Chapter 2

Mathematical Model

2.1 System Description

2.1.1 Introduction

Figure 2.1 shows schematically the satellite system being considered. There are two main bodies joined by a tether. The platform may have an arbitrary three dimensional inertia distribution. The subsatellite is considered a point mass since in most proposed applications it is significantly smaller and less massive than the platform. The tether is treated as a continuum with longitudinal flexibility and may be deployed or retrieved at any specified rate. The tether reel mass is also included and is treated as a point mass. There were two reasons for including the reel mass in the formulation. The first was to conserve the total mass of the system. For example, during retrieval the tether mass decreases while the reel mass increases at the same rate. Secondly, since motion of the attachment point is to be considered, the presence of this mass may have an effect on the dynamics.

Degrees of freedom of the system include platform pitch, tether pitch, longitudinal tether vibration and controlled motion of the tether attachment point. In addition, the center of mass of the entire system follows an orbit of arbitrary eccentricity and altitude. Motion out of the plane of the orbit is not considered. It has been shown that for small oscillations, inplane and out of plane motions decouple and so may be studied separately [6].
Figure 2.1: Platform based Tethered Satellite System (TSS).
2.1.2 Reference Frames

Four reference frames are introduced in order to establish the orientation of the system with respect to an inertial reference (Figure 2.2). The inertial frame $F_i$ is fixed to the earth's center with $z_i$ axis passing through the perigee. The orbital frame $F_c$ is fixed to the center of mass of the orbiting system with $z_c$ axis along the local vertical. The platform frame $F_p$ is fixed to the center of mass of the platform with axes along the principal axes of the platform. Finally the tether frame $F_t$ is fixed at the attachment point with $z_t$ axis along the tether.

2.1.3 Position Vectors

Using the reference frames described above, the location of any mass element can be represented as a sum of position vectors. From the inertial frame, the vector $R_c$ locates the system center of mass. From the origin of the orbital frame, $R_p$ locates the platform center of mass and $R_s$ positions the subsatellite. With reference to the platform frame, $r_p$ locates a platform mass element $dm_p$ and $d$ establishes the location of the tether attachment point. From the tether frame, $r_t$ locates a tether mass element $dm_t$. Thus for example, the position of the reel mass at any time is given by $R_c + R_p + d$ (Figure 2.3).

Note, the vector $r_t$ must be a function of the mass element being considered as well as a function of time. That is, given a particular mass element of the tether, $r_t$ changes to follow its longitudinal oscillations. To account for this the vector $r_t$ is expressed as,

$$r_t(z_t, t) = [z_t + w(z_t, t)]k_t,$$

where

$$w(z_t, t) = \varepsilon(t) \sum_{n=1}^{N} \phi_n(z_t).$$
Here \( \phi_n(z_t) \) are independent functions satisfying the geometric boundary condition,

\[
r_t(0, t) = 0,
\]

For the relatively short tether lengths considered in this study, the strain variation can be approximated as linear [8]. This gives

\[
w(z_t, t) = c(t)z_t.
\]

Thus the position of a mass element before deformation, say \( z_t k_t \), is located after deformation by

\[
r_t(z_t, t) = z_t[1 + c(t)]k_t,
\]

and for \( z_t = \bar{l} \),

\[
1 = \bar{l}[1 + c]k_t.
\]

2.1.4 Generalized Coordinates

Before obtaining the equations of motion it is necessary to choose a set of generalized coordinates for the problem. Generalized coordinates are independent angles or displacements used to specify the orientation of a system.

There are five generalized coordinates chosen for this problem. The true anomaly \( \theta \) is measured in radians from the line joining the earth's center to the perigee of the orbit (Figure 2.2). The radial distance \( r \) is measured in meters from the earth's center to the center of mass of the system. These two coordinates keep track of the orbital position of the system in an arbitrary orbit. The platform pitch angle \( \alpha_p \) is measured in radians from the local vertical to the \( z_p \) axis. The tether pitch angle \( \alpha_t \) is measured in radians from the local vertical to the \( z_t \) axis. Note, the rotations are considered positive in the clockwise sense. The variable \( \epsilon \) described above is used to monitor the difference
Figure 2.2: Reference frames and generalized coordinates.
Figure 2.3: Position vectors.
between the actual tether length \( l \) and its nominal unstretched value \( \bar{l} \). \( \epsilon \) is defined by the expression:

\[
l = \bar{l}(1 + \epsilon).
\]

Notice,

\[
\epsilon = \frac{l - \bar{l}}{\bar{l}} = \frac{\text{change in length}}{\text{original length}}
\]

which is the expression for strain in a stretched wire.

The offset of the tether attachment point is considered a specified quantity and so is not included in the list of generalized coordinates.

2.1.5 Constraints

The number of variables in the formulation can be reduced by utilizing various equality constraints. From mass considerations, the following relationships are clear.

\[
M = M_s + M_p + M_r + M_t.
\]

\[
M_r + M_t = \text{constant.} \tag{2.1}
\]

From the geometry in Figure 2.2, the following relationship holds,

\[
R_s = R_p + d + l. \tag{2.2}
\]

Finally the definition of the center of mass of a system gives another constraint. Setting the first moment of mass about \( C \) to zero leads to the following vector equation,

\[
M_s R_s + \int (R_p + r_p) \, dm_p + M_r (R_p + d) + \int (R_p + d + r_t) \, dm_t = 0.
\]

This simplifies to

\[
R_p = -\frac{1}{M} [M_s d + M_t l + \rho \int_0^l r_t \, dz_t], \tag{2.3}
\]
with the use of equations (2.2), and (2.1) and the following relations:

\[
\begin{align*}
\int R_p \, dm_p &= R_p \int dm_p = R_p M_p; \\
\int r_t \, dm_t &= \rho \int_0^i r_t \, dz_i; \\
\int r_p \, dm_p &= 0. 
\end{align*}
\]

The last equation is true since the \( F_p \) frame has its origin at the center of mass of the platform.

2.2 Nonlinear Equations of Motion

2.2.1 Kinetic Energy

The kinetic energy of a general mechanical system is given by

\[
T = \frac{1}{2} \int \dot{r} \cdot \dot{r} \, dm,
\]

where \( r \) locates the mass element \( dm \) and integration is over all such elements.

Recall that the system studied here consists of a platform, tether, reel mass and subsatellite. Integrating over each of these separately, the kinetic energy can be written as

\[
T = \frac{1}{2} \int (\ddot{R}_c + \dot{R}_p + r_p') \cdot (\dot{R}_c + \dot{R}_p + r_p') \, dm_p \\
+ \frac{1}{2} \int (\ddot{R}_c + \dot{R}_p + \dot{d} + \dot{r}_i) \cdot (\dot{R}_c + \dot{R}_p + \dot{d} + \dot{r}_i) \, dm_i \\
+ \frac{1}{2}\lambda (\ddot{R}_c + \dot{R}_p + \dot{d} + \dot{i}) \cdot (\dot{R}_c + \dot{R}_p + \dot{d} + \dot{i}).
\]
Rearranging terms gives

\[
T = \frac{1}{2} M(\dot{R}_c \cdot \dot{R}_c) + \frac{1}{2} M(\dot{R}_p \cdot \dot{R}_p) + M_{sr}(\dot{R}_p \cdot \dot{d}) \\
+ \frac{1}{2} M_{sr}(\dot{d} \cdot \dot{d}) + \frac{1}{2} \int r_p \cdot \dot{r}_p \; dm_p + M_s(R_p \cdot \dot{I}) \\
+ M_s(\dot{d} \cdot \dot{I}) + \frac{1}{2} M_s(\dot{I} \cdot \dot{I}) + \int R_p \cdot \dot{r}_p \; dm_p \\
+ \int R_p \cdot \dot{r}_t + \dot{d} \cdot \dot{r}_t + \frac{1}{2} \dot{r}_t \cdot \dot{r}_t \; dm_t \\
+ \dot{R}_c \cdot [M \dot{R}_p + M_{sr} \dot{d} + M_s \dot{I} + \int \dot{r}_t \; dm_t + \int \dot{r}_p \; dm_p].
\]

The last term in the above expression is equal to \(-l \ddot{l} \dot{p}\). This can be verified by taking the time derivative of equation (2.3) and applying Leibnitz’s rule for differentiating the integral with \(\dot{I}\) as an upper limit. Each of the remaining terms is written in terms of the generalized coordinates of the problem. To illustrate, the term \(\frac{1}{2} M_s(\dot{I} \cdot \dot{I})\) is rewritten here.

Since

\[
l = l k_t,
\]

differentiating with respect to time in the inertial frame gives

\[
\dot{l} = l \dot{k}_t + l(\omega_t \times k_t).
\]

(2.4)

Now from the geometry of the problem,

\[
k_t = [-\sin(\theta - \alpha_t), 0, -\cos(\theta - \alpha_t)],
\]

and

\[
\omega_t = [0, \dot{\theta} - \dot{\alpha}_t, 0].
\]

Using these in equation (2.4) gives

\[
\dot{l} = [-l \sin(\theta - \alpha_t) - l(\dot{\theta} - \dot{\alpha}_t) \cos(\theta - \alpha_t), 0, -l \cos(\theta - \alpha_t) + l(\dot{\theta} - \dot{\alpha}_t) \sin(\theta - \alpha_t)].
\]
Thus,

\[ \frac{1}{2} M_s (\dot{\alpha} \cdot \dot{\alpha}) = I^2 + l^2 (\dot{\theta} - \dot{\alpha})^2. \]

Continuing in this way, the kinetic energy for the system can be written as

\[
T = \frac{1}{2} M (r^2 + r^2 \dot{\theta}^2) + \frac{M_{\text{rot}} M_p}{2M} \left[ \dot{d}_x^2 + \dot{d}_z^2 + (d_x^2 + d_z^2) (\dot{\theta} - \dot{\alpha})^2 \right] \\
- 2d_x) \dot{d}_z (\dot{\theta} - \dot{\alpha}) + 2d_z \dot{d}_x (\dot{\theta} - \dot{\alpha}) \right] + \frac{M_{\text{rot}} M_p}{2M} \left[ I^2 + l^2 (\dot{\theta} - \dot{\beta})^2 \right] \\
+ \frac{M_p M_s}{M} \left[ \dot{d}_z \dot{l} (\dot{\theta} - \dot{\alpha}) \cos(\alpha_t - \alpha_p) + \dot{l} \dot{d}_z (\dot{\theta} - \dot{\alpha}) \sin(\alpha_t - \alpha_p) \right] \\
- \dot{d}_z \dot{l} \cos(\alpha_t - \alpha_p) + \dot{l} \dot{d}_z (\dot{\theta} - \dot{\alpha}) \sin(\alpha_t - \alpha_p) \\
- d_x \dot{l} (\dot{\theta} - \dot{\alpha}) \cos(\alpha_t - \alpha_p) - l \dot{d}_z (\dot{\theta} - \dot{\alpha}) \cos(\alpha_t - \alpha_p) \\
- \dot{d}_z \dot{l} (\dot{\theta} - \dot{\alpha}) \sin(\alpha_t - \alpha_p) + \dot{l} \dot{d}_z \sin(\alpha_t - \alpha_p) \right] \\
+ \frac{M_p}{M} \rho \left[ \left( \frac{1}{2} \dot{l}^2 + \dot{l} \right) (d_z (\dot{\theta} - \dot{\alpha}) \cos(\alpha_t - \alpha_p) + d_z (\dot{\theta} - \dot{\alpha}) \sin(\alpha_t - \alpha_p) \right] \\
- \dot{l} \dot{d}_z (\dot{\theta} - \dot{\alpha}) \sin(\alpha_t - \alpha_p) + \dot{l} \sin(\alpha_t - \alpha_p) \right] \\
- d_z (\dot{\theta} - \dot{\alpha}) \cos(\alpha_t - \alpha_p) - \dot{d}_z (\dot{\theta} - \dot{\alpha}) \sin(\alpha_t - \alpha_p) \right] \\
- \frac{M_p}{M} \rho \left[ \left( \frac{1}{2} \dot{l}^2 + \dot{l} \right) \dot{l} + \frac{1}{2} l^2 \dot{\theta} \dot{\beta} \right] + \frac{1}{2} \rho \left[ \frac{1}{3} \dot{r}^2 l^3 + \dot{l}^2 (1 + \epsilon)^2 \right] \\
+ \frac{1}{3} l^2 \dot{\theta} \dot{\beta}^2 + \epsilon \dot{l} \dot{\beta} (1 + \epsilon) \right] + \frac{1}{2} I_{\text{y}} (\dot{\theta} - \dot{\alpha})^2 - \frac{p^2}{2M} \left[ \frac{1}{4} \dot{r}^2 l^4 \right] \\
+ \dot{p} \dot{\theta} - \frac{1}{4} l^2 \dot{\theta} \dot{\beta}^2 + \dot{\beta} \dot{\theta} \dot{\beta} \right] - \rho \dot{\theta} (-r \cos \alpha_t - r \dot{\theta} \sin \alpha_t) \\
+ \frac{1}{2M} l^2 \rho^2 + \rho \dot{l} \left[ - \frac{M_{\text{rot}}}{2M} l (d_x (\dot{\theta} - \dot{\alpha}) \cos(\alpha_t - \alpha_p) \right] \\
+ d_x (\dot{\theta} - \dot{\alpha}) \sin(\alpha_t - \alpha_p) + d_z \sin(\alpha_t - \alpha_p) - \dot{d}_z \cos(\alpha_t - \alpha_p) \right] \\
- \frac{M_p}{2M} \dot{r} \left[ \frac{1}{2} \dot{l}^2 + \dot{l} \right]. \]
2.2.2 Potential Energy

The potential energy of the system can be divided into the gravitational contribution associated with the masses \((U_g)\), as well as the stored energy due to the elongation of the tether \((U_s)\).

Gravitational Potential Energy

The gravitational potential energy of a general mecanical system is given by

\[
U_g = -\int \frac{GM_e}{|r|} \, dm,
\]

where \(G\) is the universal gravitational constant; \(M_e\) is the mass of the earth; and \(r\) locates the mass element \(dm\) of the system. Integration is over all such elements.

As before, integrating over the four regions of the system studied here gives

\[
-\frac{1}{GM_e} U_g = \int \frac{dm_p}{|R_c + R_p + r_p|} + \frac{M_s}{|R_c + R_s|} + \frac{M_t}{|R_c + R_p + d|} + \int \frac{dm_t}{|R_c + R_p + d + r_t|}. \tag{2.5}
\]

Now,

\[
\frac{1}{|R_c + R_s|} = |rk_c + R_s|^{-1} = [|(rk_c + R_s) \cdot (rk_c + R_s)]^{-\frac{1}{2}} = \left[ r^2 + 2r(R_s \cdot k_c) + R_s \cdot R_s \right]^{-\frac{1}{2}} = \frac{1}{r} \left[ 1 + \frac{2(R_s \cdot k_c)}{r} + \frac{R_s \cdot R_s}{r^2} \right]^{-\frac{1}{2}} \approx \frac{1}{r} - \frac{R_s \cdot k_c}{r^2} + \frac{3(R_s \cdot k_c)^2 - R_s \cdot R_s}{2r^3},
\]

where the binomial expansion is used, keeping only terms to order \(1/r^3\).
After rewriting each of the quotients in (2.5) this way, the first terms from each quotient add to give the orbital potential energy (that is, the energy due to the position of the center of mass of the system). The second term in each quotient, which is of order $1/r^2$, vanishes due to the center of mass constraint. This leaves

$$-rac{U_g}{GM_e} \approx \frac{M}{r} + \frac{1}{r^3} \left[ \left( \frac{3}{2} (k_c \cdot R_s)^2 - \frac{1}{2} (R_s \cdot R_s) \right) M_s ight. \\
+ \int \left( \frac{3}{2} (k_c \cdot (R_p + r_p))^2 - \frac{1}{2} (R_p + r_p) \cdot (R_p + r_p) \right) dm_p \\
+ \int \left( \frac{3}{2} (k_c \cdot (R_p + d + r_t))^2 - \frac{1}{2} (R_p + d + r_t) \cdot (R_p + d + r_t) \right) dm_t \\
+ \left. \left( \frac{3}{2} (k_c \cdot (R_p + d))^2 - \frac{1}{2} (R_p + d) \cdot (R_p + d) \right) M_s \right] .$$

Collecting terms in common dot product gives

$$-rac{U_g}{GM_e} \approx \frac{M}{r} + \frac{1}{r^3} \left[ \frac{3}{2} M (k_c \cdot R_p)^2 + \frac{1}{2} M (R_p \cdot R_p) \\
- \frac{3}{2} M_{rst} (k_c \cdot d)^2 - 3M (k_c \cdot d)(k_c \cdot R_p) + \frac{1}{2} M_{rst} (d \cdot d) \\
+ M (d \cdot R_p) + \frac{3}{2} M_s (k_c \cdot l)^2 - \frac{1}{2} (l \cdot l) \\
+ \int \left( \frac{3}{2} (k_c \cdot r_p)^2 - \frac{1}{2} (r_p \cdot r_p) \right) dm_p \\
+ \int \left( \frac{3}{2} (k_c \cdot r_t)^2 - \frac{1}{2} (r_t \cdot r_t) \right) dm_t \right] .$$

Introducing the generalized coordinates results in the third-order approximation gives

$$U_g \approx -\frac{GM_e M}{r} - \frac{GM_e}{r^3} \left[ \frac{3}{2M} \left( M_{rst}^2 (d_x \sin \alpha_p + d_x \cos \alpha_p)^2 \\
+ M_x^2 l^2 \cos^2 \alpha_t + \frac{1}{4} \rho^2 l^2 \cos^2 \alpha_t + m_x \rho^2 l \cos^2 \alpha_t \\
- 2M_{rst} l \cos \alpha_t (d_x \sin \alpha_p + d_x \cos \alpha_p) (M_s + \rho \frac{l}{2}) \right) \right] .$$
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\[ + \frac{1}{2M} (M_{zt}^2(d_x^2 + d_y^2) + M_s^2 l^2 + \frac{1}{4} \rho l^2 l^2 \bar{\rho}) \\
+ M_{zt} l (d_x \sin(\alpha_t - \alpha_p) - d_x \cos(\alpha_t - \alpha_p))(2M_s + \rho l) + M_s l^2 \rho \]

\[ - \frac{3}{2} M_{zt} (d_x \sin \alpha_p + d_x \cos \alpha_p)^2 + 3(d_x \sin \alpha_p) \\
+ d_x \cos \alpha_p)(M_{zt} (d_x \sin \alpha_p + d_x \cos \alpha_p) - M_s l \cos \alpha_t - \frac{1}{2} \rho l \cos \alpha_t) \\
- \left( \frac{1}{2} M_{zt} (d_x^2 + d_y^2) + (d_x \sin(\alpha_t - \alpha_p) - d_x \cos(\alpha_t - \alpha_p))(M_s l \\
+ \frac{1}{2} \rho l) \right) + \frac{3}{2} M_s l^2 \cos^2 \alpha_t - \frac{1}{2} M_s l^2 + \frac{I_{yy}}{2} + \frac{I_{zz}}{2} \\
- I_{xx} + \frac{3}{2} (I_{xx} - I_{zz}) \cos^2 \alpha_p + \frac{1}{2} \rho l l^2 \cos^2 \alpha_t - \frac{1}{6} \rho l l^2 \right].

Strain Energy

The strain energy in a deformed, elastic body is given by

\[ U_s = \frac{1}{2} \int \sigma \delta dV, \]

where \( \sigma \) is the stress in a differential element of volume \( dV \), and \( \delta \) represents the strain in a differential element of volume \( dV \).

It is assumed that the properties of the tether are the same along its length. Now, by definition,

\[ E = \frac{\sigma}{\delta}, \]

where \( E \) is Young's Modulus. Hence

\[ U_s = \frac{1}{2} \int E \delta^2 dV. \]

As shown by Nayfeh and Mook [9], given an element of the tether of initial length \( dz_t \)
with one end located at \( z_t k_t \) and the other at \([z_t + w(z_t, t)] k_t \),

\[
\delta |_{z_t} = \frac{\Delta \text{length}}{\text{length}} |_{z_t}
\]

\[
\Rightarrow \delta |_{z_t} = \lim_{\Delta z_t \to 0} \frac{w(z_t + \Delta z_t, t) - w(z_t, t) + \Delta z_t - \Delta z_t}{\Delta z_t}
\]

\[
= \frac{\partial w(z_t, t)}{\partial z_t}
\]

\[
= \frac{\partial [\varepsilon z_t]}{\partial z_t}
\]

\[
= \varepsilon.
\]

So,

\[
U_s = \frac{1}{2} \int E\varepsilon^2 \, dV
\]

\[
= \frac{1}{2} EA \int_0^L \varepsilon^2 \, dz_t
\]

\[
= \frac{1}{2} EA \varepsilon^2 \tilde{I}.
\]

The total potential energy of the system is now given to third order by,

\[
U = U_x + U_s.
\]

2.2.3 Lagrange's Method

In 1788 Lagrange published his book, *Méchanique Analytique* [7], in which he describes an energy approach to obtain equations of motion. The method requires writing the kinetic energy \((T)\), and potential energy \((U)\), in terms of a set of generalized coordinates \((q_i)\). The equations of motion are then given by the following ordinary differential
equations:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial q_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_j,
\]

where \( Q_j \) represents the effect of any external forces on the coordinate \( q_j \). The result is one second order, ordinary differential equation related to each of the generalized coordinates. Substitution of the above energy expressions into equation (2.6) leads to the desired equations of motion. The equations for \( r \) and \( \theta \) turn out to be the classical Keplerian equations with small perturbation terms due to the finite dimensions of the system. It is assumed that the effect of these terms on the system dynamics is negligible. Modi and Misra[10] have shown that even after a full year of longitudinal oscillations of a 1 km tethered system, the coupling effect on \( r \) and \( \theta \) is small. Since this paper is concerned with disturbances which are quickly controlled, it is assumed that the effect of these terms is negligible. This assumption allows us to use Kepler's relations to change the independent variable of the problem from time to the true anomaly \( \theta \), which is more convenient for satellite problems.

Using:

\[
\begin{align*}
 r^2 \dot{\theta} &= h_K; \\
 r &= \frac{h_K^2}{GM_e(1 + e \cos \theta)};
\end{align*}
\]

where \( h_K \) is constant for a given orbit (angular momentum per unit mass), one obtains the following substitution for time derivatives:

\[
\begin{align*}
 \frac{d}{dt} &= \frac{d\theta}{dt} \frac{d}{d\theta} \\
 \frac{d^2}{dt^2} &= \frac{d^2}{dt} \left( \frac{d^2}{d\theta^2} - F \frac{d}{d\theta} \right);
\end{align*}
\]
where

\[ F = \frac{2e \sin \theta}{(1 + e \cos \theta)}. \]

Nondimensionalizing with respect to \( M_s^2 \frac{d^2 \theta}{dt^2} \), the resulting nonlinear, nonautonomous and coupled equations are:

Platform Pitch Equation:

\[
\frac{M_{sr} M_p}{2M M_s} \left[ -4(D_x \dot{D}_x + D_z \dot{D}_z)(1 - \alpha_p) - 2(D_x^2 + D_z^2)(-F - \ddot{\alpha}_p + F\dot{\alpha}_p) + 2(D_x F \dot{D}_x)D_z + 2(D_z F \dot{D}_z)D_x \right]
\]
\[+ \frac{M_p}{M} [-D_x(\ddot{L} - F \dot{L}) \cos(\alpha_t - \alpha_p) - 2D_x \dot{L} \sin(\alpha_t - \alpha_p)(1 - \dot{\alpha}_t)] \]
\[+ D_z(\ddot{L} - F \dot{L}) \sin(\alpha_t - \alpha_p) + 2D_z \dot{L} \cos(\alpha_t - \alpha_p)(1 - \dot{\alpha}_t) \]
\[+ D_x \dot{L}(-F - \ddot{\alpha}_t + F\dot{\alpha}_t) \sin(\alpha_t - \alpha_p) + D_x \dot{L}(1 - \dot{\alpha}_t)^2 \cos(\alpha_t - \alpha_p) \]
\[+ D_x \dot{L}(-F - \ddot{\alpha}_t + F\dot{\alpha}_t) \cos(\alpha_t - \alpha_p) + D_x \dot{L} \sin(\alpha_t - \alpha_p)(1 - \dot{\alpha}_t)^2] \]
\[+ \rho \frac{M_p}{M M_s} \left[ \frac{1}{2} (\ddot{\xi} - F \dot{\xi}) \ddot{L}^2 + 2\dot{L} \dot{\ddot{L}} \right] \]
\[+ (1 + e)(\ddot{\xi} - F \dot{\xi}) \ddot{L} + \dot{\ddot{L}}^2)(-D_x \cos(\alpha_t - \alpha_p) - D_x \sin(\alpha_t - \alpha_p)) \]
\[+ \frac{1}{2} \dot{\ddot{L}}^2 + L \dot{L} + (1 - \dot{\alpha}_t)(D_x \sin(\alpha_t - \alpha_p) - D_x \cos(\alpha_t - \alpha_p)) \]
\[+ \frac{1}{2} (-F - \ddot{\alpha}_t + F\dot{\alpha}_t) L \dot{L} + (1 - \dot{\alpha}_t) \dot{L} \ddot{L} + \frac{1}{2} \dot{L}^2 \dot{\xi}(D_x \cos(\alpha_t - \alpha_p) \]
\[+ D_x \sin(\alpha_t - \alpha_p)) + \frac{1}{2}(1 - \dot{\alpha}_t)^2 L \dot{L} \dot{\ddot{L}} \sin(\alpha_t - \alpha_p) \]
\[+ D_x \cos(\alpha_t - \alpha_p)) - \frac{I_{yy}}{M_l^2}(1 + e \cos \theta)^{-1} \left( F - \ddot{\alpha}_p + F\dot{\alpha}_p \right) \]
\[+ \frac{1}{M_s(1 + e \cos \theta)} \left[ \frac{3}{2} M(2M_{sr}^2 (D_x \sin \alpha_p - D_x \cos \alpha_p) + L \dot{L}) \right] \]
\[+ D_x \sin \alpha_p - 2M_{sr} \cos \alpha_t (D_x \cos \alpha_p + D_x \sin \alpha_p) (M_s L + \frac{1}{2} \rho L \dot{L}) \]
\[+ \frac{M_{sr}}{2M} (2M_s L + \rho L \dot{L}) (D_x \cos(\alpha_t - \alpha_p) + D_x \sin(\alpha_t - \alpha_p)) \]
\[ + 3M_{\text{srt}}(D_x \sin \alpha_p - D_z \cos \alpha_p)(D_x \cos \alpha_p + D_z \sin \alpha_p) \]
\[ + (M_\pi + \frac{1}{2} \rho \bar{L})L(D_x \cos(\alpha_t - \alpha_p) + D_z \sin(\alpha_t - \alpha_p)) \]
\[ - \frac{3}{M_\pi \bar{L}^2}(I_{xx} - I_{zz}) \cos \alpha_p \sin \alpha_p \]
\[ - 3(D_x \cos \alpha_p + D_z \sin \alpha_p) \cos \alpha_t(M_\pi L + \frac{1}{2} \rho \bar{L} L) \]
\[ + \rho \ddot{\bar{L}}[- \frac{M_{\text{srt}}}{2MM_\pi}L(-D_x \cos(\alpha_t - \alpha_p) - D_z \sin(\alpha_t - \alpha_p)) \]
\[ + \rho \ddot{\bar{L}} \frac{M_{\text{srt}}}{2MM_\pi}L(D_x \cos(\alpha_t - \alpha_p) - D_z \sin(\alpha_t - \alpha_p)) \]
\[ + \frac{M_{\text{srt}}}{2MM_\pi}L(\bar{D}_x \cos(\alpha_t - \alpha_p) - D_z \sin(\alpha_t - \alpha_p)(\dot{\beta} - \dot{\alpha}) \]
\[ + D_z \cos(\alpha_t - \alpha_p)(\dot{\beta} - \dot{\alpha}) + \dot{D}_z \sin(\alpha_t - \alpha_p)) = \tau; \]

**Tether Pitch Equation:**

\[ \frac{M_{\text{srt}}}{2M} \left[ -2L^2(-F - \ddot{\alpha}_t + F \dot{\alpha}_t) - 4L \dddot{\bar{L}}(1 - \dot{\alpha}_t) \right] \]
\[ + \frac{M_p}{M} \left[(\ddot{\bar{D}}_x - F \ddot{D}_x)L \cos(\alpha_t - \alpha_p) + (\ddot{\bar{D}}_z - F \ddot{D}_z)L \sin(\alpha_t - \alpha_p) \right] \]
\[ - 2L \ddot{D}_z \sin(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p) + 2 \ddot{D}_x \cos(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p) \]
\[ - D_x L(-F - \ddot{\alpha}_p + F \dot{\alpha}_p) \sin(\alpha_t - \alpha_p) + D_z L(-F - \ddot{\alpha}_p + \dot{\alpha}_p F) \cos(\alpha_t - \alpha_p) \]
\[ - D_x L \cos(\alpha_t - \alpha_p)(1 - \alpha_p)^2 - D_z L \sin(\alpha_t - \alpha_p)(1 - \alpha_p)^2 \]
\[ - \frac{M_p}{2MM_\pi} \rho \bar{L}[(-\ddot{\bar{D}}_x - F \ddot{D}_x) \cos(\alpha_t - \alpha_p) - (\ddot{\bar{D}}_z - F \ddot{D}_z) \sin(\alpha_t - \alpha_p) \]
\[ + 2 \ddot{D}_x \sin(\alpha_t - \alpha_p)(\dot{\beta} - \dot{\alpha}) - 2 \ddot{D}_z \cos(\alpha_t - \alpha_p)(\dot{\beta} - \dot{\alpha}) + D_x(-F - \ddot{\alpha}_p) \]
\[ + F \dot{\alpha}_p \sin(\alpha_t - \alpha_p) - D_z(-F - \ddot{\alpha}_p + F \dot{\alpha}_p) \cos(\alpha_t - \alpha_p) \]
\[ + D_z \cos(\alpha_t - \alpha_p)(1 - \alpha_p)^2 + D_z \sin(\alpha_t - \alpha_p)(1 - \alpha_p)^2 \]
\[ + \frac{1}{M} \rho \left[(-F - \ddot{\alpha}_t F \dot{\alpha}_t)L^2 \ddot{\bar{L}} + 2(1 - \dot{\alpha}_t) \dot{\bar{L}} \ddot{\bar{L}} + (1 - \dot{\alpha}_t)L^2 \dddot{\bar{L}} \right] \]
Chapter 2. Mathematical Model

\[- \frac{\rho}{2M} [2L^2 \dot{L}(1 - \dot{\alpha_i}) + \frac{2}{3} L^2 \ddot{L}(-F - \ddot{\alpha}_i + F\dot{\alpha}_i) + \frac{4}{3}(1 - \dot{\alpha}_i)\dot{\alpha}_i] + \frac{\rho^2}{2MM_s} [2L^2 \ddot{L}(1 - \dot{\alpha}_i) + L \dddot{\alpha}_i(1 - \dot{\alpha}_i) + \frac{1}{2} L^2 \dddot{L}(-F - \ddot{\alpha}_i + F\dot{\alpha}_i)]
\]
\[- \frac{1}{M_s(1 + e \cos \theta)} [-\frac{3}{2MM_s}(-2 \cos \alpha_t \sin \alpha_t(M_s^2 L^2 + \frac{1}{4} \rho^2 L^2 \ddot{L}^2)
\]
\[+ M_s \rho L^2 \ddot{L} + 2(D_x \sin \alpha_p - D_z \cos \alpha_p)M_{srt} \sin \alpha_t(M_s L + \frac{1}{2} \rho L \ddot{L})]
\[+ \frac{M_{srt}}{2MM_s} [(2M_s L + \rho L \ddot{L})(D_x \cos(\alpha_t - \alpha_p) + D_z \sin(\alpha_t - \alpha_p))]
\[+ 3(D_x \sin \alpha_p - D_z \cos \alpha_p) \sin \alpha_t(M_s L + \frac{1}{2} \rho L \ddot{L}) - (M_s L)
\[+ \frac{1}{2} \rho L \ddot{L})(D_x \cos(\alpha_t - \alpha_p) + D_z \sin(\alpha_t - \alpha_p)) - 3M_s L^2 \cos \alpha_t \sin \alpha_t
\]
\[- \rho L^2 \cos \alpha_t \sin \alpha_t] - \rho \frac{\ddot{L}}{Lb}(-\dot{r} \sin \alpha_t + r \dot{\theta} \cos \alpha_t)
\[- \rho \ddot{L}[\frac{M_{srt}}{2M}(-D_x \dot{\theta} - \dot{\alpha}_i) \sin(\alpha_t - \alpha_p)]
\[+ D_x (\dot{\theta} - \dot{\alpha}_i) \cos(\alpha_t - \alpha_p) + D_z \cos(\alpha_t - \alpha_p) + D_z \sin(\alpha_t - \alpha_p)]) = 0;
\]

Tether Length Equation:

\[\frac{M_{srt}}{M} [\dot{L} - F \ddot{L}] - L \ddot{L}(1 - \dot{\alpha}_i)^2 \] + \[\frac{M_p}{M}[(\ddot{D}_x - F \dot{D}_x) \ddot{L} \sin(\alpha_t - \alpha_p)] = (2.9)
\]
\[- (\ddot{D}_x - F \dot{D}_x) \dot{L} \cos(\alpha_t - \alpha_p) + 2 \ddot{D}_x \dot{L} \cos(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p)
\]
\[+ D_x \ddot{L} \cos(\alpha_t - \alpha_p)(-F - \ddot{\alpha}_i + f\dot{\alpha}_i) + D_x \ddot{L} \sin(\alpha_t - \alpha_p)(-F - \ddot{\alpha}_i + F\dot{\alpha}_i)
\]
\[- D_x \ddot{L} \sin(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p)^2 + 2 \ddot{D}_x \ddot{L} \sin(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p)
\]
\[+ D_x \ddot{L} \cos(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p)^2 + \frac{M_p}{2MM_s} \rho \ddot{L}^2[(\ddot{D}_x - F \dot{D}_x) \sin(\alpha_t - \alpha_p)
\]
\[- (\ddot{D}_x - F \dot{D}_x) \cos(\alpha_t - \alpha_p) + 2 \ddot{D}_x \cos(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p)
\]
\[+ D_x \cos(\alpha_t - \alpha_p)(-F - \ddot{\alpha}_i + F\dot{\alpha}_i) - D_x \sin(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p)^2
\]
\[+ 2 \ddot{D}_x \sin(\alpha_t - \alpha_p)(1 - \dot{\alpha}_p) + D_x(-F - \ddot{\alpha}_i + F\dot{\alpha}_i) \sin(\alpha_t - \alpha_p)
\[ + D_z \cos(\alpha_t - \alpha_p)(1 - \hat{\alpha}_t)^2] - \frac{1}{M}\rho[L^2(\ddot{\varepsilon} - F\dot{\varepsilon}) + 3\ddot{L}\ddot{L} - 3\sigma^2(\ddot{\alpha}_t)^2] + \frac{2}{M}\left[\frac{1}{2}(\ddot{\varepsilon} - F\dot{\varepsilon})\ddot{L}^2 \right.
\]
\[ + \frac{3}{2}\ddot{L}(\ddot{L} - F\dot{\varepsilon})\ddot{L} + L\ddot{L}^2 - L\ddot{L}^2(1 - \hat{\alpha}_t)^2 - \frac{\rho}{M}\left[\frac{1}{2}(\ddot{\varepsilon} - F\dot{\varepsilon})\ddot{L}^2 \right.
\]
\[ + 2\ddot{L}^2 \ddot{L} + (\ddot{L} - F\dot{\varepsilon})\ddot{L}L - \frac{1}{3}\ddot{L}L^2(1 - \hat{\alpha}_t)^2 - \frac{\rho^2}{2M}\left[\frac{1}{2}(\ddot{\varepsilon} - F\dot{\varepsilon})\ddot{L}^4 \right.
\]
\[ + (\ddot{L} - F\dot{\varepsilon})L^2L + 2\ddot{L}^3 \ddot{L} + L\ddot{L}^2 - \frac{1}{2}\ddot{L}^3L(1 - \hat{\alpha}_t)^2] - \frac{1}{M\varepsilon(1 + \varepsilon\cos\theta)}\left[-\frac{3}{2}\rho\alpha_t(2M^2\ddot{L}L + \frac{1}{2}\rho\ddot{L}^2 + 2M\rho\ddot{L}^2) \right.
\]
\[ - 2M\alpha_t(D_x \sin\alpha_t - D_z \cos\alpha_t)(M\ddot{L} + \frac{1}{2}\rho\ddot{L}^2)\cos\alpha_t] + \frac{1}{2M}\left[2M^2\ddot{L} \ddot{L} \right.
\]
\[ + \frac{1}{2}\ddot{L}L^2\rho^2 + (2M\alpha_t\dot{L}\rho\ddot{L} + M\alpha_t\rho\ddot{L}^2)(D_x \sin(\alpha_t - \alpha_p) - D_z \cos(\alpha_t - \alpha_p)) \right.
\]
\[ + 2M\rho\ddot{L}^2] - (M\ddot{L} + \frac{1}{2}\rho\ddot{L}^2)\cos\alpha_t(D_x \sin(\alpha_t - \alpha_p) - D_z \cos(\alpha_t - \alpha_p) - 3M\ddot{L}L\cos^2\alpha_t \right.
\]
\[ - M\ddot{L}L + \rho\ddot{L}^2 \cos^2\alpha_t - \frac{1}{3}\rho\ddot{L}^2] + \rho\ddot{L}(-\frac{1}{2M}\ddot{L}L - \frac{1}{2M}\ddot{L}L) \right.
\]
\[ + \rho\ddot{L}(-\frac{1}{2M}\ddot{L}L - \frac{1}{4MM}\rho\dot{L}\ddot{L}) + \rho\ddot{L}(-\frac{1}{2M}\ddot{L}L - \frac{1}{2M}\ddot{L}L) \right.
\]
\[ - \frac{1}{4MM}\rho\ddot{L}^2 - \frac{1}{2MM}\rho\dot{L}\ddot{L} + \rho\ddot{L}(-\frac{1}{2M}\ddot{L}L - \frac{1}{2M}\ddot{L}L) \right.
\]
\[ - \frac{1}{M}\ddot{L}L^2\rho^2 + \dot{L}\left[-\frac{M\alpha_t}{2MM}\ddot{L}(\dot{\theta} - \ddot{\alpha})\cos(\alpha_t - \alpha_p) \right.
\]
\[ + D_x(\dot{\theta} - \ddot{\alpha})\sin(\alpha_t - \alpha_p) + D_x(\sin(\alpha_t - \alpha_p) - \dot{\theta}\cos(\alpha_t - \alpha_p)) \right.
\]
\[ + \frac{1}{2M}(\ddot{L}L + \ddot{L}L) - \frac{1}{2MM}\rho\ddot{L}(\frac{1}{2}\dot{L}\ddot{L}^2 + \ddot{L}\dot{L}) - \frac{1}{2MM}\rho\ddot{L}(\frac{1}{2}\dot{L}\ddot{L}^2 + \ddot{L}\dot{L}) = 0. \]

2.3 Linearized System

Since the goal is to use the Linear Quadratic Regulator approach to control the system, it is desired to see how closely a linearized version of equations 2.7-2.9 can approximate the system dynamics. Linearizing about the state \( \alpha_p = \alpha_t = \epsilon = 0 \) gives

\[ [M]\ddot{q} = [C]q + [K]q + [B]\ddot{u} + \ddot{P}, \]
i.e.
\[ \ddot{q} = [M]^{-1}[C]q + [M]^{-1}[K]q + [M]^{-1}[\bar{B}]\bar{u} + [M]^{-1}\bar{P}, \] (2.10)

where
\[ q = \begin{bmatrix} \alpha_p \\ \alpha_t \\ \epsilon \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} \bar{D}_x \\ \tau \\ \bar{D}_z \end{bmatrix}. \]

Here, \( \tau \) is the nondimensional torque produced by the platform momentum wheel. Details of the matrices involved are given in Appendix A. Note that if \( \bar{D}_x \) and \( \bar{D}_z \) are known then the offset positions and velocities can be determined by integration. Thus they need not be included in \( u \) and are time dependent functions in the above matrices.

Now letting
\[ \bar{x} = \begin{bmatrix} \dot{q} \\ \ldots \\ q \end{bmatrix}, \]
equation (2.10) becomes
\[ \dot{\bar{x}} = [A]\bar{x} + [B]u + P, \] (2.11)

where:
\[ [A] = \begin{bmatrix} [I] & [0] \\ \vdots & \vdots & \vdots \\ [M]^{-1}[C] & [M]^{-1}[K] \end{bmatrix}; \]
\[ [B] = \begin{bmatrix} [0] \\ \vdots \\ [\bar{B}] \end{bmatrix}; \quad P = \begin{bmatrix} [0] \\ \vdots \\ \bar{P} \end{bmatrix}. \]
Chapter 2. Mathematical Model

To assess the accuracy of the linearization, both the linear and nonlinear sets of equations were integrated numerically. Figure 2.4 compares the response after a severe disturbance in each of the degrees of freedom. Note that the behaviour is closely approximated by the linear equations.
**OFFSETS**

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</thead>
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<tr>
<td>$D_X$</td>
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</tr>
<tr>
<td>$D_Z$</td>
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**MASS PARAMETERS**

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</thead>
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</tr>
<tr>
<td>$M_s$</td>
<td>100 KG</td>
</tr>
<tr>
<td>$M_r$</td>
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<tr>
<td>$\rho$</td>
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</table>

**INITIAL CONDITIONS**

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</thead>
<tbody>
<tr>
<td>$\alpha_p(0)$</td>
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<tr>
<td>$\alpha_l(0)$</td>
<td>$1^0$</td>
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<tr>
<td>$\varepsilon(0)$</td>
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<tr>
<td>$I_b$</td>
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</tbody>
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**ORBIT PARAMETERS**

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</tr>
<tr>
<td>$h$</td>
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</tbody>
</table>

**LEGEND**

- **NONLINEAR**
- **LINEAR**

Figure 2.4: Comparison between nonlinear and linear responses to a fixed initial disturbance