5. OTHER DIFFERENTIAL EQUATIONS

The methods considered in this thesis for the solution of the linear differential equations with Dirichlet boundary conditions (1.1), (3.1), and (4.1) may be extended to other differential equations. Consider first a quasi-linear boundary-value problem

\[ y'' = f(y', y, x), \quad a \leq x \leq b, \]  
\[ y(a) = \theta, \quad y(b) = \vartheta. \]  

When equation (5.1) is linearized by any of the common linearization methods, second-order approximations to isolated solutions can be found by the methods described in Section 2.1. Isolated solutions are solutions which are locally unique.

Higher-order equations can also be solved using these methods. An \( m^{\text{th}} \) order differential equation will be expressed as \( m \) first-order differential equations for the purpose of deriving the numerical scheme. Difference formulas are applied to each differential equation. A staggered mesh \( \pi_n \) is used; that is, each differential equation is discretized on different sub-intervals of the interval \([a,b]\) in such a way that \((m - 1)\) consecutive substitutions can be performed to produce an \((m + 1)\)-banded system \(A_y = f\). For example, a linear fourth-order differential equation

\[ y^{(4)} = [c_1(x) \ y]'' + [c_2(x) \ y]'' + [c_3(x) \ y] + c_4(x) \ y + c_5(x) \]  

(5.2)

can be rewritten as the system

\[ y' = c_1(x) \ y + p, \]  
(5.3a)
\[ p' = c_2(x) y + q, \quad (5.3b) \]
\[ q' = c_3(x) y + r, \quad (5.3c) \]
\[ r' = c_4(x) y + c_5(x). \quad (5.3d) \]

A midpoint scheme on a uniform but staggered mesh can be applied to the four first-order equations on the following intervals: equations (5.3a) and (5.3c) are discretized over

\[ [x_i, x_{i+1}], \quad i = 1, \ldots, n \]

and equations (5.3b) and (5.3d) are discretized over

\[ [x_{i-1/2}, x_{i+1/2}], \quad i = 2, \ldots, n. \]

So then

\[ \frac{y_{i+1} - y_i}{h_i} = \frac{1}{2} c_1(x_{i+1/2}) \left[ y_{i+1} + y_i \right] + p_{i+1/2}, \quad (5.4a) \]
\[ \frac{p_{i+1/2} - p_{i-1/2}}{\frac{1}{2}(h_i + h_{i-1})} = \frac{1}{2} c_2(x_i) y_i + q_i, \quad (5.4b) \]
\[ \frac{q_{i-1} - q_i}{h_i} = \frac{1}{2} c_3(x_{i+1/2}) \left[ y_{i+1} + y_i \right] + r_{i+1/2}, \quad (5.4c) \]
\[ \frac{r_{i-1/2} - r_{i-1/2}}{\frac{1}{2}(h_i + h_{i-1})} = \frac{1}{2} c_4(x_i) y_i + c_5(x_i). \quad (5.4d) \]

An \((m + 1)\)-point scheme in the unknowns \(y_i\) is determined by \((m - 1)\) successive substitutions. For the system (5.4), equation (5.4a) is substituted into (5.4b) to eliminate the variable \(p\). Then the resulting equation is substituted into (5.4c) and \(q\) is eliminated. Finally the new equation is substituted into equation (5.4d) to eliminate \(r\).

This yields a five-point formula with only the mesh function \(y_x\) for unknowns. When a difference scheme is applied to (5.3) on a nonuniform mesh, the discretizations become more complicated. The midpoint scheme on a nonuniform mesh gives the following discretization of the system (5.3):

\[ \frac{y_{i-1} - y_i}{h_i} = \frac{1}{2} c_1(x_{i+1/2}) \left[ y_{i+1} + y_i \right] + p_{i+1/2} \quad (5.5a) \]
\[ \frac{p_{i+1/2} - p_{i-1/2}}{\frac{1}{2}(h_i + h_{i-1})} = c_2(\bar{x}_i) \bar{y}_i + \bar{q}_i \]  
\[ (5.5b) \]

\[ \frac{\bar{q}_{i+1} - \bar{q}_i}{\bar{x}_{i+1} - \bar{x}_i} = c_3(\bar{x}_i) \bar{y}_i + \bar{r}_i \]  
\[ (5.5c) \]

\[ \frac{\bar{r}_{i+1} - \bar{r}_i}{\bar{x}_{i+1} - \bar{x}_i} = c_4(\bar{x}_i) \bar{y}_i + c_5(\bar{x}_i) \]  
\[ (5.5d) \]

where \( \bar{x}_i \) is the midpoint of \([x_{i-1/2}, x_{i+1/2}]\), \( \bar{x}_i \) is the midpoint of \([\bar{x}_{i+1}, \bar{x}_i]\), \( \bar{x}_i \) is the midpoint of \([\bar{x}_{i+1}, \bar{x}_i]\).

The methods used in this thesis can also be applied to boundary-value problems with boundary conditions that are more complicated than the Dirichlet boundary conditions considered in the previous chapters. Separated linear boundary conditions of the form

\[ k_1 y'(a) + k_2 y(a) = \theta_1 \]
\[ \ell_1 y(b) + \ell_2 y'(b) = \theta_2 \]

can very simply be included in the system

\[ A y = g \]  
\[ (1.2) \]

without affecting the tridiagonal nature of the system. However, incorporating boundary conditions that are not separated into the system \((1.2)\) will cause the system to lose its tridiagonal form. Still, a similar stability and consistency analysis can be shown to hold here as for \((1.2)\), and second-order accurate solutions are obtained on a nonuniform mesh when centered second-order difference schemes are used.
6. CONCLUSION

We have derived a class of difference schemes to find second-order approximate solutions to boundary-value problems by discretizing on an arbitrary mesh $\pi_n$. The derivation of these schemes involves, first, rewriting the boundary-value problem as a system of first-order equations. Each equation is discretized by a one-step centered difference scheme over staggered sub-intervals on the mesh $\pi_n$. The first equation in the system, involving the derivative of the mesh function (the principal unknown), is discretized at the mesh points; the next equation, which has the derivative of an auxiliary unknown, is discretized at the midpoints of the mesh. Any further equations are discretized at points mid-way between the discretization points of the preceding equation. This use of staggered mesh sub-intervals allows the substitution of one equation into the next so that, ultimately, one equation involving only the principal mesh function results. For an $m^{th}$-order boundary-value problem, an $(m + 1)$-point difference formula is produced in this way. Similarly, an $(m + 1)^2$-point formula can be obtained for boundary-value problems posed in two dimensions. The difference formulas can also be applied to linearized forms of nonlinear equations.

Singularly perturbed problems without turning points can also be solved using similar procedures, but some restrictions are imposed. First of all when rewriting the
singly perturbed boundary-value problem as a system of first-order equations, the transformation must be done in such a way that the fast and slow solution modes are weakly coupled or completely decoupled. The equation for the slow solution can be discretized using the usual difference schemes, for example, the midpoint or trapezoidal schemes. For the fast solution a fast-solution discretization method must be used, such as exponential fitting. There are two advantages to using this method over other singular perturbation methods. First, for a mesh where the maximum step size is much larger than the parameter $\varepsilon$, the mesh need not resolve the solution layer to produce accurate results elsewhere; and second, explicit upwinding need not be applied because the method of exponential fitting does automatic upwinding. However, this method is only applicable to a limited class of problems.

The three-point formulae derived in Chapter 2 for second-order linear boundary-value problems with Dirichlet boundary conditions are shown to be theoretically second-order and stable. The theory is supported by the numerical results included in this thesis. The second-order results are also shown to hold for linear boundary-value problems posed in two (or more) dimensions. For the three-point formulae derived for linear second-order singularly perturbed problems on an arbitrary mesh we have shown that the order results are obtained regardless of the uniformity of the mesh. Numerical results support all of these findings. Thus the methods presented here will be efficient and efficacious for solving differential equations that arise in mathematical modelling.