EXTENDED GROUP ANALYSIS OF THE WAVE EQUATION

By

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B. A. Sc. (Electrical Engineering) University of British Columbia, 1988

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

in
THE FACULTY OF GRADUATE STUDIES
DEPARTMENT OF MATHEMATICS
and
INSTITUTE OF APPLIED MATHEMATICS

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA
March 1990
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Abstract

A comprehensive study of potential symmetries admitted by partial differential equations is given using the wave equation $u_{tt} = c^2(x)u_{xx}$ as a given prototype equation, $R$. Methods are given for the construction of various conserved forms for $R$. Potential symmetries for $R$ are nonlocal symmetries realized as local symmetries of auxiliary systems obtained from conserved forms of $R$. The existence of potential symmetries for $R$ can be determined algorithmically and automatically by the use of a symbolic manipulation program. Most importantly, the potential symmetries obtained through one auxiliary system may or may not include and/or extend those obtained through another auxiliary system. The work in this thesis significantly extends the previously known classes of potential symmetries admitted by $R$ and results in a better understanding of the limits in the construction of potential symmetries for $R$. 
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Acknowledgement

I thank Professor George Bluman, my thesis supervisor, for his introduction of the subject and his guidance and valuable suggestions during the preparation of the manuscript.

I am indebted to Greg Reid for his computer assistance and his software packages that saved me tens of days of calculations.

Finally, I would like to thank Professor Brian Seymour for his reading of the final manuscript of the thesis.