THE NUMERICAL TREATMENT OF ILL-POSED PROBLEMS USING
THE METHOD OF CONJUGATE GRADIENTS

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Abstract

This thesis examines the use of the method of Conjugate Gradients as an iterative method to be applied to linear system of equations that are ill-conditioned. An overview of the particular problems associated with the solution of ill-conditioned linear systems is given and the method of Conjugate Gradients described. It is shown that, at each iteration, the method of Conjugate Gradients weights the contribution from the singular vectors using a polynomial that is characterized as the solution to a weighted least squares problem. The form of the polynomials is shown to approximate a series of interpolating polynomials that constitute an efficient filtering technique required for computing solutions to ill-conditioned systems. The performance of the method of Conjugate Gradients is shown to compare favourably with other accepted methods for ill-conditioned linear systems. An application from the field of image processing is given and the efficient computation of a smooth reconstructed image from a defocussed picture is demonstrated.
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