an inflection point. Howard [24] proved the semi-circle theorem which states that, for any unstable mode, the complex wave speed, \(c\), must lie inside a semi-circle in the upper half-plane whose diameter is given by the maximum amplitude of the background velocity. Miles [41] and Howard [24] showed that a sufficient condition for stability in a stratified parallel shear flow is that the gradient Richardson number \(\text{Ri} = \frac{-g\rho_\alpha' (z)}{(\rho_\alpha U'(z))^2}\), where \(U(z)\) is the horizontal background velocity profile, be greater than 1/4 everywhere in the fluid. It is possible, however, to have unstable flows for \(\text{Ri} > 1/4\) when the effects of viscosity and thermal diffusion are included (see Gage [16]). Also, for viscous, thermally dissipative flows, unstable waves whose phase speed lie outside the semi-circle predicted by Howard [24],
for inviscid flows, have been found (Gage [16]).

Since boundaries are always present in the physical world, it is of interest to know how they affect the stability of a flow. Hazel [21] studied the effects of moving the boundaries in from infinity on an inviscid flow with hyperbolic tangent background velocity and density profiles. Two effects were observed. When the total domain height is greater than or equal to ten times the thickness of the shear layer, longer wavelengths are destabilized and are the only wavelengths affected by the presence of the horizontal boundaries. This is reasonable as we only expect waves of lengths comparable to the domain height to be affected. In addition, as the domain height is decreases further, shorter wavelengths become more stable. If the boundaries are moved in closer than some critical distance apart, the latter effect dominates and the flow becomes stable for all wavenumbers and Richardson numbers. Smyth [51] studied the effect of imposing boundaries on the profile studied by Holmboe (figure 1.2). Transition from Kelvin-Helmholtz to Holmboe instabilities occurs at a lower value of J when boundaries are present. When the boundaries are not symmetrically placed, pure Holmboe instabilities are not found. Instead one mode of the Holmboe instability has a larger growth rate. For a viscous flow, Yonemitsu [70] examined how the presence of a lower boundary affects Holmboe instabilities. The wave that protrudes into the upper layer is little affected by the presence of the boundary unless the boundary is very close to the shear layer. The wave that protrudes into the lower layer, however, is significantly altered by the presence of the boundary indicating that perhaps the two modes are independent.

In section 2.2 linear theory was developed under the assumption that perturbations are two-dimensional. At some point in the transition from laminar to turbulent flow the perturbations do indeed become three-dimensional, but one must consider whether they are initially two-dimensional during the time when linear theory is valid. In 1933, Squire [57] showed that, for an unstratified flow, a three-dimensional disturbance at a
given Reynolds number is equivalent to the problem of a two-dimensional disturbance at a lower Reynolds number. Since a flow becomes more stable with decreasing values of the Reynolds number, this implies that all instabilities are initially two-dimensional for a homogeneous flow. Koppel [35] derived a generalized version of Squire’s result, showing that a three-dimensional disturbance at given Prandtl, Reynolds and Richardson numbers is equivalent to a two-dimensional disturbance at the same Prandtl number, a lower Reynolds number and a higher Richardson number. Thus any three-dimensional problem can be transformed into a two-dimensional problem. Since, for Holmboe’s instability, there is a region where the flow is destabilized as the Richardson number increases, it may be possible that the most unstable modes are three-dimensional. This was first suggested by Browand & Wang [4]. Two approaches have been used to examine this possibility. Smyth & Peltier [55] solved the three-dimensional problem and showed that, for a class of dissipative, stratified, parallel shear flows, the perturbations evolve directly into three-dimensional flows without going through the two-dimensional stage. Caulfield [7] calculated maximum growth rates to show that primary three-dimensional instabilities are possible in a generalized version of the inviscid three-layer flow studied by Taylor [62]. In chapter 3 we use the approach used by Caulfield [7] to examine the possibility of the initial perturbations being three-dimensional for the inviscid piecewise linear background flow studied by Lawrence et al. [38].
Figure 2.1: Two-layer flow considered by Helmholtz [22] and Kelvin [25]
Chapter 3

Linear Stability Analysis

3.1 Derivation of Linear Equations

We wish to examine the linear stability of the flow governed by (2.17) and (2.18). As described in Section 2.2, we split the flow field into a parallel mean component and a perturbation field, i.e.,

\[
\psi(x, z, t) = \Psi(z, t) + \psi'(x, z, t),
\]

\[
\rho(x, z, t) = \bar{\rho}(z, t) + \rho'(x, z, t),
\]

where \( \psi' \) and \( \rho' \) are \( O(\epsilon) \ll 1 \). Substituting these expressions into the governing equations (2.17) and (2.18), and collecting terms of equal magnitude, we obtain equations for both the background flow and the perturbations.

\[
\bar{\rho}_t = \frac{1}{\text{RePr}} \bar{\rho}_{zz},
\]

\[
\bar{u}_t = \frac{1}{\text{Re}} \bar{u}_{zz},
\]

and

\[
(\Delta \psi')_t + \bar{u}(\Delta \psi')_x + w' \bar{u}_{zz} = J\rho'_x + \frac{1}{\text{Re}} \Delta \Delta \psi',
\]

\[
\rho'_t + \bar{u}\rho'_x + w'\bar{\rho}_z = \frac{1}{\text{RePr}} \Delta \rho',
\]

where \( u(x, z, t) = \bar{u}(z, t) + u'(x, z, t) \).

Equations (3.3) indicate that the mean velocity and density profiles diffuse over time due to viscosity and the diffusivity of the stratifying agent. For the purpose of solving
the linear problem (3.4), however, we will use the initial mean profiles $U(z) = \bar{u}(z,0)$ and $\rho_s(z) = \bar{\rho}(z,0)$. Hence $\psi'$ and $\rho'$ are the perturbations from the initial mean state.

Using the method of normal modes, we write

$$\psi'(x, z, t) = \Re \left\{ \phi(z) e^{i \alpha (x-ct)} \right\},$$

$$\rho'(x, z, t) = \Re \left\{ \bar{\rho}(z) e^{i \alpha (x-ct)} \right\}.$$ (3.5)

Substituting these into the linear equations (3.4) we obtain the eigen-problem:

$$(U - c)(\phi_{zz} - \alpha^2 \phi) - U_{zz} \phi = J \rho - \frac{i}{\alpha \Re} (\phi_{zzzz} - 2\alpha^2 \phi_{zz} + \alpha^4 \phi),$$

$$(U - c)\bar{\rho} - \rho_{zz} \phi = -\frac{i}{\alpha \Re \Pr} (\bar{\rho}_{zz} - \alpha^2 \bar{\rho}).$$ (3.7)

Notice that the above set of equations reduce to the Orr-Sommerfeld equation when no stratification is present. Furthermore, the Taylor-Goldstein equation is obtained if there is no diffusion. Similar equations have been derived by Koppel [35], and Baldwin & Roberts [1].

### 3.2 Piecewise Linear Profiles

The starting point of our investigation is to examine the stability of an inviscid flow with two layers of constant density and a layer of constant horizontal shear (figure 3.1). This background flow has the advantage of admitting an exact dispersion relation to the normal mode analysis. Also, other than the density interface displacement $d/h$, the only additional parameter is the bulk Richardson number $J$. Finally, results of inviscid theory with piecewise linear profiles often compare favourably with experimental results. For the reasons given above, an inviscid flow with piecewise linear background profiles is an obvious first approximation to our problem.
Chapter 3. Linear Stability Analysis

We define the background velocity, \((\bar{u}, \bar{w}) = (U, 0)\), and density profiles as

\[
U = \begin{cases} 
\frac{U_1}{h} z + \frac{U_1 - U_2}{h} & h/2 < z < H/2, \\
\frac{U_1 - U_2}{h} z + \frac{U_1 + U_2}{2} & -h/2 < z < h/2, \\
\frac{U_2}{h} & -H/2 < z < -h/2,
\end{cases}
\]

\[\rho_a = \begin{cases} 
\rho_1 & z > -d, \\
\rho_2 & z < -d,
\end{cases}
\]

where \(h\) is the shear layer thickness, \(H\) is the total depth of the flow, and \(d\) is the displacement of the density interface from the center of the shear layer (figure 3.1). If we nondimensionalize using \(L = h/2\), \(\delta U = (U_1 - U_2)/2\), \(\bar{U} = (U_1 + U_2)/2\), \(\delta \rho = (\rho_2 - \rho_1)/2\), and \(\rho_0 = (\rho_2 + \rho_1)/2\) then the nondimensional background profiles are

\[
U = \begin{cases} 
1 & 1 < z < H/2, \\
z & -1 < z < 1, \\
-1 & -H/2 < z < -1,
\end{cases}
\]

\[\rho_a = \begin{cases} 
-1 & z > -\varepsilon, \\
1 & z < -\varepsilon,
\end{cases}
\]

where \(H\) is now the nondimensional depth of the flow, and \(\varepsilon = 2d/h\). Since the flow is inviscid, (3.3) indicates that there is no diffusion of the mean flow. Furthermore, (3.7) reduces to the nondimensional Taylor-Goldstein equation:

\[
(U - c)(\phi_{zz} - \alpha^2 \phi) - U_{zz} \phi + 2J \delta(z + \varepsilon) \frac{\phi}{U - c} = 0,
\]

where \(\delta(z)\) is the delta function.

Assuming the rigid wall boundary condition \(\phi = 0\) at \(z = \pm H/2\) and using the techniques outlined in Drazin & Reid [12] (see appendix A for details of the calculations),
we obtain the following dispersion relation.

\[ c^4 + a_1 c^3 + a_2 c^2 + a_3 c + a_4 = 0, \]

where \( a_i = a_i(\varepsilon, \alpha, J, H) \) and are given in appendix A. As \( H \to \infty \) we retrieve the results of Lawrence et al. [38]. Also, this is simply a special case of the dispersion relation derived by Smyth [51] for piecewise linear profiles with non-symmetric horizontal boundaries.

### 3.2.1 Results: Piecewise Linear Profiles

#### Symmetric Case (\( \varepsilon = 0 \))

For \( \varepsilon = 0 \) and \( H = \infty \), we reproduce the results of Holmboe [23] (figure 3.2). When \( J > 0.07 \), Holmboe instabilities are the only instabilities that occur. In the unstable region, there are two modes of equal growth rates and equal but opposite phase speeds (figures 3.3 and 3.4). The growth rate of the Holmboe instability increases with increasing \( J \) until it reaches a maximum value (figure 3.3) after which it decreases as \( J \) increases. When \( J < 0.07 \), the flow is unstable to both Kelvin-Helmholtz and Holmboe instabilities. An enlargement of the stability diagram in this region is shown in figure 3.5. When \( \varepsilon = 0 \), (3.11) reduces to

\[ c^4 + a_2 c^2 + a_4 = 0 \]

(3.12)

Solutions to (3.12) are oscillatory, i.e., \( c_r \neq 0 \), when \( a_2^2 - 4a_4 < 0 \). In this case there are four solutions, two stable and two unstable, of the form \( c = \pm c_r \pm ic_i \). The superposition of the two unstable modes is Holmboe’s instability. When \( a_2^2 - 4a_4 > 0 \) and \( a_2 > 0 \), there are two stationary (\( c_r = 0 \)) unstable solutions to (3.12) with different growth rates. These are Kelvin-Helmholtz instabilities. The contour \( a_2^2 - 4a_4 = 0 \) gives the stability boundary for the Holmboe instabilities.

Results in figures 3.2 and 3.3 show the growth rates for the dominant Kelvin-Helmholtz
instability only. It is clear from figure 3.5 that the Kelvin-Helmholtz instability dominates for $J$ close to 0 whereas the Holmboe instability dominates when $J$ is near 0.07. It is not obvious at which value of the bulk Richardson number, $J_T$, the flow changes from having a Kelvin-Helmholtz instability as the most unstable mode to a Holmboe instability. To determine the value of $J_T$, the maximum growth rate for a fixed value of $J$ must be found for both Kelvin-Helmholtz and Holmboe regimes. In appendix B the $\alpha$–$J$ curves along which the maximum growth rates occur are calculated analytically for each region. These results are used to determine the maximum growth rates in each regime (figure 3.6). It is evident that transition from Kelvin-Helmholtz to Holmboe instabilities occurs at $J_T \approx 0.046$.

By varying the value of $H$, we examine the effect of horizontal boundaries on the stability of the flow (figure 3.2). Decreasing $H$ reduces the growth rate of the most amplified mode (figure 3.3). Moving the horizontal boundaries closer together, however, also causes the flow to become unstable at small values of $\alpha$. This mechanism was initially observed by Hazel [21]. The presence of horizontal boundaries does not significantly affect the stability of the flow, however, until $H \approx 10$. Since Kelvin-Helmholtz instabilities occur at smaller wave numbers than Holmboe instabilities, they are stabilized more rapidly as $H$ decreases. This results in the transition from Kelvin-Helmholtz to Holmboe instabilities being shifted to a smaller value of $J$ as the distance between boundaries decreases. When $H \leq 4$, the flow is unstable to Holmboe instabilities only. As the boundaries continue to move together, the flow becomes stable at small $J$ (see figure 3.2, $H = 3$). These results are consistent with those of Smyth [51].

**Non-Symmetric Case ($\varepsilon \neq 0$)**

When $\varepsilon > 0$ (figures 3.3, and 3.2), the flow has no pure Kelvin-Helmholtz or Holmboe instabilities. When $H = \infty$, there are two unstable modes, one moving to the right and
one moving to the left. As \( \varepsilon \) increases, the growth rate of the right mode increases while its phase speed decreases. The left mode exhibits the opposite behaviour. Lawrence et al. [38] showed that, for fixed \( \alpha \) and \( J \), \( c^{(r)}_r + c^{(l)}_l = -\varepsilon \), where \((r)\) and \((l)\) correspond to right and left moving waves, respectively. For a fixed value of \( J \), the range of \( \alpha \) for which the flow is unstable decreases for the right moving wave and increases for the left moving wave as \( \varepsilon \) increases. On the stability diagram, the limb corresponding to the right modes tilts towards the \( J \)-axis and thus the most unstable right mode for a fixed value of \( J \) occurs at smaller values of \( \alpha \) with increasing \( \varepsilon \). The limb corresponding to the left mode tilts towards the \( \alpha \)-axis and its most unstable mode occurs at larger \( \alpha \) with increasing \( \varepsilon \) (figure 3.2). This is the behaviour described by Lawrence et al. [38].

We now examine the effect of horizontal boundaries (i.e., a finite value of \( H \)) on the stability of non-symmetric flows. To date, this effect has not yet been studied in any detail (Smyth [51] studied the case corresponding to \( \varepsilon = 0.05 \) and \( \alpha = 0.2 \) with \( H = \infty \) and \( H = 8 \)). As with the \( \varepsilon = 0 \) case, the presence of horizontal boundaries causes the flow to become unstable at small wave numbers. The boundaries have little effect on the fastest growing mode until \( H \) drops below 10, at which point growth rates of the fastest growing right and left modes decrease with decreasing \( H \). Since right moving instabilities occur at smaller wave numbers than left moving instabilities, they are stabilized more quickly by the presence of boundaries. The result is that growth rates of right moving waves decrease more rapidly than those of left moving waves. This effect becomes even more pronounced as \( \varepsilon \) increases and causes the left mode to dominate when \( H \leq 5 \) (figure 3.3).

### 3.2.2 Three-Dimensional Perturbations: Piecewise Linear Profiles

Until now, we have assumed that the perturbations are two-dimensional. Here, we discuss the relationship between two-dimensional and three-dimensional linear instabilities.
The most general form of a three-dimensional disturbance \((\mathbf{u}'', \rho'', p'')\), where \(\mathbf{u}'' = (u'', v'', w'')\), is that of a traveling wave whose amplitude varies with \(z\) and which propagates at an angle \(\vartheta\) with respect to the \(x\)-axis (see White [68]), i.e.,

\[
(\mathbf{u}'', \rho'', p'') = \Re \left\{ (\tilde{u}(z), \tilde{\rho}(z), \tilde{p}(z)) e^{i \alpha \left( x \cos \vartheta + y \sin \vartheta - ct \right)} \right\}
\]  

Starting with a background flow of the form \(\mathbf{u} = (U(z), 0, 0)\) with density distribution \(\rho_a(z)\) in hydrostatic equilibrium with the pressure \(p_a(z)\), we superimpose perturbations of the form (3.13). The equations of motions for an inviscid flow with the Boussinesq approximation under the linear assumption reduce to

\[
\left( U - \frac{c}{\cos \vartheta} \right) \left( \ddot{w} - \alpha^2 \ddot{w} \right) - U_{zz} \ddot{w} - \frac{J \rho_a \dot{w}}{\cos^2 \vartheta \left( U - \frac{c}{\cos \vartheta} \right)} = 0,
\]  

which, for the background profile described by (3.9), becomes

\[
\left( U - \frac{c}{\cos \vartheta} \right) \left( \ddot{w} - \alpha^2 \ddot{w} \right) - U_{zz} \ddot{w} + \frac{2J \delta(z + \varepsilon) \ddot{w}}{\cos^2 \vartheta \left( U - \frac{c}{\cos \vartheta} \right)} = 0.
\]  

The above is simply the Taylor-Goldstein equation for two-dimensional perturbations, (3.10), with bulk Richardson number \(J/\cos^2 \vartheta\) and complex phase speed \(c/\cos \vartheta\). Finding the growth rate for the three-dimensional problem is equivalent to finding the growth rate of a related two-dimensional problem and multiplying it by \(\cos \vartheta\), i.e.,

\[
\alpha c_i(U, \vartheta, \alpha, J) = \cos \vartheta \cdot \alpha c_i(U, 0, \alpha, J/\cos^2 \vartheta).
\]  

This is simply a special case of Koppel’s [35] result for the more general case of a stratified shear flow with viscosity and dissipation. Koppel’s result is given in a form similar to the above by Smyth & Peltier [55].

The assumption that perturbations are initially two-dimensional is valid providing the fastest growing two-dimensional mode is more unstable than the fastest growing
three-dimensional mode, \( \text{i.e.} \),

\[
\cos \vartheta \cdot \alpha^* c_i(U, 0, \alpha^*, J / \cos^2 \vartheta) \leq \alpha^* c_i(U, 0, \alpha^*, J)
\]  

(3.17)

where \( \alpha^* \) is the wave number at which the maximum growth rate occurs for a given value of the bulk Richardson number. For \( J > 0 \), this is equivalent to having

\[
\frac{\cos \vartheta}{\sqrt{J}} \cdot \alpha^* c_i(U, 0, \alpha^*, J / \cos^2 \vartheta) \leq \frac{1}{\sqrt{J}} \cdot \alpha^* c_i(U, 0, \alpha^*, J).
\]

(3.18)

In order to validate the two-dimensional assumption, we must show that \( \alpha^* c_i / \sqrt{J} \) decreases with increasing \( J \) along the curve of maximum growth rate for a given \( J \). In appendix B we derive analytic expressions for \( \alpha^* - J \) curves for the case originally studied by Holmboe (\( H = \infty \) and \( \varepsilon = 0 \)). These are used to find the maximum growth rates in each region (figure 3.6). In the Kelvin-Helmholtz regime, the maximum growth rate decreases with increasing \( J \) and thus (3.18) is automatically satisfied. In the Holmboe regime, however, there is a region where the maximum growth rate increases for increasing \( J \). Thus it may be possible that (3.18) is violated in this region. Figure 3.6 shows, however, that \( \alpha^* c_i / \sqrt{J} \) does indeed decrease with increasing \( J \). This implies that the initial instabilities are two-dimensional.

For the case studied by Lawrence et al. [38] (\( H = \infty \), \( \varepsilon > 0 \)), we were not able to find exact expressions for the \( \alpha^* - J \) curves along which the growth rate is maximized. We can, nevertheless, use the results used to produce the stability diagrams (figure 3.2) to determine these curves for right and left moving modes. These are shown in figure 3.7 as are the curves \( \alpha^* c_i / \sqrt{J} \) for \( \varepsilon = 0, 0.25, \) and \( 0.5 \). For larger values of \( \varepsilon \), the maximum growth rate of the right mode decreases with increasing \( J \) and thus (3.18) is automatically satisfied. For smaller values of \( \varepsilon \), however, and for the left mode, there are regions where the maximum growth rate increases with \( J \). Nevertheless, \( \alpha^* c_i / \sqrt{J} \) decreases with increasing \( J \) and thus we expect the initial instabilities to be two-dimensional.
Although we have shown that, for the inviscid case, the two-dimensional assumption is valid, these results should be used with caution. Smyth & Peltier [55] have shown that it is possible for the most unstable initial instabilities to be three-dimensional when viscosity and thermal dissipation are included. We will discuss this possibility further in section 3.3.2.

3.3 Smooth Profiles

Although results for the piecewise linear profile case often agree with experimental results, two parameters are neglected: the Reynolds number and the Prandtl number. Furthermore, initially smooth background profiles are required in the nonlinear numerical simulations used in chapter 4. We now consider background flows with smooth profiles and include the effects of viscosity and diffusion. The initial mean velocity, \((\bar{u}, \bar{w}) = (U, 0)\), and density profiles are depicted in figure 3.8 and are

\[
\begin{align*}
U(z) &= \frac{U_1 - U_2}{2} \tanh \frac{2z}{h} + \frac{U_1 + U_2}{2}, \\
\rho_0(z) &= -\frac{\rho_2 - \rho_1}{2} \tanh \frac{2}{\eta} (z + d) + \frac{\rho_1 + \rho_2}{2},
\end{align*}
\]

(3.19)

where \(h\) is the shear layer thickness and \(\eta\) is the density interface thickness. Wang [66] and Yonemitsu [70] have found that the hyperbolic tangent function is a good approximation to the mean velocity fields measured in their laboratory experiments of salt stratified flows with thin density interfaces. Also, for large values of \(R = h/\eta\), the above density profile is a good representation of thin density interfaces.

If we nondimensionalize (3.19) using \(L = h/2, \delta U = (U_1 - U_2)/2, \bar{U} = (U_1 + U_2)/2,\)
\[ \delta \rho = (\rho_2 - \rho_1)/2, \text{ and } \rho_0 = (\rho_2 + \rho_1)/2, \text{ as described in section 2.1.3, then the nondimensional background profiles are} \]

\[
\begin{align*}
U(z) &= \tanh z, \\
\rho_a(z) &= -\tanh R(z + \varepsilon),
\end{align*}
\]

where, as before, \( \varepsilon \) is the nondimensional distance of the density interface displacement.

With these background profiles, (3.7) become

\[
\begin{align*}
(U - c)(\phi_{zz} - \alpha^2 \phi) - U_{zz} \phi &= \frac{\Im}{\alpha \text{Re}}(\phi_{zzzz} - 2\alpha^2 \phi_{zz} + \alpha^4 \phi), \\
(U - c)\tilde{\rho} - \rho_{zz} \tilde{\phi} &= \frac{-i}{\alpha \text{RePr}}(\tilde{\rho}_{zz} - \alpha^2 \tilde{\rho}),
\end{align*}
\]

where, as defined for the piecewise linear case, \( J = 2(\rho_2 - \rho_1)gh/(\rho_2 + \rho_1)(U_1 - U_2)^2 \) is the bulk Richardson number. We solve (3.21) numerically using the method described in the next section.

### 3.3.1 Numerical Method

In this section we describe the method used to solve (3.21) for a viscous flow with diffusion and initial background profiles given by (3.20). For the inviscid problem with piecewise linear profiles, it was possible to have an unbounded domain (i.e., \(-\infty < z < \infty\)). Since we solve the present problem numerically, we must restrict ourselves to a finite domain. The boundary conditions used are \( \phi = \phi_{zz} = \tilde{\rho} = 0 \) at \( z = \pm H/2 \), where \( H \) is the total height of the domain (figure 3.8). These are equivalent to setting \( u_z' = w' = \rho' = 0 \) at \( z = \pm H/2 \).

Following the method of Klaassen & Peltier [30] we use a truncated Fourier sine series to approximate \( \phi \) and \( \tilde{\rho} \) and seek solutions of the form

\[
\begin{align*}
\phi(z) &= \sum_{n=1}^{N} \phi_n \sin \frac{n\pi(z + H/2)}{H}, \\
\tilde{\rho}(z) &= \sum_{n=1}^{N} \rho_n \sin \frac{n\pi(z + H/2)}{H}.
\end{align*}
\]
Substituting \((3.22)\) into \((3.21)\) we obtain the following.

\[
\begin{align*}
\sum_{n=1}^{N} \left\{ -(U - c)D_n \phi_n S_n - U_{zz} \phi_n S_n - J \rho_n S_n \right\} &= -i \frac{1}{\alpha \text{Re}} \sum_{n=1}^{N} D_n^2 \phi_n S_n, \\
\sum_{n=1}^{N} \left\{ (U - c) \rho_n S_n - \rho_{zz} \phi_n S_n \right\} &= \frac{i}{\alpha \text{Pr} \text{Re}} \sum_{n=1}^{N} D_n \rho_n S_n,
\end{align*}
\]

(3.23)

where \(D_n = (n \pi / H)^2 + \alpha^2\) and \(S_n = \sin \frac{n \pi (z + H/2)}{H}\). In order to solve for the coefficients \(\phi_n\) and \(\rho_n\) we use Galerkin's method. We multiply the above equations by \(\sin m \pi (z + H/2) / H\) for \(m = 1 \ldots N\) and integrate from \(z = -H/2\) to \(z = H/2\). We obtain \(2N\) equations that can be written as

\[
\begin{align*}
\sum_{n=1}^{N} \frac{D_n t_1(m, n) + t_2(m, n)}{D_m} \phi_n + \frac{i}{\alpha \text{Re}} D_m \phi_m + \frac{J}{D_m} \rho_m &= c \phi_m, \\
\sum_{n=1}^{N} \{-t_3(m, n) \phi_n + t_1(m, n) \rho_n\} + \frac{i}{\alpha \text{Pr} \text{Re}} D_m \rho_m &= c \rho_m
\end{align*}
\]

(3.24)

for \(m = 1, \ldots, N\), and where

\[
\begin{align*}
t_1(m, n) &= \frac{2}{H} \int_{-H/2}^{H/2} U \sin \frac{m \pi (z + H/2)}{H} \sin \frac{n \pi (z + H/2)}{H} \, dz, \\
t_2(m, n) &= \frac{2}{H} \int_{-H/2}^{H/2} U_{zz} \sin \frac{m \pi (z + H/2)}{H} \sin \frac{n \pi (z + H/2)}{H} \, dz, \\
t_3(m, n) &= \frac{2}{H} \int_{-H/2}^{H/2} \rho_{zz} \sin \frac{m \pi (z + H/2)}{H} \sin \frac{n \pi (z + H/2)}{H} \, dz,
\end{align*}
\]

(3.25, 3.26, 3.27)

We have reduced the system of ordinary differential equations \((3.21)\) to a simple matrix eigen-problem \((3.24)\) which is of the form

\[
AV = cV;
\]

(3.28)

where \(A\) is a \(2N \times 2N\) matrix defined by

\[
\begin{align*}
A_{n,m} &= \frac{1}{D_m} (D_n t_1(m, n) + t_2(m, n)) + \frac{i D_m}{\alpha \text{Re}} \delta_{nm}, \\
A_{n,m+N} &= \frac{J}{D_m} \delta_{nm}, \\
A_{n+N,m} &= -t_3(m, n), \\
A_{n+N,m+N} &= t_1(m, n) + \frac{i D_m}{\alpha \text{Pr} \text{Re}} \delta_{nm},
\end{align*}
\]

(3.29, 3.30, 3.31, 3.32)
\( \delta_{mn} \) is the Kronecker delta function, and

\[
V = \begin{pmatrix}
\hat{\phi}_1 \\
\vdots \\
\hat{\phi}_N \\
\rho_1 \\
\vdots \\
\rho_N
\end{pmatrix}.
\] (3.33)

It should be noted that although the above method finds \( 2N \) values of the complex phase speed \( c \) and the corresponding eigenvalues, it is only the two most unstable modes that are of interest. These correspond to the values of \( c \) with the two largest values of \( \alpha c_i \).

It was determined that \( N = 100 \) is usually sufficiently large to resolve eigenvalues and eigenvectors and is the value used, unless otherwise stated.

### 3.3.2 Results: Smooth Profiles

We are interested in how displacing the density interface affects the stability of the flow. This corresponds to varying the parameter \( \varepsilon \). Lawrence et al. [38] have studied this effect for the inviscid case originally examined by Holmboe [23] where the only additional parameter is the bulk Richardson number \( J \). Their results have been reproduced in section 3.2.1 where the effects of boundaries were also studied. For smooth profiles we have the following additional parameters: the ratio \( R \) between the shear layer thickness and the density interface thickness, the Reynolds number \( \text{Re} \), and the Prandtl number \( \text{Pr} \).

Yonemitsu [70] studied the viscous case without diffusion, but only looked for the most unstable mode (as opposed to two unstable modes, one moving to the right and another moving to the left, which corresponds with the results of Lawrence et al. [38]).
Smyth et al. [53] have shown that an initial mean flow with arbitrary initial values of $R$ and $Pr$ will eventually reach a state where $R = \sqrt{Pr}$. For this reason, we will set $Pr = R^2$, linking the effects of $Pr$ and $R$ on the stability of the flow. Below, we discuss the effect of specific parameters on the linear stability of the flow.

**Varying $R$**

The most significant difference between the piecewise linear profiles (3.9) examined in section 3.2 and the smooth profiles (3.20) examined here is that we now allow the density interface to have a finite thickness, which we change by adjusting the value of the parameter $R$. Since the expected value of $R$ can vary greatly depending on the stratifying agent and the temperature of the fluid (table 3.1), it is of interest to see how this parameter affects the stability of the flow. In our study of the effect of $R$ on the stability of the flow with varying $\varepsilon$, we use the values $R = 3, 4, 6, \text{ and } 8$ (with $Pr = 9, 16, 36, \text{ and } 64$, respectively). Although salt stratified flows have much larger values of the Prandtl number than examined here, we are unable to numerically resolve such a flow as there is insufficient diffusion to compute with a feasible value of $N$ (see section 3.3.1). Nevertheless, with the range of Prandtl numbers chosen, we are able to observe trends in how the stability of the flow changes from thermally stratified to salt stratified flows.

Results of Hazel [21] and those of section 3.2.1 indicate that, when $H = 10$, horizontal boundaries do not significantly affect the stability of an inviscid flow. A similar result for the viscous case is discussed below. Based on the above, we set $H = 10$ to insure that the horizontal boundaries do not unduly influence the stability of the flow. We also use $Re=300$ which Smyth [51] found to be sufficiently large to give good agreement with his inviscid solutions, facilitating comparison with results of section 3.2.1.

For $\varepsilon = 0$ we have computed the maximum growth rate in the Holmboe regime for the above values of $R$ (figure 3.9). As $R$ decreases, the maximum growth rate decreases.
Chapter 3. Linear Stability Analysis

<table>
<thead>
<tr>
<th>Temperature</th>
<th>$\rho$ (kg/m$^3$)</th>
<th>$\mu$ (kg/m·s)</th>
<th>$\kappa$ (m$^2$/s)</th>
<th>Pr</th>
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<td>°C</td>
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Table 3.1: Values of Prandtl number for 1) diffusion of heat in pure water, 2) diffusion of NaCl (0.01 molar) in water, and 3) diffusion of salts in sea water. In all cases $\rho$ was calculated using the international equation of state of sea water (see appendix 3 of Pond & Pickard [17]) with salinities of 0, 0.5844, and 35 pss, respectively. $\mu$ was determined from table 16 in Sverdrup et al. [60] using salinities of 0, 0.5844, and 35 ppt, respectively. Values for the coefficient of thermal diffusivity ($\kappa$ for case 1) are from Fischer et al. [15]. For cases 2 and 3, $\kappa$ corresponds to the diffusivity of NaCl in water. Values were taken from CRC Handbook of Chemistry and Physics [67] and are for molarity of 0.01 and 0.6 for cases 2 and 3, respectively. Where necessary, linear interpolation was used to determine coefficients that are not listed in referenced tables.
It is well known that Holmboe instabilities do not exist when $R = 1$ and so there is some critical value of $R$ below which Holmboe instabilities do not occur. Using the method of least squares, we fit a curve of the form $(\alpha \epsilon)_{\text{max}} = a/(b - R) + c$, where $a$, $b$, and $c$ are constants, to the data in figure 3.9. This gives us a critical value of $R = 2.39$. This agrees with the results of Smyth [51] who found a critical value of $R \approx 2.4$ by computing the stability diagrams for several values of $R$.

We start by examining the results for $\epsilon = 0$ (figure 3.10). We observe the same behaviour found by Smyth [51] for the inviscid case with smooth profiles. Increasing $R$ increases the region of instability for both right and left travelling waves: at a given value of $J$, the range of $\alpha$ for which the flow is unstable increases as $R$ increases; at a given value of $\alpha$, the range of $J$ for which the flow is unstable also increases with increasing $R$. In addition, the growth rates of the fastest growing Holmboe instabilities (figure 3.11) increase with $R$. Growth rates of Holmboe instabilities when $R$ is finite are smaller than the piecewise linear case which corresponds to $R \to \infty$. When $J$ is small, stratification is weak and we expect the flow to behave like the homogeneous case for which the value of $R$ has no effect. Consequently the stability of the flow for small $J$ does not change significantly as $R$ varies. For larger values of $J$, the wave number at which the most unstable mode occurs increases with $R$. Also, transition from Kelvin-Helmholtz instabilities to Holmboe instabilities occurs at smaller values of $J$ as $R$ increases.

When $\epsilon > 0$, the region of instability for both left and right modes increase as $R$ increases (figures 3.10). Also, the growth rate of the fastest growing mode increases with increasing $R$ (figure 3.11). These results are the same as the $\epsilon = 0$ case. The wave number at which the most unstable mode occurs, however, changes with both $\epsilon$ and $R$. If we fix $\epsilon$, then the wave number at which the fastest growing mode occurs increases with increasing $R$; conversely, if we fix $R$ and vary $\epsilon$ the behaviour depends on the value of $R$, as described below.
For a piecewise linear inviscid flow, Lawrence et al. [38] found that as $\varepsilon$ increases, the most unstable right travelling wave occurs at a smaller wave number, whereas the most unstable left travelling wave occurs at a larger wave number. On the stability diagrams, this corresponds to the tilting of the limb corresponding to the right moving wave (the right limb) towards the $J$-axis, and the limb corresponding to the left moving wave (the left limb) towards the $\alpha$-axis (figure 3.2). This behaviour remains unchanged for large finite values of $R$ (figure 3.10, $R = 6$ and 8). As $R$ decreases below some critical value of approximately 4, however, this behaviour starts to change (figure 3.10, $R = 4$). For $R = 4$, there is very little tilting of the right limb and the wave number at which the most unstable right moving wave occurs remains almost constant as $\varepsilon$ increases. When $R = 3$, however, the wave number of the most unstable right mode increases with increasing $\varepsilon$ (figure 3.10, $R = 3$). This change in behaviour with changing density interface thickness was documented by Caulfield [7] for three layer inviscid flows. It is seen that for all values of $R$, the left limb tilts towards the $\alpha$-axis. Also, the left limb lifts off the $\alpha$-axis as $\varepsilon$ increases (figure 3.10). This effect becomes more noticeable as $R$ decreases. This did not occur in the findings of Lawrence et al. [38] since their case corresponds to $R \to \infty$. The wave number of the most unstable left mode increases as $\varepsilon$ increases, except when $R = 3$ and $0 < \varepsilon < 0.25$. In this case, $\alpha^*$ decreases as $\varepsilon$ increases. This is a result of the large separation of the left mode from the $\alpha$-axis. The separation of the left limb from the $\alpha$-axis affects the range of $J$ for which both left and right moving unstable modes exist. As $\varepsilon$ decreases, the range of $J$ for which both modes exist decreases.

Varying $H$

Above, it was assumed that $H = 10$ is sufficiently large for the boundaries to have little affect on the stability of the flow. In section 3.2.1 we found this to be true for an inviscid flow with piecewise linear profiles. We now examine the validity of this assumption for
smooth profiles. Having studied the effect of $R$ in the previous section, we now set $R = 3$ since it is sufficiently large to allow Holmboe instabilities, yet small enough to converge without requiring an excessive number of Fourier modes. Also $\Pr = R^2 = 9$ is a physically realistic value of the Prandtl number, corresponding approximately to the diffusion of heat in water at 10°C (see table 3.1). As above we set $Re = 300$.

When $\varepsilon = 0$ (figure 3.12), the effect of $H$ on the stability of the flow differs markedly from the piecewise linear case. For $H \geq 10$ the boundaries do not significantly affect the stability of the flow except at small values of both $\alpha$ and $J$. This is not surprising in light of the results from the piecewise linear case. The fastest growing mode is not affected by the presence of horizontal boundaries until $H < 10$, at which point its growth rate decreases as $H$ decreases (figure 3.13). An interesting difference between the effect of $H$ on the piecewise linear profiles and smooth profiles is how the transition between Kelvin-Helmholtz instabilities and Holmboe instabilities changes as $H$ varies. For the piecewise linear profiles, it was observed that as $H$ decreases the transition occurs at smaller values of $J$ until only Holmboe modes are present at $H = 4$. This was explained by the Kelvin-Helmholtz instabilities being stabilized first as they occur at smaller wave numbers than the Holmboe instabilities. In contrast to the piecewise linear case where Holmboe instabilities exist for all $J > 0$, for smooth profiles there is a range of $J$ between 0 and some critical value $J_c$ (which depends on the value of $R$) for which Holmboe instabilities do not exist. In addition, for smooth profiles, instabilities near the transition between Kelvin-Helmholtz and Holmboe instabilities occur at smaller wave numbers and have the smallest growth rates. These are the first modes to be stabilized by the boundaries. This manifests itself by the separation of the two regions of instability with a stable region in between. As $H$ decreases, the region separating the two modes increases while the growth rates of the two modes decrease.

When $\varepsilon = 0.25$ with $H = 20$ (figure 3.13), the growth rate of fastest growing left mode
is much smaller than that of the fastest growing right mode. As $H$ decreases, the growth rates of the two modes decrease. Overall, the growth rate of the right mode decreases more rapidly than that of the left mode as $H$ decreases. Nevertheless, the right mode always remains the faster growing of the two modes. This is in contrast to the piecewise linear case where for small $H$ the left mode dominates. For the piecewise linear case with $\varepsilon = 0.25$ and large $H$ (figure 3.3), the difference between growth rates of left and right modes is not great, and thus the rapid drop in growth rate of the right mode as $H$ decreases allows it to fall below the growth rate of the left mode. This is not possible for the case examined here since the initial difference between the growth rates of right and left modes is so large.

**Varying Re**

In this section we examine the effect of varying the Reynolds number, $Re$, on the linear stability of a mean flow described by (3.20). We study the effect of $Re$ by computing the stability diagram for $Re = 300$ and $100$ with $R = 3$. These are shown in figure 3.14.

As $Re$ decreases, viscosity becomes more important and we expect an overall stabilization of the modes with decreasing $Re$. Since viscosity has the greatest effect on small wave lengths, it is primarily the instabilities at large wave numbers which are stabilized as $Re$ decreases. This has the effect of decreasing the wave number at which the most unstable mode occurs. When $\varepsilon = 0$, growth rates of the most unstable modes decrease with decreasing $Re$ and are affected primarily in the transition between Kelvin-Helmholtz and Holmboe instabilities. When $\varepsilon > 0$, the fastest growing right mode decreases as $Re$ decreases. There is a slight increase in the growth rate of the most amplified left mode as $Re$ decreases. Although this is surprising, the difference in growth rates is not significant. It is difficult to draw any conclusions about the left mode without further investigation.
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Varying $\varepsilon$: summary

We now discuss the overall effect of varying $\varepsilon$. It was seen that precise behaviour of the flow stability with increasing $\varepsilon$ depends on the thickness of the density interface (which is specified by the parameter $R$). Nevertheless, there are trends that were observed in all the cases examined. As with the piecewise linear case, there are two types of instabilities when $\varepsilon = 0$. For small $J$, the flow is unstable to Kelvin-Helmholtz instabilities, whereas for larger $J$ it is unstable to Holmboe instabilities. When $\varepsilon > 0$, there are no pure Kelvin-Helmholtz or Holmboe instabilities. The Kelvin-Helmholtz instability is replaced by an instability with positive phase speed. The growth rate of this instability increases with $\varepsilon$. For larger $J$, there are still two unstable modes, one moving to the right and another to the left. As $\varepsilon$ increases, however, the growth rate of the right moving wave increases whereas that of the left moving wave decreases. This is consistent with the results of Lawrence et al. [38]. On the stability diagram this corresponds to a splitting into two limbs. The limb corresponding to right travelling waves gets larger. Thus as $\varepsilon$ increases, the maximum value of $J$ for which the flow is unstable increases. The limb corresponding to left travelling waves shrinks. Hence, the interval of $J$ over which there are two unstable modes with opposite phase speeds decreases. It was also seen that as $\varepsilon$ increases, the phase speed of the right mode decreases whereas that of the left mode increases. The most significant difference between the smooth profile case and the piecewise linear case examined in section 3.2 is that when $\varepsilon > 0$, the difference in growth rates between right and left modes is much larger for the smooth profile case studied here. We expect this difference to be largely due to the different background profiles used in the two studies. This is supported by the similarity between our results for a smooth background flow with the effects of viscosity and diffusion included and with $\varepsilon = 0$ and those of Smyth [51] who studied the stability of an inviscid flow with the same smooth background profiles.
3.3.3 Three-Dimensional Instabilities: Smooth Profiles

We wish to examine the possibility of the instabilities initially being three-dimensional when viscosity and diffusion are included. In this case, the relationship between two-dimensional and three-dimensional instabilities for an inviscid flow (3.16) becomes

$$\alpha c_i(U, \vartheta, \alpha, \text{Re}, \text{J}, \text{Pr}) = \cos \vartheta \cdot \alpha c_i(U, 0, \alpha, \text{Re} \cos \vartheta, \text{J} / \cos^2 \vartheta, \text{Pr}).$$  (3.34)

This is Koppel’s result [35] which reduces to Squire’s theorem [57] when J = 0. In order for the initial instabilities to be two-dimensional, we require

$$\cos \vartheta \cdot \alpha^* c_i(U, 0, \alpha^*, \text{Re} \cos \vartheta, \text{J} / \cos^2 \vartheta, \text{Pr}) \leq \alpha^* c_i(U, 0, \alpha^*, \text{Re}, \text{J}, \text{Pr}),$$  (3.35)

where $\alpha^*$ is the wave number at which the growth rate is maximized for given values of J, Re, and Pr. We can conclude from the above that the initial instabilities may be three-dimensional if the growth rate of the fastest growing mode increases with increasing J.

Above, we examined the effect on the stability of the flow of decreasing the Reynolds number. It was observed that this tends to decrease the growth rates of the right mode when $\varepsilon > 0$, and both modes when $\varepsilon = 0$. Therefore the only way that the three-dimensional instabilities can dominate is if the growth rate of the fastest growing mode increases with increasing J. When $\varepsilon = 0$, there is a region where the most amplified growth rate increases with increasing J. This indicates that three-dimensional instabilities may grow faster than two-dimensional instabilities. In order to verify this, one would have to solve the three-dimensional linear stability problem to see if the initial perturbations are indeed three-dimensional. Smyth & Peltier [55] have done this for a range of Re with $R = 3$ and $\varepsilon = 0$. They were able to show that, for smaller values of Re, the flow is initially unstable to three-dimensional perturbations. This includes the value of Re = 300 examined here. When $\varepsilon > 0$, the growth rate of the fastest growing right mode
decreases with increasing $J$ (figure 3.11) implying that the most unstable instabilities are
two-dimensional.

The behaviour of the left mode when $\varepsilon > 0$ is different than that of the right mode.
First, we observed a slight increase in the growth rate of the most amplified left mode
when the Reynolds number decreased from 300 to 100. Also, there is a region where the
growth rate of the left mode increases with increasing $J$ (figure 3.11). Thus it may be
possible that the left mode is most unstable to three-dimensional perturbations. Further
investigation is required to determine whether or not this is true. Although the left
mode may be unstable to three-dimensional instabilities, its small growth rate compared
to that of the right mode when $\varepsilon > 0$ make it unlikely that we will initially see three-
dimensional instabilities. Therefore, for the remainder of this study, we will concentrate
on the development of two-dimensional instabilities. It may be possible, however, that
the eventual growth of the left mode is one mechanism through which the perturbations
become three-dimensional at later times.

3.3.4 Eigenfunctions

So far, we have only examined the linear growth rates to determine the overall stability
of the flow. In order to better understand the mechanism of the instability in the linear
regime, we now consider the eigenfunctions found from linear stability analysis. Since,
for this study, we are interested in flows that have two unstable modes, a right moving
wave and a left moving wave, we must examine not only the eigenfunctions of each mode
separately, but also the superposition of the two. We restrict our attention to the study
of the modes found by linear analysis of viscous flows with hyperbolic tangent profiles
as these are the starting point of the simulations described in chapter 4. Since linear
stability analysis only determines the eigenfunctions up to a constant, we normalize the
eigenfunctions so that they have the same amount of perturbation kinetic energy in each
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In appendix C we derive the equation governing the evolution of the horizontally averaged perturbation kinetic energy, given by (C.14). In the linear regime, we can neglect higher order terms in the perturbations and obtain the following equation for the development of the perturbation kinetic energy.

\[
\frac{1}{2} \left( \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial z} \right) = -\bar{u} \bar{w}' \frac{\partial w'}{\partial t} - \left( \frac{\partial \rho'}{\partial t} \right) \frac{\partial}{\partial z} \left( \bar{w} \Delta w' + \bar{w}' \Delta w \right) \frac{1}{Re} \left( \bar{w} \Delta u' + \bar{w}' \Delta w' \right) + \frac{1}{Re} \left( \bar{u} \Delta u' + \bar{w} \Delta w' \right). \tag{3.36}
\]

The left hand side of (3.36) is the rate of change of perturbation kinetic energy which is equal to the sum of the following terms (see Bradshaw [3] for homogeneous viscous flows and Smyth & Peltier [54] for stratified inviscid flows).

1. \(-\bar{u} \bar{w}' \frac{\partial w'}{\partial t}\) represents the extraction of energy from the mean flow by the perturbation,

2. \(-\left( \frac{\partial \rho'}{\partial t} \right) \frac{\partial}{\partial z} \) represents the change in perturbation kinetic energy associated with the transfer of perturbation kinetic energy by the pressure fluctuations,

3. \(-\bar{w} \Delta u' + \bar{w}' \Delta w'\) represents the conversion of perturbation kinetic energy into potential energy,

4. \(\bar{w} \Delta u' + \bar{w}' \Delta w'\) can be divided into a viscous diffusion term and a viscous dissipation term.

In order to gain a better understanding of the development of the instabilities, we examine terms on the right hand side of (3.36) by calculating the correlations \(\bar{u} \bar{w}', \bar{w} \bar{w}', \bar{w} \bar{w}'\), and \(\bar{w} \Delta u' + \bar{w}' \Delta w'\). In addition, we examine the eigenfunctions \(\hat{\rho}, \hat{w}, \hat{u}, \hat{\omega},\) and \(\hat{\rho}\) determined from linear stability analysis. Although we do not graph the complex amplitude of the stream function perturbation \(\phi\), it can be inferred from the graph of \(\hat{w}\) since the two are related by \(\phi = i \hat{w} / \alpha\).
Eigenfunctions of Individual Modes

We start by examining the eigenfunctions for the Holmboe instability corresponding to $R = 3$, $Pr = 9$, $Re = 300$, $J = 0.3$, and $\varepsilon = 0$. This corresponds to the case studied by Smyth & Peltier [54] for an inviscid fluid and gives us a basis for comparison for the $\varepsilon \neq 0$ case. The eigenfunctions and correlation terms in (3.36) for the right and left modes of the Holmboe instabilities are shown in figure 3.15. Since $\varepsilon = 0$ the growth rates of the two modes are equal and the phase speeds are equal and opposite. The horizontal velocity perturbation amplitude, $\bar{u}(z)$, is concentrated in a narrow region centered near the critical level (the height $z_c$ such that $U(z_c) = c_r$) of the mode. Also, the density perturbation amplitude, $\bar{\rho}(z)$, is concentrated near $z = 0$, the center of the density interface, and is due to the strong gradient of $\rho_s$ at this point. The vorticity perturbation amplitude $\bar{\omega}$ has two peaks, one just above (below) the density interface and another just above the critical level (below) for the right (left) mode.

For both left and right modes the maximum value of the perturbation kinetic energy, $\bar{u}'^2 + \bar{w}'^2$, occurs near their critical levels. Since $\bar{u}'\bar{w}' < 0$, energy is being extracted from the mean flow. This indicates that the modes are unstable. It can be shown (see Klaassen & Peltier [29]) that $\bar{u}'\bar{w}'$ can be expressed in terms of the slope of the phase of $\bar{\omega}$, i.e.,

$$\bar{u}'\bar{w}' = -\frac{1}{2\alpha} r^2 \frac{d\varphi}{dz}, \quad (3.37)$$

where $\bar{\omega} = r(z) \exp(i\varphi(z))$. A positive slope of $\varphi$ indicates that the mode is growing, if it is negative then it is decaying. For the eigenfunctions shown in figure 3.15 the slope of the phase of $\bar{\omega}$ is positive and thus the perturbations are unstable. This is in agreement with the positive growth rate $\alpha c_i = 0.030561$ and the correlation $\bar{u}'\bar{w}'$. The minimum of $\bar{u}'\bar{w}'$ occurs near the critical level and so the perturbations extract energy from the mean flow most efficiently at this level.

The correlation $\bar{u}'\bar{p}'$ represents a vertical flux in the perturbation kinetic energy. If
If \( \overline{w'\rho'} > 0 \) then the kinetic energy is directed upward. It is directed downward when \( \overline{w'\rho'} < 0 \). For the right mode, \( \overline{w'\rho'} < 0 \) and hence perturbation kinetic energy is being moved downward from the critical level towards the center of the density interface. For the left mode, the perturbation kinetic energy is being moved upward from the critical level to the density interface.

The quantity \( \overline{w'\rho'} \) corresponds to a vertical displacement of mass. When \( \overline{w'\rho'} > 0 \) mass is moved against the gravitational restoring force, causing an increase in the potential energy of the flow. Potential energy is converted into perturbation kinetic energy when mass is displaced in the direction of the gravitational restoring force (\( \overline{w'\rho'} < 0 \)). For both the right and left modes \( \overline{w'\rho'} > 0 \) indicating that perturbation kinetic energy is being converted to potential energy. From the above we can describe the growth of the perturbations as follows. The perturbations extract kinetic energy from the mean flow at the critical level. The perturbation kinetic energy is then transported towards the center of the density interface where a portion of it is converted to potential energy.

When \( \varepsilon = 0.25 \) (not shown), the general mechanism for growth and the shapes of the eigenvalues are similar to the \( \varepsilon = 0 \) case. The horizontal velocity perturbation \( \tilde{u} \) has a jet concentrated near the critical level. Also, the density perturbation amplitude, \( \tilde{\rho} \), is concentrated near the center of the density interface which now occurs at \( z = -\varepsilon \). There are, however, some differences in the relative amplitudes associated with the two modes which did not occur for the \( \varepsilon = 0 \) case. The magnitude for the density perturbation of the left mode is greater than that of the right mode. The vorticity \( \tilde{\omega} \) has two peaks, one near the density interface and one near the critical level, but instead of being equal in amplitude, as in the \( \varepsilon = 0 \) case, the peak near the density interface is greater. Also, the left mode is starting to develop vorticity near the critical level of the right mode.

As with the \( \varepsilon = 0 \) case, the perturbation kinetic energy, \( \overline{u'^2} + \overline{w'^2} \), is largest near the critical levels. The magnitude of \( \overline{w'\omega'} \) is larger for the right mode, indicating that it is
extracting energy from the mean flow more efficiently and hence will grow more quickly. Finally, $|w'p'|$ and $|\bar{w}'\rho'|$ are larger for the right mode than the left. This indicates that the right mode is responsible for a larger portion of the vertical displacement of perturbation kinetic energy. Also, most of the conversion from perturbation kinetic energy to potential energy occurs through the right mode.

When $\varepsilon = 0.5$ (see figure 3.16), the differences between the right and left modes are even more pronounced. The overall shapes of the eigenfunctions for the right mode are not significantly different from the $\varepsilon = 0$ case. The eigenfunctions of the left mode, however, have been altered dramatically by the displacement of the density interface. This is similar to results of Yonemitsu [70] who found that for non-symmetrically placed horizontal boundaries only one mode was affected. It is clear, from $\bar{w}'\rho'$, $w'p'$ and $\bar{w}'\rho'$ that the right mode is the main mechanism through which energy is exchanged between the various reservoirs (mean kinetic, perturbation kinetic, and potential energies). For the left mode, the most efficient extraction of energy from the mean flow no longer occurs at the critical level. Instead it has been shifted upward towards the density interface.

Superposition of Right and Left Modes

To better understand how right and left modes interact in the linear regime, we superimpose the two modes and examine the evolution of the perturbation kinetic energy throughout one cycle of its period. In appendix D it is shown that the period of the horizontally averaged kinetic energy is $2\pi/\alpha(c_r^{(r)} - c_r^{(l)})$. We examine the evolution of the terms in (3.36) for one period of the kinetic energy of the superimposed modes in increments of $\Delta \chi = \pi/4$ where $\chi = \alpha(c_r^{(r)} - c_r^{(l)})t$. For convenience, we suppress the linear growth components of the perturbations.

The evolution of the terms in (3.36) throughout a cycle is shown in figure 3.17 for the $\varepsilon = 0$ case. In the linear regime, growth and decay of the perturbation kinetic energy is
controlled by $\overline{uw'}$ (which determines the transfer of kinetic energy from the mean flow to the perturbation). The perturbation extracts energy most efficiently at $\chi = 0$. This corresponds to a local maximum in the perturbation kinetic energy. $\chi = \pi$ corresponds to the time when the most amount of perturbation kinetic energy is being returned to the mean flow, indicating a local minimum in the perturbation kinetic energy. In between these local extrema there is a period of decay for $0 < \chi < \pi$ and a period of growth for $\pi < \chi < 2\pi$ after which the cycle repeats itself. Although the perturbations spend equal amounts of time in the growth and decay portions of the cycle, the amplitudes of $\overline{uw'}$ are largest when the perturbations are extracting kinetic energy from the mean flow. Thus if growth of the modes were included, we would observe a net gain in perturbation kinetic energy over one cycle. This oscillation in the growth pattern of Holmboe waves was first predicted by Holmboe [23] and later verified by Smyth et al. [53], and Smyth & Peltier [54]. In chapter 4, we also observe this result numerically, and examine how the exchange of energy changes in the nonlinear regime.

Examination of the correlations $\overline{uw'}$, $\overline{wp'}$, and $\overline{w'p'}$, enable us to determine how energy is transferred between the three energy reservoirs: mean kinetic energy, perturbation kinetic energy, and potential energy. When energy is transferred from the mean flow to the perturbations (i.e., $\overline{uw'} < 0$), the critical levels act as sources for the perturbation kinetic energy. Perturbation kinetic energy is then transported vertically away from the critical level towards the density interface. The density interface then acts as a sink, converting perturbation kinetic energy into potential energy. This is as described above from the results of the eigenfunctions for the individual modes. The opposite behaviour occurs when the perturbations return kinetic energy to the mean flow. The critical levels act as sinks and the density interface as a source. Potential energy at the density interface is converted back into perturbation kinetic energy which is transported towards the critical levels where the perturbations return energy to the mean flow. There is a phase
difference, however, between $\overline{\nu'\nu'}$ and $\overline{\theta'\theta'}$. At $\chi = 3\pi/4$ perturbation kinetic energy is still being transferred to potential energy even though it is also returning energy to the mean flow. Similarly at $\chi = 3\pi/2$, potential energy is being converted to perturbation kinetic energy as is mean kinetic energy.

Figure 3.18 shows the evolution of the perturbation kinetic energy and the correlations for $\varepsilon = 0.25$. There is essentially the same oscillation between growth and decay as described for $\varepsilon = 0$. The perturbation kinetic energy is maximum when $\chi = 0$ and minimum when $\chi = \pi$ with corresponding periods of decay and growth in between these local extrema. As with the $\varepsilon = 0$ case, the growth and decay of the perturbation kinetic energy is in phase with $\overline{\nu'\nu'}$, which determines the exchange of energy between the mean flow and the perturbations. Eigenfunctions shown in figure 3.16 indicate that we can, for the most part, use the line $z = -\varepsilon$ to separate contributions to left and right modes, particularly if we focus on the behaviour near their respective critical levels. In figure 3.18 we observe that the right mode extracts energy from the mean flow more efficiently than the left mode. During the time when the perturbations are returning energy to the mean flow, the right mode returns less than the left mode. These two features indicate that the right mode grows more rapidly than the left mode. This is indeed the case, as determined by the values of $\alpha c_i$ predicted from linear analysis. Upon examination of $\overline{\theta'\theta'}$, it is evident that more perturbation kinetic energy is being moved around in the upper layer than the lower layer. Thus the right mode is responsible for much of the vertical transport of perturbation kinetic energy. We believe that the right mode is probably responsible for most of the exchange between perturbation kinetic energy and potential energy, although this is difficult to determine from the evolution of $\overline{\theta'\theta'}$.

The evolution of $\overline{\nu'^2 + \theta'^2}$, and the correlations in equation 3.36 for $\varepsilon = 0.5$ are shown in figure 3.19. As with the two previous cases, $\varepsilon = 0$ and 0.25, the perturbation kinetic energy oscillates between a maximum value at $\chi = -\pi/4$ and a minimum at $\chi = 3\pi/4$. 
There is a phase difference, however, between $u'w'$ and $u'^2 + \omega'^2$, whereas for $\varepsilon = 0$ and 0.25 they were not. The most efficient transfer of energy from the mean flow to the perturbations occur at $\chi = 0$. The perturbations return the most energy to the mean flow when $\chi = \pi$. Also, the perturbations spend more time throughout the cycle extracting energy from the mean flow than in the previous cases. If we continue to use $z = -\varepsilon$ to separate the behaviour associated with the right mode from that of the left mode, we observe that right and left modes no longer exhibit the same behaviour throughout the cycle. There is a slight phase shift between the two modes. The right mode start to release energy to the mean flow and resumes its extraction of energy from the mean flow at different times than the left mode. As with the $\varepsilon = 0.25$ case, the right mode extracts energy more efficiently from the mean flow and returns less than the left mode. Also, the right mode is the primary means through which perturbation kinetic energy is transported vertically towards the center of the shear layer.
Figure 3.1: Background flow for piecewise linear velocity and density profiles
Figure 3.2: Stability diagrams for inviscid case

Linear stability diagrams for piecewise linear profiles with $\varepsilon = 0, 0.25,$ and $0.5$ for $H = \infty$, $10, 5, 4,$ and $3$. Contours are of constant growth rate $\alpha c_i$. For $\varepsilon > 0$, solid contours correspond to right travelling waves and dashed contours to left travelling waves. When $\varepsilon = 0$, growth rates of left and right modes coincide in the Holmboe regime whereas in the Kelvin-Helmholtz regime only dominant modes are shown. Stability boundaries ($\alpha c_i = 0$) are the outer most contour in each limb.
Figure 3.3: Growth rates of most unstable modes for inviscid case

Growth rate of most unstable mode at a given J for stability diagrams shown in figure 3.2. Maximum growth rate in the Holmboe regime ($\varepsilon = 0$) is indicated by the horizontal dotted line.
Figure 3.4: Phase speeds of most unstable modes for inviscid case
Phase speed of most unstable mode at a given $J$ for stability diagrams shown in figure 3.2.
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Figure 3.5: Linear stability diagrams for small J when $\varepsilon = 0$

Growth rates (solid contours) and phase speeds (dotted lines) for the case studied by Holmboe [23]. When $J < 0.07$ there are two regions of instability: Kelvin-Helmholtz instabilities ($c_r = 0$) and Holmboe instabilities (the superposition of right and left moving waves).
Figure 3.6: $\alpha c_i/\sqrt{J}$ for Kelvin-Helmholtz and Holmboe waves

Growth rate, $\alpha c_i$ and $\alpha c_i/\sqrt{J}$ along $\alpha$-$J$ curve for which growth rate is maximized for $H = \infty, \varepsilon = 0$. Results for both Kelvin-Helmholtz (dotted line) and Holmboe (solid line) instabilities are shown.
Figure 3.7: $\alpha c_i/\sqrt{J}$ for $\varepsilon = 0, 0.25$, and 0.5

Growth rate, $\alpha c_i$ and $\alpha c_i/\sqrt{J}$ along $\alpha$-J curve for which growth rate is maximized for $H = \infty$, with $\varepsilon = 0, 0.25$, and 0.5. Both right (solid line) and left (dashed line) modes are shown.
Figure 3.8: Background flow for hyperbolic tangent velocity and density profiles
Figure 3.9: Maximum growth rate in Holmboe regime with varying $R$

Maximum computed growth rate in Holmboe regime (+) for $R = 3, 4, 6, \text{ and } 8$ (Pr=9, 16, 36, and 64, respectively) with $\varepsilon = 0$, Re=300, and $H = 10$. The solid line is the approximation $(\alpha c)_{\max} = 0.2234/(0.5544 - R) + 0.1218$ determined using the method of least squares.
Linear stability diagrams for $\varepsilon = 0$, 0.25, and 0.5 with $Re=300$, $H = 10$, and $R = 3, 4, 6, \text{and} 8$ ($Pr = 9, 16, 36, \text{and} 64$, respectively). Contours are of constant growth rate $\alpha c_i$. For $\varepsilon > 0$, the solid contours correspond to right travelling waves and the dashed contours to left travelling waves. When $\varepsilon = 0$, growth rates of left and right modes coincide in the Holmboe regime whereas in the Kelvin-Helmholtz regime only dominant modes are shown. Stability boundaries ($\alpha c_i = 0$) are the outer most contour in each limb.
Figure 3.11: Maximum growth rates for smooth profiles: varying $R$

Maximum growth rate at a given $J$ for stability diagrams shown in figure 3.10. Maximum growth rate in the Holmboe regime ($\varepsilon = 0$) is indicated by the horizontal dotted line.
Figure 3.12: Linear stability diagrams for smooth profiles: varying $H$

Linear stability diagrams for $R = 3$, Pr=9, Re=300, $\varepsilon = 0$, and 0.25 with $H = 20$, 10, 5, 4, and 3, ($N = 150, 100, 100, 100$, and 100, respectively). For $\varepsilon > 0$, the solid contours correspond to right travelling waves and the dashed contours to left travelling waves. When $\varepsilon = 0$, growth rates of left and right modes coincide in the Holmboe regime whereas in the Kelvin-Helmholtz regime only dominant modes are shown. Stability boundaries ($\alpha c_i = 0$) are the outer most contour in each limb.
Figure 3.13: Maximum growth rates for smooth profiles: varying $H$

Maximum growth rate at a given $J$ for stability diagrams shown in figure 3.12. Maximum growth rate in the Holmboe regime ($\varepsilon = 0$) is indicated by the horizontal dotted line.
Figure 3.14: Smooth profiles: varying Re

Linear stability diagrams for $R = 3$, Pr = 9, $H = 10$, $Re = 100$, and 300 with $\varepsilon = 0$, 0.25, and 0.5. Here, $N = 50$ and 100, for $Re = 100$ and 300, respectively. Also shown is the maximum growth rate at a given $J$. When $\varepsilon > 0$, solid lines correspond to right travelling waves and dashed lines to left travelling waves. When $\varepsilon = 0$, growth rates of left and right modes coincide in the Holmboe regime whereas in the Kelvin-Helmholtz regime only dominant modes are shown. Stability boundaries ($\alpha_c = 0$) are the outer most contour in each limb. Maximum growth rate in the Holmboe regime ($\varepsilon = 0$) is indicated by the horizontal dotted line.
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Right mode, \( \alpha c_i = 0.030561 \), \( c_r = 0.598220 \), and \( z_c = 0.69037 \).

Left mode, \( \alpha c_i = 0.030561 \), \( c_r = -0.598220 \), and \( z_c = -0.69037 \).

Figure 3.1.5: Eigenfunctions and correlations when \( \varepsilon = 0 \)

Eigenfunctions and correlations when \( \varepsilon = 0 \), \( R = 3 \), \( Re = 300 \), \( Pr = 9 \), \( H = 10 \), \( J = 0.3 \),
and \( \alpha = 0.54 \), a) \( \rho \), b) \( \tilde{w} \), c) \( \tilde{u} \), d) \( \tilde{\omega} \), e) \( \tilde{p} \), f) \( \frac{(\tilde{u}^2 + \tilde{w}^2)}{2} \), g) \( \tilde{u} \tilde{w} \), h) \( \tilde{w} \tilde{p} \), i) \( \tilde{w} \tilde{p} \), j) \( \tilde{u} \Delta \tilde{u} + \tilde{w} \Delta \tilde{w} \).

\( z = -\varepsilon \) is indicated by the dashed horizontal line and the critical levels are the horizontal dotted line. For a) through e), both amplitude (thick line) and phase (thin line) are shown. In a) the phase is only given when the amplitude is nonzero.
Right mode, $\alpha c_i = 0.110458$, $c_r = 0.296927$, and $z_c = 0.30615$.

Left mode, $\alpha c_i = 0.006855$, $c_r = -0.943582$, and $z_c = -1.76975$.

Figure 3.16: Eigenfunctions and correlations when $\varepsilon = 0.5$

Eigenfunctions and correlations when $\varepsilon = 0.5$, $R = 3$, $Re = 300$, $Pr = 9$, $H = 10$, $J = 0.3$, and $\alpha = 0.67$. 
Figure 3.17: Evolution of perturbation kinetic energy and correlations with $\varepsilon = 0$.

Evolution of perturbation kinetic energy and correlations when both modes are present for $\varepsilon = 0$, $R = 3$, $Re = 300$, $Pr = 9$, $H = 10$, $J = 0.3$, and $\alpha = 0.54$. 
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\[
\begin{align*}
\chi = \pi & \quad (u'^2 + w'^2)/2 \quad u'w' \quad w'\rho' \quad w'\rho' \quad u'\Delta u' + w'\Delta w' \\
\chi = \frac{5\pi}{4} & \quad 0 \quad -0.024 \quad 0.024 \quad -0.012 \quad 0.012 \quad -0.18 \quad 0.18 \quad 4\times10^{-5} \quad 4\times10^{-5} \\
\chi = \frac{3\pi}{2} & \quad 0 \quad 0.024 \quad -0.024 \quad 0.012 \quad -0.012 \quad 0.18 \quad -4\times10^{-5} \quad 4\times10^{-5} \\
\chi = \frac{\pi}{4} & \quad 0 \quad 0.024 \quad -0.024 \quad 0.012 \quad -0.012 \quad 0.18 \quad -4\times10^{-5} \quad 4\times10^{-5}
\end{align*}
\]
Figure 3.17 continued.
Figure 3.18: Evolution of perturbation kinetic energy and correlations with $\varepsilon = 0.25$

Evolution of perturbation kinetic energy and correlations when both modes are present for $\varepsilon = 0.25$, $R = 3$, $Re = 300$, $Pr = 9$, $H = 10$, $J = 0.3$, and $\alpha = 0.52$. 
Figure 3.18 continued.
Figure 3.19: Evolution of perturbation kinetic energy and correlations with $\varepsilon = 0.5$

Evolution of perturbation kinetic energy and correlations when both modes are present for $\varepsilon = 0.5$, $R = 3$, $Re = 300$, $Pr = 9$, $H = 10$, $J = 0.3$, and $\alpha = 0.67$. 
Figure 3.19 continued.
Chapter 4

Nonlinear Numerical Simulations

In this chapter we wish to study the nonlinear evolution of perturbations in a flow with initial background profiles given by (3.20). When variations in density are small, (2.17) and (2.18), derived in chapter 2, govern the two dimensional evolution of the flow:

\[
(\Delta \psi)_t + u(\Delta \psi)_x + w(\Delta \psi)_z = J\rho'_x + \frac{1}{Re} \Delta \Delta \psi, \tag{4.1}
\]

and

\[
\rho_t + u\rho_x + w\rho_z = \frac{1}{Re Pr} \Delta \rho. \tag{4.2}
\]

Since we wish to calculate the perturbations from the initial mean state, we write

\[
\psi(x, z, t) = \Psi(z, 0) + \psi'(x, z, t),
\]
\[
\rho(x, z, t) = \rho_0(z) + \rho'(x, z, t), \tag{4.3}
\]
\[
u(x, z, t) = U(z) + u'(x, z, t),
\]
\[
w(x, z, t) = w'(x, z, t),
\]

where \(\psi', \rho', u'\) and \(w'\) are the perturbations from the initial mean state. Substituting (4.3) into (4.1) and (4.2) we obtain the following equations to be solved.

\[
(\Delta \psi'_t)_t + (U + u')(\Delta \psi')_x + w'(U_{zz} + (\Delta \psi')_z) = J\rho'_x + \frac{1}{Re} (\Delta \Delta \psi' + U_{zzz}), \tag{4.4}
\]

and

\[
\rho'_t + (U + u')\rho'_x + w'(\rho_0 + \rho')_z = \frac{1}{Re Pr} (\Delta \rho' + \rho_{ssz}). \tag{4.5}
\]

It is assumed that the flow is periodic in the direction of the flow (x-direction) with \(\psi' = \Delta \psi' = \rho' = 0\) at the horizontal boundaries \(z = \pm H/2\).
The initial perturbations are of the form $(\psi', \rho') = \Re\{((\phi, \tilde{\rho})\exp(i\tilde{\alpha}x))\}$, where $\tilde{\alpha}$ is the real wave number which determines the period of the flow, and $(\phi, \tilde{\rho})$ are the complex amplitudes determined using the normal modes approach to linear stability analysis. As discussed in chapter 3, the most unstable modes for the right and left moving waves occur at different wave numbers when $\varepsilon \neq 0$ (see figure 4.1 for the case $J = 0.3$, $R = 3$, $Re = 300$, $Pr = 9$, and $H = 10$). If these modes were used to initialize the flow, the computational domain would have to be very large in order to maintain periodicity in the $x$-direction. Since it is believed that it is the linear growth rate that determines the relative stability of the modes, we use the following criteria for selecting the wave number, $\tilde{\alpha}$, of the flow. We choose $\tilde{\alpha}$ so that the ratio of the left to right growth rates at this wave number equals the ratio of the maximum left growth rate to the maximum right growth rate. The values of these are indicated in figure 4.1 for the above mentioned case. The growth rates at the selected wave numbers are close to the maximum values.

Although both modes have the same period in the $x$-direction, the complex phase speed, $c$, and complex amplitudes, $(\phi, \tilde{\rho})$, are different for left and right moving waves. The precise form of the initial perturbations were discussed in section 3.3.4.

The initial amplitudes of the perturbations are determined by setting the initial amount of wave kinetic energy in a given mode, $K'$, sufficiently small (0.0005 for the simulations examined here) to guarantee that the nonlinear simulation spends some time in the linear regime. The wave kinetic energy $K'$ is defined as

$$K'(t) = \left\langle \overline{\tilde{u}^2 + \tilde{w}^2} \right\rangle. \quad (4.6)$$

Since we are calculating the perturbations from the initial mean flow we write $u(x, z, t) = \tilde{u}(z, t) + \tilde{u}'(x, z, t)$ where $\tilde{u}(z, t) = U(z) + \overline{\tilde{u}}(z, t)$, and $\tilde{u}'(x, z, t) = u'(x, z, t) - \overline{\tilde{u}}(z, t)$. Here $\overline{\tilde{u}'}$ represents the change in the mean flow due to diffusion. $\left\langle \cdot \right\rangle = \left( \int_0^L dx \right) / L$ and $\left\langle \cdot \right\rangle = \int_{-H/2}^{H/2} dz$ indicate horizontal averaging and vertical integration respectively, where
Chapter 4. Nonlinear Numerical Simulations

$L = 2\pi/\bar{\alpha}$ is the length of the domain in the $x$-direction.

Equations (3.3) indicate that the mean velocity and density profiles diffuse over time due to the presence of viscosity and thermal diffusion. This means that the values of the bulk Richardson number $J$ and the Reynolds number, $Re$, will increase over time (due to the increase in the shear layer thickness, $h$). Hence, when the simulation enters the nonlinear regime, the actual values of $J$ and $Re$ will have increased from their initial value. We will continue to describe, however, the flow by the initial values of these parameters.

4.1 Numerical Method

In order to numerically solve (4.4) and (4.5) we use second order centered differencing in space and a fourth order Runge-Kutta scheme in time. We let $\Delta x$ and $\Delta z$ denote the step sizes in the $x$- and $z$-directions respectively. Using the following centered differencing approximations

\[
\frac{\partial (\cdot)_{i,j}}{\partial x} \approx D_x(\cdot)_{i,j} = \frac{(\cdot)_{i+1,j} - (\cdot)_{i-1,j}}{2\Delta x}
\]

\[
\frac{\partial (\cdot)_{i,j}}{\partial z} \approx D_z(\cdot)_{i,j} = \frac{(\cdot)_{i,j+1} - (\cdot)_{i,j-1}}{2\Delta z}
\]

\[
\frac{\partial^2 (\cdot)_{i,j}}{\partial x^2} \approx D_{xx}(\cdot)_{i,j} = \frac{(\cdot)_{i+1,j} - 2(\cdot)_{i,j} + (\cdot)_{i-1,j}}{(\Delta x)^2}
\]

\[
\frac{\partial^2 (\cdot)_{i,j}}{\partial z^2} \approx D_{zz}(\cdot)_{i,j} = \frac{(\cdot)_{i,j+1} - 2(\cdot)_{i,j} + (\cdot)_{i,j-1}}{(\Delta z)^2}
\]

\[
\Delta \approx \Delta_h = D_{xx} + D_{zz}
\]

we can write the semi-discretization of (4.4) as

\[
\Delta_h \dot{\Psi}'_{i,j} = -(U_j + D_z \Psi'_{i,j}) D_x \Delta_h \Psi'_{i,j} + D_x \psi'_{i,j} (U_{zzj} + D_z \Delta_h \psi'_{i,j}) + \]
\[ JD_x \Psi'_{i,j} + \frac{1}{Re} (\Delta_t \Delta_x \Psi'_{i,j} + U_{zzz,j}) \]
\[ = F(\Psi', \Psi'_{i,j}) \]  
(4.8)

and (4.5) as
\[ \Psi'_{i,j} = (U_j + D_x \Psi'_{i,j}) D_x \Psi'_{i,j} + D_x \Psi'_{i,j} (\rho_{a_{i,j}} + D_x \rho_{i,j}) + \frac{1}{RePr} (\Delta_t \rho_{i,j} + \rho_{a_{i,j}}) \]
\[ = G(\Psi', \Psi'_{i,j}, U, \rho_{a_{i,j}}) \]  
(4.9)

where \( \Psi'_{i,j} \approx \psi'(i \Delta x, j \Delta z, t) \) and \( \rho'_{i,j} \approx \rho'(i \Delta x, j \Delta z, t) \) are the approximate values of the stream function and density perturbations at a given grid point, and (\( \ast \)) indicates differentiation with respect to time. Before discretizing in time, (4.8) must be written in the form
\[ \Psi'_{i,j} = f(\Psi', \Psi'_{i,j}, U)_{i,j} \]  
(4.10)

To this end, we use a method described by Strang [59] which makes use of periodicity of the function in the \( x \)-direction. The first step is to take the discrete Fourier transform of the semi-discretization (4.8) in the \( x \)-direction. Writing \( \mathcal{F} \{ \cdot \} = \ddot{\cdot} \), we obtain the following.
\[ \frac{\ddot{\Psi}'_{k,j+1}}{(\Delta z)^2} + \left\{ \frac{2 \cos(2\pi k \Delta x)}{(\Delta x)^2} - \frac{2}{(\Delta z)^2} \right\} \ddot{\Psi}'_{k,j} + \frac{\ddot{\Psi}'_{k,j-1}}{(\Delta z)^2} = \ddot{F}(\Psi', \Psi'_{i,j}, U)_{k,j} \]  
(4.11)

Now for each Fourier component, \( k \), (4.11) describes a tridiagonal system. This is easily solved, giving
\[ \ddot{\Psi}'_{k,j} = \ddot{F}(\Psi', \Psi'_{i,j}, U)_{k,j} \]  
(4.12)

Taking the inverse discrete Fourier transform of the above yields an equation of the desired form (4.10) which along with (4.9) can be discretized in time using the fourth order Runge-Kutta scheme.
4.1.1 Numerical Stability

In this section we discuss the stability of the numerical scheme described above. Canuto et al. [6] give the stability region of the fourth order Runge-Kutta scheme used for the time discretization in the numerical solution of (4.4) and (4.5). A schematic of the region is shown in figure 4.2. In order to have stability, we require that the amplification factor of the semi-discretized scheme (4.8) and (4.9) lie inside the kidney-shaped region. For simplicity, we will require that the amplification factor, $\lambda$, lie within the indicated rectangle, i.e.,

$$0 \geq \Re \{\lambda\} \geq \frac{-2.79}{2\Delta t},$$

$$0 \leq |\Im \{\lambda\}| \leq \frac{\sqrt{2}}{\Delta t},$$

(4.13)

where $\Delta t$ is the size of the time step.

To determine an approximate range for the amplification factor $\lambda$, we first consider the original equations (4.4) and (4.5). Since these are nonlinear equations in $\psi'$ and $\rho'$, we use the method of frozen coefficients to linearize the problem. We know that $|U| \leq 1$. Also, since the velocity perturbations $u'$ and $w'$ are much smaller than the mean flow we can use $|u'| \leq 1$ and $|w'| \leq 1$ (note that these bounds are much larger than the physical bounds). Using the above bounds, we can reduce (4.4) and (4.5) to the following linear system.

$$(\Delta \psi')_t + 2(\Delta \psi')_x + U_{zz} + (\Delta \psi')_z = J\rho'_x + \frac{1}{\Re}(\Delta \Delta \psi' + U_{zzz}),$$

(4.14)

and

$$\rho'_t + 2\rho'_x + \rho_a + \rho'_z = \frac{1}{\Re Pr}(\Delta \rho' + \rho_a),$$

(4.15)

Since the quantities $U_{zz}$, $U_{zzz}$, $\rho_a$, and $\rho_{a\zz}$ are independent of the grid size, we can restrict our attention to the stability of the following equations.

$$(\Delta \psi')_t = -2(\Delta \psi')_x - (\Delta \psi')_z + J\rho'_x + \frac{1}{\Re} \Delta \Delta \psi',$$

(4.16)
and

\[ \rho'_i = -2\rho'_{x} - \rho'_{z} + \frac{1}{\text{RePr}} \Delta \rho'. \] (4.17)

Using the notation described by (4.7), the semi-discretization of (4.16) and (4.17) is

\[ \Delta_h \Psi'_{i,j} = -2 D_x \Delta_h \Psi'_{i,j} - D_z \Delta_h \Psi'_{i,j} + J D_x \rho'_{i,j} + \frac{1}{\text{Re}} \Delta_h \Delta_h \Psi'_{i,j} \] (4.18)

and

\[ \hat{\rho}'_{i,j} = -2 D_x \rho'_{i,j} - D_z \rho'_{i,j} + \frac{1}{\text{RePr}} \Delta_h \rho'_{i,j}. \] (4.19)

We use von Neumann analysis to determine the amplification factors of (4.18) and (4.19). Taking the two-dimensional discrete Fourier transform of (4.18) and (4.19) we obtain

\[ \hat{\Delta}_h \hat{\Psi}_{k,l} = \left[ \frac{(\Delta_h)^2}{\text{Re}} + i \hat{\Delta}_h \hat{\dot{D}} \right] \hat{\Psi}_{k,l} + i J \frac{\sin \xi}{\Delta x} \hat{\dot{\rho}}_{k,l} \] (4.20)

\[ \hat{\dot{\rho}}_{k,l} = \left[ \frac{\Delta_h}{\text{RePr}} + i \hat{\dot{D}} \right] \hat{\dot{\rho}}_{k,l} \]

where \( \xi = 2\pi k \Delta x, \zeta = 2\pi l \Delta z, \hat{\dot{D}} = -2 \sin \xi/\Delta x - \sin \zeta/\Delta z, \) and \( \hat{\Delta}_h = 2(\cos \xi - 1)/(\Delta x)^2 + 2(\cos \zeta - 1)/(\Delta z)^2, \) the discrete Fourier transform of the Laplacian operator.

We can write (4.20) in matrix form,

\[ \begin{pmatrix} \hat{\dot{\Psi}}_{k,l} \\ \hat{\dot{\rho}}_{k,l} \end{pmatrix} = \begin{pmatrix} \frac{\Delta_h}{\text{Re}} + i \hat{\dot{D}} & i J \frac{\sin \xi}{\Delta_h \Delta x} \\ 0 & \frac{\Delta_h}{\text{RePr}} + i \hat{\dot{D}} \end{pmatrix} \begin{pmatrix} \hat{\Psi}_{k,l} \\ \hat{\dot{\rho}}_{k,l} \end{pmatrix} \] (4.21)

The amplification factors of the semi-discretization are given by the eigenvalues of the above matrix, \( \lambda_1 = \hat{\Delta}_h/\text{Re} + i \hat{\dot{D}}, \) and \( \lambda_2 = \hat{\Delta}_h/\text{RePr} + i \hat{\dot{D}}. \) The restrictions (4.13) give the following conditions for stability.

\[ \Re{\lambda} = \frac{2}{\text{Re}} \left( \frac{\cos \xi - 1}{(\Delta x)^2} + \frac{\cos \zeta - 1}{(\Delta z)^2} \right) \geq \frac{-4}{\text{Re}} \left( \frac{1}{(\Delta x)^2} + \frac{1}{(\Delta z)^2} \right) \geq \frac{-2.79}{2\Delta t}, \] (4.22)

and

\[ |\Im{\lambda}| = \left| \frac{-2\sin \xi}{\Delta x} - \frac{\sin \zeta}{\Delta z} \right| \leq \frac{2}{\Delta x} + \frac{1}{\Delta z} \leq \frac{\sqrt{2}}{\Delta t}, \] (4.23)
providing $\Pr \geq 1$ (and hence $|\Re\{\lambda_1\}| \geq |\Re\{\lambda_2\}|$).

From (4.22) and (4.23) we can deduce the following restriction on the time step, $\Delta t$,

$$\Delta t \leq \min \left\{ \frac{\sqrt{2}}{\Delta x + \Delta z}, \frac{2.79\text{Re}}{8 \left( \frac{1}{(\Delta x)^2} + \frac{1}{(\Delta z)^2} \right)} \right\}.$$

(4.24)

Although (4.24) only guarantees a stable numerical scheme for the linear equations (4.14) and (4.15) with constant coefficients, in general it is still used to determine an appropriate value of $\Delta t$ for the numerical solution to the nonlinear equations (4.4) and (4.5).

### 4.2 Results

#### 4.2.1 Methodology

For the background flow defined by (3.20), linear theory predicts that the right mode dominates when $\varepsilon > 0$. We start our investigation of the nonlinear evolution by examining the development of a flow when the initial perturbation contains only the right mode. Next, we perturb the flow with the mode corresponding to the more slowly growing left moving wave. Finally, in order to examine the interaction between the two waves, we impose both modes.

To examine the evolution of the perturbations, we examine the density and vorticity distributions. At different times throughout the simulation, we plot contours of constant density ranging from -0.72 in the upper layer to 0.72 in the lower layer. The contours are of equal increments of 0.18. We superimpose vorticity ($\omega = u_z - w_x$) shading upon the density contours, where white corresponds to -1.8 (counter-clockwise vorticity) and black to 1.8 (clockwise vorticity). These plots enable us to visualize how the perturbations evolve. They also allow us to qualitatively compare results with those of laboratory experiments.

It is desirable to compare results of the nonlinear simulation with those of linear
stability analysis. Specifically, we wish to compute the growth rates and phase speeds of the perturbations. In appendix D, we showed that $K'(t)$ oscillates in time when two modes are present. A lack of oscillations would indicate that the perturbation is composed of one mode only. To compute the growth rate of the simulations, any oscillations in $K'(t)$ are removed using a low-pass filter. The linear growth rate $\alpha c_i$ can then be determined from the resulting perturbation kinetic energy (see Smyth [51]) $\tilde{K}'$.

$$\alpha c_i = \frac{1}{2} \frac{d}{dt} \ln \tilde{K}' \quad (4.25)$$

To determine the phase speed $c_r$, we track the position of maximum/minimum values of density at the critical levels determined from linear analysis of the right/left modes. The change in position with respect to time gives the phase speed. This method of tracking the right and left moving waves has been used by Smyth et al. [53] in their study of symmetric Holmboe waves.

In appendix C, the equations governing the evolution of mean kinetic energy, perturbation kinetic energy and potential energy were derived and are given by (C.22), (C.23), and (C.29), respectively. Adopting the notation of Klaassen & Peltier [29], we have

$$\bar{K}_t = -C(\bar{K}, K') - D(\bar{K}) \quad (4.26)$$

$$K'_t = C(\bar{K}, K') + C(P, K') - D(K') \quad (4.27)$$

$$P_t = -C(P, K') - D(P) \quad (4.28)$$

where

$$C(\bar{K}, K') = -\langle \bar{u}_z \bar{w} \rangle, \quad D(\bar{K}) = \frac{1}{\text{Re}} \langle (\bar{u}_z)^2 \rangle,$$

$$C(P, K') = -\langle \bar{w} \rangle, \quad D(K') = \frac{1}{\text{Re}} \langle (\bar{u}_z^2 + (\bar{u}_x')^2 + (\bar{w}_z')^2 + (\bar{w}_x')^2) \rangle,$$

$$D(P) = \frac{-J}{\text{RePr}} \langle z \bar{\rho}_{zz} \rangle.$$
Chapter 4. Nonlinear Numerical Simulations

The notation $C(R_1, R_2)$ indicates a conversion of energy from reservoir $R_1$ to reservoir $R_2$. $D(R_1) > 0$ represents a loss of energy from reservoir $R_1$ due to diffusion. We shall examine the evolution of the three energy reservoirs, $\tilde{K}$, $K'$, and $P$ over time as well as the total energy $T = \tilde{K} + K' + P$. In addition, we will examine the terms that govern the evolution of energy, i.e., the exchange of energy between reservoirs and the loss of energy due to dissipation.

Equations (3.3) indicate that the mean velocity and density profiles diffuse over time due to the presence of viscosity and thermal diffusion. It is of interest to see how the presence of perturbations in the flow affect the development of the mean flow. To this end we compare the evolution of the computed mean flow $\bar{u}(z, t)$, and $\bar{\rho}(z, t)$ (where $\bar{\rho}(z, t) = \rho_\infty(z) + \bar{\rho}(z, t)$) with the exact solutions to (3.3). Solutions to (3.3) with initial conditions given by (3.20) and boundary conditions

$$
U_z(-\frac{H}{2}, t) = \text{sech}^2 \left(-\frac{H}{2}\right) = \bar{U}_z,
$$

$$
U_z(\frac{H}{2}, t) = \text{sech}^2 \left(\frac{H}{2}\right) = \bar{U}_z,
$$

$$
\rho_\infty(-\frac{H}{2}, t) = -\tanh R(-\frac{H}{2} + \varepsilon) = \rho_{\infty-},
$$

$$
\rho_\infty(\frac{H}{2}, t) = -\tanh R(\frac{H}{2} + \varepsilon) = \rho_{\infty+}
$$

(4.30)

can be found using separation of variables (see, for example, Boyce & DiPrima [2]). They are given by

$$
\bar{u}(z, t) = \bar{U}_z z + \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \exp \left\{ \frac{-n^2 \pi^2 t}{\text{Re} H^2} \right\} \cos n \left(\frac{\pi z}{H} + \frac{\pi}{2}\right),
$$

(4.31)

$$
\bar{\rho}(z, t) = \bar{\rho}_\infty + \Delta \rho_\infty z + \sum_{n=1}^{\infty} B_n \exp \left\{ \frac{-n^2 \pi^2 t}{\text{Re} \text{Pr} H^2} \right\} \sin n \left(\frac{\pi z}{H} + \frac{\pi}{2}\right),
$$

(4.32)

where

$$
A_n = \frac{2}{H} \int_{-H/2}^{H/2} \left\{ U(z) - \bar{U}_z z \right\} \cos n \left(\frac{\pi z}{H} + \frac{\pi}{2}\right) dz,
$$

(4.33)

$$
B_n = \frac{2}{H} \int_{-H/2}^{H/2} \left\{ \rho_\infty(z) - (\bar{\rho}_\infty + \Delta \rho_\infty z) \right\} \sin n \left(\frac{\pi z}{H} + \frac{\pi}{2}\right) dz,
$$

(4.34)
with $\tilde{\rho}_a = (\rho_{a+} + \rho_{a-})/2$ and $\Delta \rho_a = (\rho_{a+} - \rho_{a-})/H$. Truncating the above series \((4.31)\) and \((4.32)\) at \(n = 15\) is an excellent approximation for \(t > 100\).

### 4.2.2 Parameter Values

As already discussed in chapter 3, Smyth \[51\] found that \(R > 2.4\) is required in order for Holmboe instabilities to exist. Since higher values of \(R\) mean a thinner density interface, more grid points are required in the vertical direction in order to numerically resolve the flow. We choose to study the case \(R = 3\) which is sufficiently large to support Holmboe instabilities yet small enough to allow computations without requiring too fine a grid. This corresponds to \(Pr = 9\) which is the approximate value for the diffusion of heat in water at \(10^\circ C\) (table 3.1).

In the numerical simulations described, we set \(Re = 300\) which is sufficiently large to allow the instabilities to fully develop before they are damped by viscosity. Furthermore, it is of the same order of magnitude as that of many laboratory experiments. Smyth et al. \[53\] have examined in detail the nonlinear evolution of Holmboe waves when \(\varepsilon = 0\) with \(J = 0.3\), \(R = 3\), \(Pr = 9\), \(Re = 300\), and \(H = 10\). We choose the same parameter values in our study of non-symmetric Holmboe instabilities. This allows us to verify our results for \(\varepsilon = 0\) against those of Smyth et al. It also gives us a good basis for comparison for the non-symmetric case. \(J = 0.3\) corresponds approximately to the most unstable Holmboe instability (figure 3.11). Also, figure 3.12 indicates that by setting \(H = 10\) we avoid the influence of the horizontal boundaries on the development of the instabilities. We examine the cases \(\varepsilon = 0, 0.1, 0.25,\) and \(0.5\).

To determine the appropriate number of grid points we compute the evolution of the perturbed flow at several grid levels and compare the results. As discussed in section 4.1.1, the time steps selected satisfy \((4.24)\). Perturbation kinetic energies are shown
in figures 4.3, 4.4, and 4.5 for the right mode, left mode, and both mode cases, respectively. Since the numerical scheme used is dispersive, it is not surprising that, for $\varepsilon = 0$ and 0.1, there is a phase shift between the results computed on the $nx=nz=64$ grid, where $nx$ and $nz$ are the number of grid points in the $x$- and $z$-directions, respectively, and those from the $nx=nz=128$ grid. Comparing density contours (not shown), we found that, although the phase speeds differed between the two grid levels, the flows looked qualitatively the same. Since there is little phase difference between the 128 by 128 and 256 by 256 grids used for $\varepsilon = 0.25$ and 0.5, we believe that we can compute on a 128 by 128 grid with confidence up to $t = 400$ for $\varepsilon = 0$ and 0.1.

When $\varepsilon = 0.25$ and 0.5 we encountered difficulties in numerically resolving the flow. This appeared to be associated with the extreme gradients in the density field and is demonstrated in figure 4.6. The perturbation in the density gets very steep. Denser fluid from the lower layer separates from the wave and is released into the less dense upper layer. For $\varepsilon = 0.5$, this results in a decrease in the perturbation kinetic energy (see, for example, figure 4.5 for $\varepsilon = 0.5$) and occurs at different times for the $nx=nz=128$ and $nx=nz=256$ cases. Although this type of behaviour has been observed in experimental studies, we were not able to resolve the exact time or manner in which it occurs. It was observed that this ejection of denser fluid into the upper layer is most likely to occur when $\varepsilon$ is large. This is a result of the rapid growth rate of the right mode and explains why we have resolution problems when $\varepsilon = 0.5$.

When $\varepsilon = 0.25$, ejection of denser fluid into the upper layer only occurs on the coarser grid when the initial perturbations only contain one mode. We also notice that there is no sudden drop in the perturbation kinetic energy. Since the density contours remain qualitatively the same, except during the brief period of fluid breaking off from the perturbation, and the perturbation kinetic energy compares favourably between the two grid levels, we believe that we can compute with confidence on the finer grid up
to $t = 400$, when only one mode is initially present. When the flow is perturbed with both modes, however, fluid breaks off from the perturbation at both grid levels, but at different times. Although the perturbation kinetic energies do not differ significantly, the density contours quickly lose their similarities. Therefore we only have confidence in our results up until $t = 170$.

A summary of the parameters used is given in table 4.1.

<table>
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<th>$\varepsilon$</th>
<th>$\bar{\alpha}$</th>
<th>nx</th>
<th>nz</th>
<th>$\Delta t$</th>
<th>$t_{\text{max}}$</th>
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</thead>
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<td>----</td>
<td>----------</td>
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<td>128</td>
<td>0.0625</td>
<td>400</td>
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<td>128</td>
<td>0.0625</td>
<td>400</td>
</tr>
<tr>
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<td>0.52</td>
<td>256</td>
<td>256</td>
<td>0.03125</td>
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</tr>
<tr>
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<td>0.67</td>
<td>256</td>
<td>256</td>
<td>0.03125</td>
<td>140</td>
</tr>
</tbody>
</table>

Table 4.1: Parameters used for the numerical simulations when $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, and $H = 10$. The time step size is $\Delta t$. $t_{\text{max}}$ is the largest time for a given simulation for which the results are reliable.

### 4.2.3 Right Mode

When the flow is initialized with the right mode only, we expect to see the evolution of a perturbation protruding into the upper layer and moving to the right. This is indeed the case as seen by the density contours shown in figures 4.7, 4.8, and 4.9 for $230 \leq t \leq 245$ ($\varepsilon = 0$), $190 \leq t \leq 205$ ($\varepsilon = 0.25$), and $125 \leq t \leq 140$ ($\varepsilon = 0.5$), respectively. This is further supported by the lack of oscillations in the initial evolution of the wave kinetic energy (figure 4.3). When $\varepsilon = 0$, however, oscillations begin to develop at $t \approx 200$. This indicates the presence of another mode. Figure 4.7, $385 \leq t \leq 400$, shows that it is the left moving wave that is developing. This is not surprising as left and right modes have the same linear growth rate for this case and one expects the left wave to eventually obtain the same amplitude as the right wave. Another interesting observation is that,
even though right and left waves have the same linear growth rates, once the left mode starts to grow it is in the nonlinear regime and grows much more slowly than if it had started its growth in the linear regime. As $\varepsilon$ increases, the right mode dominates for a longer amount of time before the development of the left moving wave. This is due to the reduced growth rate of the left mode as well as the increased growth rate of the right mode.

An interesting phenomenon is the tilting of the wave in the opposite direction to which it is travelling (figures 4.7 and 4.8). This tilting is caused by a concentration of counter-clockwise vorticity that develops at the wave crest and has been observed in laboratory experiments by Lawrence [37]. The movement of the wave to the right is closely linked to the concentration of clockwise vorticity that proceeds the perturbation in the density interface and also moves to the right. Ripples in the vorticity to the left of the wave are a resolution problem caused by the dispersive nature of the numerical scheme and become less noticeable as the grid is refined.

We calculate the effective linear growth rate of the perturbations. Since only the right mode is present in the initial development of the perturbations, no oscillations are observed in the perturbation kinetic energy. Thus we can compute the growth rate directly without filtering. In figure 4.10 these are compared to the growth rates predicted by linear stability analysis. When $\varepsilon = 0$ we have excellent agreement between linear theory and the simulations. As $\varepsilon$ increases, the growth rate of the simulation is less than that of linear theory. The maximum amplitude achieved appears independent of the value of $\varepsilon$ (figure 4.3). Consequently, the perturbations spend less time in the linear regime as $\varepsilon$ increases because of the increasing growth rate of the right mode.

Holmboe [23] predicted from linear theory, and Smyth et al. [53] showed numerically, that the phase speeds of the two modes in a Holmboe wave are not constant but move fastest when they pass each other and slowest when they are widely separated. When
the flow is perturbed with the right mode only, this oscillation in the phase speed is not initially observed since the left wave is not present to interact with the right wave (figure 4.11). In all cases, the perturbations at both critical levels are initially moving to the right. At later times, however, the perturbation moving at the lower critical level starts to exhibit oscillations in its phase speed. This coincides with the development of the left moving wave, as indicated by the oscillations in the perturbation kinetic energy (figure 4.3), and occurs at later times as $\varepsilon$ increases. When $\varepsilon = 0$, the perturbation at the lower critical level behaves erratically between $330 \leq t \leq 370$ after which it appears to be moving to the left. At this point, the phase speed of the right moving wave also has oscillations in its phase speed as the two waves pass through each other.

In order to compare the initial phase speed with that predicted by linear theory, we compute the average phase speed over a cycle. Differences between predicted and computed phase speeds are shown in figure 4.12. In general, the initial phase speed is slightly faster than that predicted by linear theory. Also, the phase speed increases steadily with time. It appears that the phase speed may be asymptoting to a constant value, although further investigations are required to verify this conjecture.

By studying the energy budgets given by (4.26), (4.27), and (4.28), we can gain a better understanding of the mechanism that controls the growth of the instability. We will start by discussing the trends that are present for all cases. First, we notice that the total energy decreases with time (figures 4.13 and 4.14) and that this is caused by the diffusion of the mean flow. In addition, diffusion of the mean flow causes the potential energy $P$ to increase over time. During the initial linear growth of the instabilities, the decay in mean kinetic energy, $\bar{K}$ is primarily due to diffusion. As the perturbations enter the nonlinear regime, however, we see a decrease in the mean diffusion term $D(\bar{K})$ and an increase in $C(\bar{K}, K')$. At this point, the growth of the perturbation has a more significant effect on the total energy budget.
Since the flow was perturbed with only the right mode, we initially see no oscillations in any of the quantities in the energy budget. During this period, $C(\bar{K}, K') > 0$ and $C(P, \bar{K}) < 0$ with $|C(\bar{K}, K')| > |C(P, K')|$, indicating a transfer of mean kinetic energy to perturbation kinetic energy and a smaller transfer of perturbation kinetic energy to potential energy. Thus the perturbations are undergoing a net growth as indicated by the increase in $K'$. For $\epsilon = 0$ oscillations occur at later times in the quantities $\bar{K}$, $K'$, $P$, $C(\bar{K}, K')$, $C(P, K')$, $D(\bar{K})$ and $D(K')$. As discussed above, this is due to the eventual growth of the left mode and indicates that the two waves are interacting. Notice that $C(\bar{K}, K')$ remains almost always positive and hence the perturbation is constantly extracting energy from the mean flow. Conversely, $C(P, K')$ oscillates between positive and negative values. Therefore perturbation kinetic energy is converted to potential energy for some time and then the process is reversed. The cumulative transfer $\int_0^t C(P, K') dt$ is always negative, however, and so there is a net transfer of perturbation kinetic energy to potential energy. When the left mode becomes sufficiently large, there is a drop in $D(\bar{K})$ which is reflected by a flattening of the mean kinetic energy curve. A more detailed description of the energy budget when both waves are present will be discussed for the $\epsilon = 0$ case initially perturbed with both modes as this permits easy tracking of the two waves.

As $\epsilon$ increases, the oscillations occur at later times. For $\epsilon = 0.5$ (figure 4.14), there are no rapid oscillations in the short time for which the simulation was resolved. The behaviour is very different from the $\epsilon = 0$ case. We now see oscillations of much longer periods in $C(\bar{K}, K')$ and $C(P, K')$. This is reminiscent of the energy budgets observed by Klaassen & Peltier [29] for the Kelvin-Helmholtz instability. One significant difference is that, in Klaassen & Peltier [29], $C(\bar{K}, K')$ and $C(P, K')$ oscillate between positive and negative values, whereas, in our case, $C(\bar{K}, K')$ remains positive and $C(P, K')$ is negative for $t < 130$. Since the Kelvin-Helmholtz instability is characterized by the presence of
interwinding fingers of heavier and lighter fluid, Klaassen & Peltier [29] attribute the transfer of potential energy to mean kinetic energy (via the perturbation kinetic energy) to the fingers of heavier fluid being pulled back down and the figures of lighter fluid being lifted upward. In the flow examined here, the perturbation is always displacing denser fluid upward and so there is no transfer of potential energy to mean kinetic energy. When $130 \leq t \leq 140$, $C(\bar{K}, K')$ is negative. During this time, there is a decrease in the wave kinetic energy. The wave amplitude, as seen by the density contours (figure 4.9), also decreases. The transfer quantities $C(\bar{K}, K')$ and $C(P, K')$ are in phase and thus there is a continuous transfer of energy from mean kinetic energy to perturbation kinetic energy to potential energy. When $\varepsilon = 0.1$ and 0.25 (not shown) the energy budgets show characteristics of both the $\varepsilon = 0$ and $\varepsilon = 0.5$ cases. Eventually rapid oscillations appear as the left wave grows, but there are more slowly varying oscillations superimposed upon these.

One of the main concerns in the study of the evolution of instabilities is to determine how they influence the development of the mean flow. Mean velocity and density profiles at the final time in the numerical simulation and those given by (4.31) and (4.32) for the evolution of the mean flow with no perturbations are shown in figure 4.15. Thus, we are able to analyse how the presence of the perturbations affects the development of the mean flow. When $\varepsilon = 0$, the mixing of momentum (as seen from the mean velocity profile) is primarily in the upper layer. As $\varepsilon$ increases, the amount of mixing in the upper layer does not change significantly, but it spreads into the lower layer. Mixing of momentum occurs throughout the upper layer, whereas changes in density due to mixing are concentrated near the crest of the wave with a smaller change occurring near the density interface. As $\varepsilon$ increases, the change in density due to mixing increases at both these locations.
4.2.4 Left Mode

Except for the $\varepsilon = 0$ case, the flow behaviour is markedly different when it is initialized with the left mode only than when it is initialized with the right mode only. This is a direct consequence of the right mode having the largest linear growth rate when $\varepsilon > 0$. As the difference between the growth rates of the left and right modes increases, i.e., as $\varepsilon$ increases, the right mode influences the flow much more quickly and results in the onset of oscillations in the wave kinetic energy occurring at earlier times as $\varepsilon$ increases (figure 4.4). In fact, for $\varepsilon = 0.25$ and 0.5, the right mode affects the flow while it is still in the linear regime. Furthermore, density contours when $\varepsilon = 0$ and 0.1 (not shown), show that only the left mode is present in the early stage of the nonlinear regime, whereas for $\varepsilon = 0.25$ and 0.5 the right mode, having a larger growth rate, has had sufficient time to grow and is clearly present (figures 4.16 and 4.17) even at early times in the nonlinear regime. When $\varepsilon = 0$ the flow behaves exactly the same way as the right mode only case, except that the roles of the right and left moving waves are reversed. Results for $\varepsilon > 0$ indicate that both left and right modes will eventually develop in the flow. We also see that both left and right moving waves have a concentration of counter-clockwise vorticity in their peaks once their amplitudes become sufficiently large.

When $\varepsilon > 0$ the left mode only case spends longer in the linear regime than the right mode only case due to the smaller growth rates. The linear growth rates from the nonlinear simulation are given in figure 4.10 and are only slightly larger than those predicted by linear stability analysis. As is seen from the wave kinetic energy (figure 4.4), the $\varepsilon = 0.5$ case behaves differently from the smaller values of $\varepsilon$. It only spends a very short amount of time growing at the rate of the left mode. After a short transition period, it continues to grow linearly but with a growth rate of 0.045. As this value lies midway between the growth rates for right and left modes predicted from linear analysis,
we conclude that the change in growth rate is due to the presence of the rapidly growing right mode.

As with the right mode only case, the phase speed of the left mode does not oscillate initially (figure 4.18). Also, perturbations at both critical levels move to the left. It is only when the right wave affects the flow that we observe oscillations in phase speed at the critical level of the right wave. When \( \varepsilon = 0.1 \), the motion of the perturbation at the critical level corresponding to the right moving wave exhibits interesting behaviour (figure 4.18). Initially the wave moves to the left, as expected. As the wave evolves, oscillations in the phase speed become more dramatic until a turn about occurs at \( t \approx 200 \). The wave then travels to the right. Notice that the wave in the lower layer is unaffected by the turn about and continues to move to the left. Similar behaviour is observed for \( \varepsilon = 0.25 \) and 0.5 except that the turn about occurs sooner as \( \varepsilon \) increases. Although only the right mode is noticeable in the density contours in figure 4.17 (\( \varepsilon = 0.5 \)) the fact that the perturbation at the lower critical level is still moving to the left would suggest that the left moving mode is growing and would eventually become noticeable in the flow. This is further supported by the results shown in figure 4.16 where both modes are clearly present.

Figure 4.12 indicates that the initial average phase speed of the left mode is slightly smaller than that predicted by linear theory. Also, for \( \varepsilon = 0, 0.1, \) and 0.25, the phase speed increases steadily over time. This effect becomes less pronounced as \( \varepsilon \) increases and reverses when \( \varepsilon = 0.5 \) for which the phase speed decreases with time. This supports the hypothesis that the phase speeds are asymptoting to a single value.

When \( \varepsilon = 0 \), the energy budget for the nonlinear simulation initially perturbed with the left mode only is identical to the right only case shown in figure 4.13. This result is not surprising in view of the symmetry of the flow in this case. As \( \varepsilon \) increases, oscillations begin sooner since, with increasing \( \varepsilon \), the right mode becomes important more quickly.
Because the difference in linear growth rates between right and left modes increases with increasing $\varepsilon$, the interactions between the two waves is weaker for larger values of $\varepsilon$. This is probably due to the difference in amplitudes of the two waves for the times considered and is evident in the reduced amplitude of oscillation in $C(P, K')$. In general, however, the energy budgets look similar to the $\varepsilon = 0$ case.

When $\varepsilon = 0$, momentum mixing, as indicated by the difference between the evolution of the mean flow with and without perturbations, is the same as the right only case (figure 4.15) with the amounts of mixing in upper and lower layers reversed. As $\varepsilon$ increases, there is a decrease in the amount of mixing in the lower layer and an increase in the amount of mixing in the upper layer. To explain this phenomenon, we examine how the growth rates of right and left modes change with increasing $\varepsilon$. Since the growth rate of the left mode decreases as $\varepsilon$ increases, it grows less as $\varepsilon$ increases which results in less mixing in the lower layer. A similar argument holds in the upper layer which is influenced by the right moving wave. The right wave causes more mixing with increasing $\varepsilon$ since it is the dominant mode when $\varepsilon > 0$. The change in density due to mixing exhibits the same trends observed for the momentum mixing, although, as with the right mode only case, the magnitude of the change in density is much smaller than that of the change in the mean velocity field.

4.2.5 Both Modes

For the symmetric case ($\varepsilon = 0$) when the flow is initialized with both modes, growth of the wave kinetic energy (figure 4.5) is similar to those of the left and right mode only cases except that oscillations are present at all times. These oscillations are expected since both modes are present from the start and grow at the same rate. The flow looks similar to that computed by Smyth et al. [53] with a right moving wave protruding into the upper layer and a left moving wave protruding into the lower layer. One interesting
difference is that, as seen in figure 4.19, our simulations have not lost the symmetry at later times whereas those of Smyth et al. [53] did. We expect the flow to remain symmetric even at large times since the background flow is initially symmetric, as are the horizontal boundary conditions. Results when the flow is initially perturbed with only one mode further supports the results observed here. It was seen that, when $\varepsilon = 0$, the flow tends towards a state where both modes are present.

As $\varepsilon$ increases, oscillations in the perturbation kinetic energy decrease (figure 4.5), especially in the linear regime. This observation indicates that, although the perturbations are initialized with the same amount of kinetic energy in both left and right modes, the right mode quickly dominates the flow for large values of $\varepsilon$. Furthermore, the effective linear growth rate lies in between the values predicted by linear theory for the right and left modes, but is dominated by that of the right mode (figure 4.10). When $\varepsilon = 0$ we have excellent agreement between the predicted linear growth rate and that of the linear simulation.

Even for relatively small values of $\varepsilon$, i.e. $\varepsilon = 0.1$, linear theory predicts a difference in the growth rates for left and right modes (figure 4.1). This has a significant effect on the development of the perturbations. The right mode grows more quickly and initially has a larger amplitude (figure 4.20, $195 \leq t \leq 210$). The left mode, however, continues to grow and is almost as large as the right mode at the end of the simulation. The numerical simulation is starting to have resolution problems at later times. In order to compute for longer times, a finer grid should be used.

When $\varepsilon = 0.25$, there is a rapid growth of the right mode which dominates the flow in the early development of the perturbations (figure 4.21, $100 \leq t \leq 115$). Since the flow is initialized with both modes, the left moving wave is still present but, due to its smaller growth rate, takes longer to develop. At $t = 170$ (figure 4.21), we observe both right and left modes, although the right mode still has the larger amplitude. When $\varepsilon = 0.5$, the
flow initially evolves in a similar manner (figure 4.22) with only the right mode present. In fact, for the time in which we have confidence in our results, the flow looks similar to the right mode only case. This is probably a direct result of the reduced growth rate of the left mode at larger values of $\varepsilon$. Although the left mode is still present, it has not yet had a chance to grow to an observable level for the results presented here.

We discuss the behaviour of the flow for $t > 140$ when $\varepsilon = 0.5$. This is beyond the range of time for which the flow was adequately resolved, but is still instructive from a qualitative point of view. The right mode continues to grow until $t \approx 185$. At this time, the protrusion into the upper layer separates from the density interface and diffuses into the ambient fluid (figure 4.6), causing a drop in the perturbation kinetic energy (figure 4.5). After this sudden decrease in magnitude of the perturbation, the growth of the right mode appears to be stunted. The left mode, however, continues its slow growth and starts to dominate the flow (figure 4.22). Although we were not able to resolve this behaviour numerically, it occurred in several of the simulations. We believe that the numerical simulation helps us understand the overall mechanism of the instability for large values of $\varepsilon$ even though the precise details were not resolved. Furthermore, Koop et al. [34] have observed a similar phenomenon in laboratory experiments. Their flow is initially unstable to Kelvin-Helmholtz type instabilities. The billows formed mix quickly into the ambient fluid. Further downstream, weaker instabilities of a Holmboe nature appear at the interface. This leads us to believe that the phenomena observed in our simulations are physically possible.

The positions of the perturbations over time in both top and bottom layers are shown in figure 4.23. For $\varepsilon = 0$ our results are in agreement with those observed by Smyth et al. [53] and predicted by Holmboe [23] for the linear regime. During this stage, the waves speed up as they pass through each other, and slow down when they are far apart. As the perturbations enter the nonlinear regime, however, there is a shift in the behaviour of
the phase speed. The phase speed still fluctuates during a cycle, but now the waves move fastest as they approach each other, slowing down as they pass through each other. The method for tracking the phase speed breaks down as the waves start to interact in the nonlinear regime. Just after the waves pass through each other, there is a jump in the position of maximum/minimum density (figure 4.23). This jump occurs when density contours within the wave have more than one local extremum (figure 4.19, $t = 245$), and indicates an inaccurate measurement of the instantaneous phase speed. The average phase speed throughout the cycle appears to increase with time (figure 4.12). This observation is consistent with the results of the right mode only and left mode only cases, but the interaction between the two waves slows down their average phase speed. In light of the above discussion, further investigations are required to better resolve this behaviour.

When $\varepsilon = 0.1$, the flow initially behaves the same as the $\varepsilon = 0$ case. It has a right moving perturbation in the upper layer and a left moving perturbation in the lower layer. It also exhibits oscillations in the phase speed as the two waves interact. For $40 \leq t \leq 100$, the right moving wave appears to be unaffected by the presence of the left moving wave and does not have any oscillations in its phase speed throughout the cycle. On the other hand, the left moving wave has large changes in its phase speed throughout the cycle. This is most likely a result of the difference in amplitude of the two modes at this stage of their development. The right mode has had sufficient time to grow and is clearly present in the density contours, whereas the left mode has not yet grown enough to be seen (not shown). After $t = 100$ the two waves behave as in the nonlinear regime for $\varepsilon = 0$, speeding up as they approach each other. For $t > 330$, the two waves appear to be moving in a non-smooth way. This behaviour is due to the resolution problem already discussed above. As for the $\varepsilon = 0$ case, the average phase speed of the right moving wave increases over time (figure 4.12) and is initially the same as for the right mode only case.
As the left wave grows, however, its presence starts to affect the phase speed of the right wave which is slower than the right mode only case. The phase speed of the left wave initially increases but then decreases as the influence of the right wave strengthens.

For larger values of $\varepsilon$ (i.e., $\varepsilon = 0.25$ and $0.5$), perturbations in the upper and lower layers exhibit the same characteristics in the initial linear regime as the two previous cases. Oscillations in the phase speeds as the waves pass through each other are only present for a very short time, however, after which the right moving wave behaves very much like the case of the right wave only, with no oscillations in the phase speed. The perturbation at the left critical layer looses its distinctive motion to the left. It bounces between moving to the left and moving to the right in no particular fashion, indicating a complicated interaction between the slowly growing left moving wave and the rapidly growing right moving wave. For $t > 140$ in the $\varepsilon = 0.25$ case, the left moving wave has grown large enough for the two waves to interact in the nonlinear regime as described above. The reason it takes longer to reach this state than the $\varepsilon = 0.1$ case is the left mode has a much slower growth rate and hence takes longer to achieve a large enough amplitude to affect the motion of the right mode. For $\varepsilon = 0.5$, the simulation does not last long enough for the left mode to grow sufficiently large to affect the evolution of the right mode, but the above results suggest that the flow would eventually exhibit the same behaviour as the other cases with smaller values of $\varepsilon$.

For both $\varepsilon = 0.25$ and $0.5$, the average phase speed of the right mode increases steadily over time (figure 4.12) and appears unaffected by the left moving wave. The phase speed of the left mode, when $\varepsilon = 0.25$ appears to be slowed down by the presence of the right wave when compared to the left only results. The simulation for $\varepsilon = 0.5$ is too short to draw any conclusions about how the phase speed of the left mode varies with time.

The energy budget for $\varepsilon = 0$ is shown in figure 4.24 and was discussed in detail by Smyth et al. [53]. We will review the results here and discuss how they tie into the
previous cases. As before, we see that the total energy $T$ decreases with time due to diffusion. Although there are rapid oscillations in $\bar{K}$, $K'$, and $P$, there are no oscillations in the total energy indicating a balance in the exchange of energy between the reservoirs. The long term behaviour of $\bar{K}$, $K'$, and $P$ are as in the right mode and left mode only cases. Since $\int_0^t C(\bar{K}, K')dt > 0$ for any value of $t$, there is a net transfer of energy from the mean flow to the perturbations. The greatest transfer of energy, however, is between the potential energy and the perturbation kinetic energy. Although $C(P, K')$ oscillates between positive and negative quantities, we have that $\int_0^t C(P, K')dt < 0$ at any instant. Thus there is a net conversion of perturbation kinetic energy to potential energy. Nevertheless, $\int_0^t C(\bar{K}, K')dt > -\int_0^t C(P, K')dt$ indicating that a portion of the energy transferred to the perturbation kinetic energy from the mean flow remains in the perturbation reservoir. Diffusion of the mean kinetic energy decreases as $\bar{K}$ decreases. Also, since both modes are present at all times, there is no sudden drop in $D(\bar{K})$, as when the flow is initially perturbed with one mode only, but a steady decrease over time. Initially $D(\bar{K}) \gg C(\bar{K}, K')$ but as the flow enters the nonlinear regime, the two quantities have the same order of magnitude and both contribute to the overall decrease in $\bar{K}$. Diffusion of the perturbation kinetic energy increases as $K'$ increases whereas $D(P)$ remains constant throughout the simulation.

From figure 4.23 we can determine when the right and left wave pass through each other and when they are maximally separated (for example, figure 4.19, at $t \approx 235$ and $t \approx 240$, respectively). Studying the energy budgets we can determine how the interactions of the two waves affect the growth of the perturbations. We will examine this interaction during the linear stage and in the fully nonlinear regime (see figures 4.25 and 4.26, respectively, for enlargements of positions and energy budgets during these times). While the waves are growing linearly, the growth and decay of the perturbation kinetic energy is controlled by the transfer of mean kinetic energy to the perturbation kinetic
energy, and vice versa. Extraction of energy from $\bar{K}$ is optimal just after the waves have passed each other. This corresponds to a maximum in the perturbation kinetic energy. As the waves approach each other, there is a brief period when $C(\bar{K}, K') < 0$, corresponding to a minimum in $K'$. In general, the transfer of energy is from mean kinetic energy to perturbation kinetic energy and then from perturbation kinetic energy to potential energy (or in the reverse direction). There is, however, a slight phase shift between $C(\bar{K}, K')$ and $C(P, K')$. This was observed when studying the evolution of the eigenfunctions in chapter 3.

Once the waves enter the nonlinear regime (figure 4.26), transfer of energy from $\bar{K}$ to $K'$ has two maxima. One occurs just before the waves pass through each other and the other just after. During the earlier stages of the nonlinear regime the most efficient extraction of energy from the mean flow occurs just after the waves have passed through each other. At later times, however, it occurs as the two waves are approaching each other. When the waves are maximally separated, there is a brief period when $C(\bar{K}, K') < 0$ and thus the perturbations are giving up energy to the mean flow. As left and right moving waves approach each other, $C(P, K') > 0$. This corresponds to a growth period for $K'$ and a decay period for $P$. During the part of the cycle when the waves are moving away from each other, there is a reversal in the behaviour of $C(P, K')$ corresponding to a decrease in the perturbation kinetic energy and an increase in the potential energy. In contrast to the linear regime where oscillations in $K'$ are governed by $C(\bar{K}, K')$, in the nonlinear regime they are linked to $C(P, K')$. Also, in the linear regime, maxima in $K'$ occur just after the waves have passed through each other, minima occur as the waves are approaching each other. In the nonlinear regime, however, maxima in $K'$ occur as the waves pass through each other whereas minima occur when the waves are maximally separated.

When $\varepsilon = 0.1$ and $0.25$ (not shown), the energy budgets look similar to those of the
\( \varepsilon = 0 \) case. Therefore, the two waves need not have equal amplitudes for the nonlinear interaction to exhibit the same behaviour described above. If \( \varepsilon \) is sufficiently large, however, the energy budget can look very different even when the flow is initially perturbed with both right and left modes, as seen by examining the \( \varepsilon = 0.5 \) case. The energy budget when \( \varepsilon = 0.5 \) (figure 4.27) looks similar to that of the right mode only (figure 4.14). This similarity is not surprising as the density contours (figures 4.9 and 4.22) are virtually identical for the two cases. One difference between the two energy budgets is that when the flow is initialized with both modes, there are small oscillations in the transfers between the reservoirs, \( C(\bar{K}, K') \) and \( C(P, K') \). These oscillations indicate that the left mode, however small it may be, is interacting with the right mode. Initially, these interactions are very weak and we see a continuous transfer of energy from \( \bar{K} \) to \( K' \) with a smaller transfer from \( K' \) to \( P \). With increasing time, \( C(P, K') \) oscillates between positive and negative values. This behaviour in the energy budgets looks similar to that of the right mode only when \( \varepsilon = 0.1 \) and 0.25, suggesting that, given enough time, the left mode will grow and the energy budget will behave as those described above when the amplitudes of the two waves are similar.

Mean velocity and density profiles are shown in figure 4.28. When \( \varepsilon = 0 \), the changes due to mixing in the mean velocity and density profiles are the same in both upper and lower layers. This is as expected since the perturbations remain symmetric even near the end of the simulation (figure 4.19). As observed earlier, momentum mixing tends to occur throughout the entire domain whereas changes in density due to mixing are concentrated near the waves’ crests. When the perturbations initially contained the right or left mode only, there was a small change of density due to mixing near the center of the density interface. Changes were in opposite directions for right and left modes. When both modes are present, there is no change in density due to mixing at the density interface. Hence, any changes in mean density at the interface are due to diffusion and changes
in density due to mixing for left and right modes cancel at this level. As $\varepsilon$ increases, the amount of momentum mixing increases in the upper layer and decreases in the lower layer. This result is consistent with results of the right and left mode only cases. For the profiles shown it is difficult to determine the effect of varying $\varepsilon$ on the density changes due to mixing. Based on previous results, we expect the same type of behaviour as for the momentum. It also appears that as $\varepsilon$ increases, there is a larger change in density due to mixing at the interface. When $\varepsilon$ is larger, the left mode is not sufficiently strong to produce an opposing change in density at the interface and thus there is no cancellation of the change in density due to the right mode at this level.
Growth rates from linear stability analysis for $J = 0.3$ with $Re = 300$, $Pr = 9$, $R = 3$, and $H = 10$. The solid line corresponds to the right moving wave and the dotted line to the left moving wave. When $\varepsilon = 0$ the right and left growth rates coincide. The vertical dashed line indicates the value of $\tilde{\alpha}$ = 0.54, 0.53, 0.52, and 0.67 for $\varepsilon = 0$, 0.1, 0.25, and 0.5, respectively, the wave number used in the simulations.
Figure 4.2: Absolute stability region of fourth order Runge-Kutta scheme

Figure 4.3: Perturbation kinetic energy for right mode

Perturbation kinetic energy when flow is initialized with the right mode only with varying grid size for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, and $H = 10$. 
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Figure 4.1: Perturbation kinetic energy for left mode

Perturbation kinetic energy when flow is initialized with the left mode only with varying grid size for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, and $H = 10$. 
Figure 4.5: Perturbation kinetic energy for both modes

Perturbation kinetic energy when flow is initialized with both modes with varying grid size for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, and $H = 10$. 
Figure 4.6: Effect of grid size when $\varepsilon = 0.5$

Density contours for both modes for $\varepsilon = 0.5$ with $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, and $H = 10$. $x$ has been scaled with respect to its period $2\pi/\alpha$. Top and bottom rows are results for $nx=nz=128$ and $nx=nz=256$, respectively. Breaking off of fluid parcel was not resolved numerically.
Figure 4.7: Density and vorticity for right mode only with $\varepsilon = 0$

Results of nonlinear simulations when the flow is initialized with the right mode only for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, $H = 10$, and $\varepsilon = 0$. Contours are of constant density $\rho = \rho_\alpha(z) + \rho'$ in the $x$-$z$ plane and shading is of vorticity. $x$ has been scaled with respect to its period $2\pi/\tilde{\alpha}$. 
Figure 4.8: Density and vorticity for right mode only with $\varepsilon = 0.25$

Results of nonlinear simulations when the flow is initialized with the right mode only for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, $H = 10$, and $\varepsilon = 0.25$. Contours are of constant density $\rho = \rho_a(z) + \rho'$ in the $x$-$z$ plane and shading is of vorticity. $x$ has been scaled with respect to its period $2\pi/\tilde{a}$. 
Figure 4.9: Density and vorticity for right mode only with $\varepsilon = 0.5$

Results of nonlinear simulations when the flow is initialized with the right mode only for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, $H = 10$, and $\varepsilon = 0.5$. Contours are of constant density $\rho = \rho_0(z) + \rho'$ in the $x$-$z$ plane and shading is of vorticity. $x$ has been scaled with respect to its period $2\pi/\tilde{\alpha}$. 
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Figure 4.10: Linear growth rates from nonlinear simulations

Linear growth rates from: 1. linear stability analysis (△), 2. nonlinear simulations with left or right mode only (+), 3. nonlinear simulations with both modes (◇) when $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, and $H = 10$. Upper values correspond to the right mode and lower values to the left mode.
Figure 4.11: Positions of waves for right mode only

Horizontal position of maximum density at critical level for right wave (solid line) and minimum density at critical level for left wave (dashed line) when the flow is initially perturbed with the right mode only for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, and $H = 10$. The horizontal position has been scaled with respect to its period $2\pi/\alpha^*$. 
Figure 4.12: Average phase speeds from nonlinear simulations

Average phase speed over a cycle for: 1. nonlinear simulations with right or left mode only (+), 2. nonlinear simulations with both modes (○) when \( J = 0.3, \) \( R = 3, \) \( Pr = 9, \) \( Re = 300, \) and \( H = 10. \) Phase speed predicted by linear theory is indicated by a dashed line. Positive phase speeds correspond to right moving waves and negative wave speeds to left moving waves.
Figure 4.13: Energy budgets for right mode only with $\varepsilon = 0$

Energy budget for nonlinear simulation initially perturbed with the right mode only for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, and $H = 10$ with $\varepsilon = 0$. Reservoirs: $T + 6.1$ (solid line), $\bar{K} - 2.1$ (dashed line), $K'$ (dotted line), and $P + 9.1$ (dash-dot-dash line). Transfers: $C(\bar{K}, K')$ (solid line), and $C(P, K')$ (dotted line). Diffusions: $D(\bar{K})$ (dashed line), $D(K')$ (dotted line), and $D(P)$ (dash-dot-dash line).
Figure 4.14: Energy budgets for right mode only with $\varepsilon = 0.5$

Energy budget for nonlinear simulation initially perturbed with the right mode only for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, and $H = 10$ with $\varepsilon = 0.5$. See figure 4.13 for description.
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Figure 4.15: Evolution of mean profiles for right mode only

Mean velocity and density profiles at 1. $t = 0$ (dashed line), 2. from numerical simulations initialized with right mode only (solid line), and 3. diffusion of mean flow without perturbations (dotted line). Parameter values are $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, and $H = 10$. 
Figure 4.16: Density and vorticity for left mode only with $\varepsilon = 0.25$

Results of nonlinear simulations when the flow is initialized with the left mode only for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, $H = 10$, and $\varepsilon = 0.25$. Contours are of constant density $\rho = \rho_0(z) + \rho'$ in the $x$-$z$ plane and shading is of vorticity. $x$ has been scaled with respect to its period $2\pi/\tilde{a}$. 
Figure 4.17: Density and vorticity for left mode only with $\varepsilon = 0.5$

Results of nonlinear simulations when the flow is initialized with the left mode only for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, $H = 10$, and $\varepsilon = 0.5$. Contours are of constant density $\rho = \rho_0(z) + \rho'$ in the $x$-$z$ plane and shading is of vorticity. $x$ has been scaled with respect to its period $2\pi/\bar{\alpha}$. 
Figure 4.18: Positions of waves for left mode only

Horizontal position of maximum density at critical level for right wave (solid line) and minimum density at critical level for left wave (dashed line) when the flow is initially perturbed with the left mode only for \( J = 0.3, \; R = 3, \; Pr = 9, \; Re = 300, \) and \( H = 10. \) The horizontal position has been scaled with respect to its period \( 2\pi/\alpha^* \).
Figure 4.19: Density and vorticity for both modes with $\varepsilon = 0$

Results of nonlinear simulations when the flow is initialized with both modes for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, $H = 10$, and $\varepsilon = 0$. Contours are of constant density $\rho = \rho_0(z) + \rho'$ in the $x$-$z$ plane and shading is of vorticity. $x$ has been scaled with respect to its period $2\pi/\alpha$. 
Figure 4.20: Density and vorticity for both modes with $\varepsilon = 0.1$

Results of nonlinear simulations when the flow is initialized with both modes for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, $H = 10$, and $\varepsilon = 0.1$. Contours are of constant density $\rho = \rho_a(z) + \rho'$ in the $x$-$z$ plane and shading is of vorticity. $x$ has been scaled with respect to its period $2\pi/\tilde{\alpha}$. 
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Figure 4.21: Density and vorticity for both modes with $\varepsilon = 0.25$

Results of nonlinear simulations when the flow is initialized with both modes for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, $H = 10$, and $\varepsilon = 0.25$. Contours are of constant density $\rho = \rho_{a}(z) + \rho'$ in the $x$-$z$ plane and shading is of vorticity. $x$ has been scaled with respect to its period $2\pi/\tilde{a}$. 
Figure 4.22: Density and vorticity for both modes with $\varepsilon = 0.5$

Results of nonlinear simulations when the flow is initialized with both modes for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, $H = 10$, and $\varepsilon = 0.5$. Contours are of constant density $\rho = \rho_a(z) + \rho'$ in the $x$-$z$ plane and shading is of vorticity. $x$ has been scaled with respect to its period $2\pi/\tilde{\alpha}$. 

Figure 4.23: Positions of waves for both modes

Horizontal position of maximum density at critical level for right wave (solid line) and minimum density at critical level for left wave (dashed line) when the flow is initially perturbed with both modes for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, and $H = 10$. The horizontal position has been scaled with respect to its period $2\pi/\alpha^*$. 

\[ \text{Top Layer } y = 0.703125 \quad \text{Bottom Layer } y = -0.703125, \varepsilon = 0 \]

\[ \text{Top Layer } y = 0.625000 \quad \text{Bottom Layer } y = 0.859375, \varepsilon = 0.1 \]

\[ \text{Top Layer } y = 0.468750 \quad \text{Bottom Layer } y = -1.09375, \varepsilon = 0.25 \]

\[ \text{Top Layer } y = 0.312500 \quad \text{Bottom Layer } y = -1.75781, \varepsilon = 0.5 \]
Figure 4.24: Energy budgets for both modes with $\varepsilon = 0$

Energy budget for nonlinear simulation initially perturbed with both modes for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, and $H = 10$ with $\varepsilon = 0$. See figure 4.13 for description.
Figure 4.25: Wave positions and energy budgets in linear regime for \( \varepsilon = 0 \)

Wave positions and energy budgets in linear regime for \( \varepsilon = 0 \) as described in figure 4.24. Times of maximum wave separation and wave intersection are indicated by vertical lines.
Figure 4.26: Wave positions and energy budgets in nonlinear regime for $\varepsilon = 0$

Wave positions and energy budgets in nonlinear regime for $\varepsilon = 0$ as described in figure 4.24. Times of maximum wave separation and wave intersection are indicated by vertical lines.
Figure 4.27: Energy budgets for both modes with $\varepsilon = 0.5$

Energy budget for nonlinear simulation initially perturbed with both modes for $J = 0.3$, $R = 3$, $Pr = 9$, $Re = 300$, and $H = 10$ with $\varepsilon = 0.5$. See figure 4.13 for description.
Figure 4.28: Evolution of mean density profiles for both modes

Mean velocity and density profiles at 1. \( t = 0 \) (dashed line), 2. from numerical simulations initialized with both modes (solid line), and 3. diffusion of mean flow without perturbations (dotted line). Parameter values are \( J = 0.3 \), \( R = 3 \), \( Pr = 9 \), \( Re = 300 \), and \( H = 10 \).
Chapter 5

Comparison with Laboratory Experiments

Although numerical simulations provide a vast amount of quantitative information about a flow, we are using it to study an isolated phenomenon. In nature, such a phenomenon is unlikely to occur on its own. Also, due to numerical restrictions, it is often difficult, if not impossible, to reproduce the high Reynolds number characteristic of natural systems. These issues lead to the question of whether or not the results of the numerical simulations are indeed physically realistic.

There have been field observations of Kelvin-Helmholtz billows, for example, in cloud formations, Colson [11], and in the Mediterranean Sea, Woods [69]. There is, however, a paucity of quantitative data on Holmboe instabilities in the field and in the laboratory. Also, it is difficult to make field observations of Holmboe instabilities due to the non-stationary nature of the instabilities. Thus, although we would like to make detailed quantitative comparisons between the numerical results and both laboratory and field observations, we restrict our attention to qualitative comparisons with laboratory observations.

In order to compare our results with those of laboratory experiments, we draw on three sources: the results of Lawrence et al. [38], Zhu [72], and Guez & Lawrence [20]. First, we examine the results of Zhu who provides one of the best sequences of photographs of a nearly symmetric Holmboe instability. This instability occurs at a relatively large value of the bulk Richardson number. Next, we compute the evolution of two flows with weak stratification and large density interface displacement and compare these with the results
of Lawrence et al. [38] and Guez & Lawrence [20]. In all three cases, the stratification is a result of having a layer of fresh water over a layer of salt water. Full descriptions of the experimental setups are given in the individual references.

5.1 Symmetric Case

The results from Zhu [72] are shown in figure 5.1. It is evident that there is a wave protruding into the upper layer which is moving to the right and one into the lower layer moving to the left. Since the mean flow is moving to the right, it appears that the right wave is moving faster than the left wave. Flow conditions for this experiment are given in table 5.1. From the data, we calculate the relative speeds of right and left moving waves with respect to the mean flow. The relative phase speeds are 0.6 cm/s and -0.6 cm/s for right and left moving waves, respectively. The fact that the two waves are moving in opposite directions at the same speed relative to the mean flow is indicative that the density interface is at the same level as the center of the shear layer (i.e., \( \varepsilon = 0 \)). Since the measured waves speeds are approximate, these wave speeds do not contradict the measured range of \( \varepsilon \) as it is still possible that the density interface displacement is small and negative. For the purpose of comparison, however, we set \( \varepsilon = 0 \) in the numerical simulation.

Due to the large values of both the Reynolds and Prandtl numbers (table 5.1), we cannot match the flow conditions in the numerical simulations since doing so would result in insufficient diffusion to adequately resolve the flow. Also, selecting \( 10 \leq R \leq 15 \) with \( \text{Re} = 460 \) requires a large number of grid points in the \( z \)-direction. For the purpose of comparison, we use the same parameters used in chapter 4 to examine the evolution of the symmetric case. The values of these parameters are \( J = 0.3, \text{Re} = 300, \text{Pr} = 9, R = 3, \) and \( \tilde{a} = 0.54 \). Parameters relating to the numerical scheme are given in table 4.1.
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<table>
<thead>
<tr>
<th>J</th>
<th>Re</th>
<th>Pr</th>
<th>$g'$</th>
<th>$U_1$</th>
<th>$U_2$</th>
<th>$h$</th>
<th>$\Lambda$</th>
<th>$\varepsilon$</th>
<th>$R$</th>
<th>$c_r^{(r)}$</th>
<th>$c_r^{(l)}$</th>
</tr>
</thead>
<tbody>
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<td>460</td>
<td>700</td>
<td>1.6</td>
<td>2.5</td>
<td>-1.7</td>
<td>5</td>
<td>14</td>
<td>[-0.1,0]</td>
<td>[10,15]</td>
<td>1.0</td>
<td>-0.2</td>
</tr>
<tr>
<td>0.06</td>
<td>25</td>
<td>700</td>
<td>0.4</td>
<td>3.7</td>
<td>1.7</td>
<td>0.58</td>
<td>3.9</td>
<td>0.5</td>
<td>10</td>
<td>0</td>
<td>*</td>
</tr>
<tr>
<td>0.03</td>
<td>25</td>
<td>700</td>
<td>0.13</td>
<td>3.1</td>
<td>1.5</td>
<td>0.6</td>
<td>4.1</td>
<td>-</td>
<td>10</td>
<td>0</td>
<td>*</td>
</tr>
</tbody>
</table>

Table 5.1: Approximate flow conditions of laboratory experiments from Zhu [72] (J = 0.45), Lawrence et al. [38] (J = 0.06), and Guez & Lawrence [20] (J = 0.03). $\Lambda$ is the observed wavelength of the instability, $g'$ is the modified gravitational acceleration. When J = 0.03 and 0.06 there are no observable left moving waves (indicated by * in the table).

The lower value of the initial bulk Richardson number used in the numerical simulations is not an unreasonable choice since J increases as the flow evolves. Also, there is some uncertainty in determining the value of J in the laboratory experiments.

In order to qualitatively compare our results with those of Zhu [72], we must output our results at the same frequency as the photographs were taken. During the 0.5 second between photographs, a wave has travelled $(c_r - \bar{U}) \cdot 0.5 / \Lambda = 0.02$ of its total wavelength. In non-dimensional time, we must output our results at intervals of $t^*_v = 0.02 \cdot \Lambda^*/c_r^* = 0.02 \cdot 2\pi/c_r^* \cdot \bar{a} = 0.389$. Here * indicates the non-dimensional quantities used in the numerical simulation.

Results of the numerical simulation for J = 0.3 with $\varepsilon = 0$ are shown in figure 5.2. As in the experimental observations of Zhu [72], there is a wave moving to the right which protrudes into the upper layer and one moving to the left protruding into the lower layer. There are, however, several qualitative differences between the two results. The wave peaks are sharper in the laboratory results of Zhu [72] (reproduced in figure 5.1) than in the results of the numerical simulations. A possible explanation for this difference is that the density interface is much thinner for the experimental flow than in the numerical
simulation. Also, it is difficult to determine at what stage of the perturbation evolution the flow shown in the photographs occurs. As seen in figure 4.19, the shape of the waves changes throughout the evolution of the perturbation.

Another observable difference is that in the experiments, the amplitude of the left mode appears larger than the amplitude of the right mode, whereas in the numerical simulations they have the same amplitude. There are two possible explanations for this difference. First, as shown in figure 4.20, the two waves can have significantly different amplitudes even when the density interface displacement is small. Thus it may be possible that the density interface shift is a small negative number. This possibility is supported by the range of $\varepsilon$ given in table 5.1 for the results of Zhu [72]. Second, in chapter 4 it was discovered that the relative amplitudes of right and left modes during early stages of their nonlinear evolution depends on the relative initial amplitude of the two modes. Since the Holmboe instability observed by Zhu [72] occurred in a facility designed to study an exchange flow through a channel with an underwater sill, the Holmboe instability is just one of several mechanisms that are taking place (see figure 7.1a of Zhu [72]). It is possible that one of the other mechanisms has an influence on the amplitude of the two modes. In spite of these differences, we feel that the comparison between the numerical and experimental results are favourable.

5.2 Non-Symmetric Case

Experimental results from Guez & Lawrence [20] and Lawrence et al. [38] are shown in figures 5.3 and 5.4 respectively. Approximate flow conditions are given in table 5.1. These laboratory observations are clear examples of flows exhibiting “one-sidedness”. In both cases, perturbations take the form of fluid being dragged from the lower layer into the upper layer. This behaviour is characteristic of the non-symmetric case with $\varepsilon > 0$. 

Since, for both cases, the bulk Richardson number is small, billowing is observed and increases as J decreases.

Although this set of experiments occurs at a lower Reynolds number, the Prandtl number and the ratio \( R \) are still too large to numerically model. The lower value of the Reynolds number does, however, allow us to easily compute the evolution of the flow for several values of \( R \) (and hence several values of Pr) to see if any trends are present. For this purpose, we use the flow parameters from the experimental results of Guez & Lawrence [20]. Since the value of \( \varepsilon \) is not given for this experiment, we set \( \varepsilon = 0.5 \). This value is given by Lawrence et al. [38]. Since experiments of Guez & Lawrence [20] were performed in the same facility as those of Lawrence et al. [38], it is not unreasonable to assume that the density interface displacements are approximately equal.

When \( J = 0.03 \) and \( \varepsilon = 0.5 \), the right moving instability is the only unstable mode and is the only mode contained in the initial perturbation. At low Reynolds numbers, the initial amplitude of the perturbation has a significant effect on how the flow develops. If the initial amplitude is too small, then diffusion acts to stabilize the flow before the perturbation has a chance to fully develop. This phenomenon was observed by Klaassen & Peltier [29] in their study of Kelvin-Helmholtz instabilities. In the experiments of Guez & Lawrence [20], the flow is forced at a frequency corresponding to \( \alpha = 0.45 \), allowing us to initialize the perturbations with a relatively large amount of kinetic energy (\( K' = 0.005 \)).

To compare our results with the photographs in figure 5.3, we wish to output the results at intervals equal to the time required for the wave to travel one wavelength. This time interval, \( t_o \), is given by

\[
\begin{align*}
  t_o &= \frac{\Lambda}{U + c_r} \\
  &= \frac{2\pi h/2}{\alpha(U + c^*_r \cdot \delta U)}
\end{align*}
\]

where \( c^*_r \) is the non-dimensional phase speed obtained from linear stability analysis. In
non-dimensional time, we obtain

\[
 t^* = \frac{\pi \frac{h}{2}}{\alpha (\tilde{U} + \tilde{c}_r \cdot \delta U)} \cdot \frac{\delta U}{h/2}
 = \frac{2\pi \delta U}{\alpha (\tilde{U} + \tilde{c}_r \cdot \delta U)}
 = 4.75 \tag{5.2}
\]

Results from the numerical simulations with \( R = 3, 6, \) and \( 10 \) (and hence \( Pr = 9, 36, \) and \( 100 \), respectively) are shown in figures 5.5, 5.6 and 5.7. In all cases, \( nx = 64, nz = 128, \) and \( \Delta t = 0.0625 \). Although the results are shown at intervals of 4.75, as calculated above, the billows nevertheless move across the frame as time increases. This movement is explained by recalling that, as discussed in Chapter 4, the phase speed of a right moving wave increases with time when \( \varepsilon \geq 0 \).

For all values of \( R \), the flow behaves qualitatively the same. A finger of heavier fluid is drawn up into the upper layer and forms a billow. The finger becomes thinner with a sharper tip as \( R \) (and hence \( Pr \)) increases. For \( R = 6 \) and \( 10 \), the billow has separated by \( t = 57 \). As the billow wraps around itself, it is stretched and becomes thinner. Eventually, the upper portion of the billow becomes so thin that it diffuses into the ambient fluid. At this point, the billow divides into two disconnected parts. When \( R = 3 \), the billow remains connected at all times since the initial thickness of the billow is much larger than that of the other two cases. It is observed that the diffusion rate increases as \( R \) decreases. This change in diffusion rate can be related to the decreasing value of the Prandtl number and is consistent with the results of Klaassen & Peltier [28] who studied the effect of the Prandtl number on Kelvin-Helmholtz billows. For \( R = 10 \) we observe a resolution problem at the selected grid size. Since the results for \( R = 6 \) are qualitatively the same as for the \( R = 10 \) case and do not change when the grid is refined, we use this case for comparison with the experimental results of Guez & Lawrence [20].

Although the results from the numerical simulations depicted in figure 5.6 and those
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of the laboratory experiments of Guez & Lawrence [20] reproduced in figure 5.3 are qualitatively the same, there are several differences between them. In the numerical simulations the tips of the figures in the billows are not as sharp as those of the laboratory experiments. This difference between experimental and numerical observations is a result of the values of $R$ and $Pr$ used in the numerical simulations being much smaller than those of the experiments. This difference becomes less pronounced as $R$ and $Pr$ are increased.

Another noticeable difference between the two results, is that there is not as much wrapping around of the billows in the numerical simulations as in the laboratory experiments. There are two possible causes of this discrepancy. Firstly, Klaassen & Peltier [28] found that as the Prandtl number increases, there is more wrapping around of fingers in a Kelvin-Helmholtz instability before the displaced fluid diffuses into the ambient flow. Since we have used $Pr = 36$, which is much smaller than the actual value, to model the flow, perhaps the large amount of diffusion suppresses the wrapping around of the billow. Secondly, there is some uncertainty in determining the flow conditions for the laboratory experiments. It is possible that the actual value of the bulk Richardson number is slightly smaller than 0.03. Since the amount of billowing is sensitive the the value of the bulk Richardson number, we expect more curling up of the billows as $J$ decreases.

Breakdown of the billows occurs at approximately the same time in both the numerical simulations and the laboratory experiments. The cause of this breakdown is different, however, for the two cases. In the laboratory experiment the billows collapse when the flow becomes three-dimensional. Since the numerical simulations restrict the flow to two-dimensions, transition to three-dimensional flow cannot possibly be the cause of the breakdown of the billows. Instead, it is due to the top part of the billow diffusing into the ambient fluid. Because of this difference, it does not make physical sense to compare the two flows beyond this point. Nevertheless, it is of interest to note that the two flows remain qualitatively the same.
Next we examine the experimental results of Lawrence et al. [38] when $J = 0.06$ (figure 5.4). Since the results of the nonlinear simulations with $R = 6$ compared favourably with the experimental results of Guez & Lawrence [20] and both experiments were conducted in the same facility, we use the same value to compare with the results of Lawrence et al. [38]. The only difference is that no forcing is present in the laboratory study of Lawrence et al. [38]. Nevertheless, we found that a fairly large initial amplitude of the perturbation was still required in order for the perturbation to become fully developed. Having a large initial amplitude poses no problems since only the right moving mode is an unstable mode when $J = 0.06$ with $\varepsilon = 0.5$. When both right and left modes are unstable, more care must be taken in choosing the initial amplitudes of the perturbations, as is discussed below.

Results of the numerical simulations with $J = 0.06$ are shown in figure 5.8 (here $t_0^* = 4.95$). As with the $J = 0.03$ case, the fingers of the billows are not as thin as in the experimental results of Lawrence et al. [38] (figure 5.4). Also, the fingers do not curl up as much in the numerical simulations. The possible explanations for these differences are the same as the $J = 0.03$ case. Although the results of the numerical simulations and the laboratory experiments differ at later times, qualitative agreement as the perturbation initially develops is excellent.

Lawrence et al. [38] examined the development of perturbations for a range of Richardson numbers (figure 5.4). Comparing numerical results with those of Lawrence et al. [38] is difficult when $J$ is large and both right and left moving modes are unstable. The problem lies in selecting the correct initial amplitude of the perturbation. If the amplitude is not sufficiently large, then diffusion damps the perturbation before it has a chance to grow. If the initial amplitude is too large, the perturbation does not spend much time in the linear regime, resulting in the left mode being of approximately the same amplitude
as the right mode. The experimental observations indicate, however, that the initial am-
plitude is such that the perturbation spends sufficient time in the linear regime for the
right mode to dominate. Finding the correct initial amplitude is a matter of trial and
error. We have not attempted to numerically reproduce these experiments.
Figure 5.1: Experiments of Zhu [72]

Sequence of photographs showing Holmboe waves (figure 7.1b of Zhu [72]). The upper layer is from left to right, the lower layer from right to left. The grid markings are 5 cm apart and the photographs were taken at 0.5 second intervals. See table 5.1 for flow conditions.
Figure 5.2: Results from numerical simulations for $J = 0.3$

Density contours of results from numerical simulations with $J = 0.3$, $Re = 300$, $R = 3$, $Pr = 9$, $\varepsilon = 0$, with $\tilde{\alpha} = 0.54$. $x$ has been scaled with respect to its period $2\pi/\tilde{\alpha}$. 
Figure 5.3: Experiments of Guez & Lawrence [20]

Sequence of photographs showing development of Kelvin-Helmholtz type billows from Guez & Lawrence [20]. The flow is from left to right. The field view of each photograph is approximately 25cm with an overlap of approximately one wavelength between each successive photograph. See table 5.1 for flow conditions.
Figure 5.4: Experiments of Lawrence et al. [38]

Sequence of photographs showing development of shear instabilities at various Richardson numbers (figure 6 from Lawrence et al. [38]). The flow is from left to right. The field view of each photograph is approximately 25cm with an overlap of approximately 5cm between each successive photograph. As the bulk Richardson number increases, less billowing is observed. See table 5.1 for flow conditions when $J = 0.06$. 
Figure 5.5: Results from numerical simulations for $J = 0.03$ with $R = 3$
Density contours of results from numerical simulations with $J = 0.03$, $Re = 25$, $R = 3$, $Pr = 9$, $\varepsilon = 0.5$, with $\tilde{\alpha} = 0.45$. Contours are from -0.72 to 0.72 in equal increments of 0.18. $x$ has been scaled with respect to its period $2\pi/\tilde{\alpha}$.
Figure 5.6: Results from numerical simulations for $J = 0.03$ with $R = 6$

Density contours of results from numerical simulations with $J = 0.03$, $Re = 25$, $R = 6$, $Pr = 36$, $\varepsilon = 0.5$, with $\tilde{\alpha} = 0.45$. Contours are from -0.72 to 0.72 in equal increments of 0.18. $x$ has been scaled with respect to its period $2\pi/\tilde{\alpha}$. 
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Figure 5.7: Results from numerical simulations for $J = 0.03$ with $R = 10$

Density contours of results from numerical simulations with $J = 0.03$, $Re = 25$, $R = 10$, $Pr = 100$, $\varepsilon = 0.5$, with $\bar{\alpha} = 0.45$. Contours are from -0.72 to 0.72 in equal increments of 0.18. $x$ has been scaled with respect to its period $2\pi/\bar{\alpha}$. 
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