Chapter 4

Linear Conserved Forms of the Wave Equation

4.1 Introduction to Linear Conserved Forms

From the last chapter on nonlinear conserved forms of the wave equation

\[ u_{tt} = c^2(x)u_{xx}, \]  \hspace{1cm} (4.1)

we see that if an auxiliary system is nonlinear it is likely that it is not useful in finding new symmetries of the wave equation (4.1). In this chapter we will consider some new linear conserved forms (Bluman[13]) of the wave equation (4.1).

We begin with a general auxiliary linear system for the wave equation (4.1). From this general linear system we can construct linear conserved forms of the wave equation (4.1).

Consider the linear system

\[ v_x = A(x,t)u_t + B(x,t)u, \] \hspace{1cm} (4.2a)
\[ v_t = C(x,t)u_x + D(x,t)u, \] \hspace{1cm} (4.2b)

where \((A, B, C, D)\) are some functions of \(x\) and \(t\).

Applying the integrability condition \(v_{xt} = v_{tx}\) to (4.2a,b) we have

\[ A_tu_t + Au_{tt} + B_tu + Bu_t = C_xu_x + Cu_{xx} + D_xu + Du_x. \] \hspace{1cm} (4.3)

Substituting the wave equation (4.1) into (4.3) and then setting to zero the coefficients of \(u, u_t, u_x\) and \(u_{xx}\) we have the following set of PDE's:

\[ B_t - D_x = 0, \] \hspace{1cm} (4.4a)
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\[ A_t + B = 0, \quad (4.4b) \]
\[ C_x + D = 0, \quad (4.4c) \]
\[ c^2(x)A - C = 0. \quad (4.4d) \]

The linear system (4.2a,b) corresponds to a conserved form of the wave equation (4.1) for any solution \((A, B, C, D)\) of (4.4a-d).

From (4.4b-d) we can express \(A, B\) and \(D\) in terms of \(C\):

\[ A = \frac{C}{c^2(x)}, \quad (4.5a) \]
\[ B = -\frac{C_t}{c^2(x)}, \quad (4.5b) \]
\[ D = -C_x. \quad (4.5c) \]

Then from (4.4a) we see that \(C\) must satisfy

\[ C_{tt} = c^2(x)C_{xx}, \quad (4.6) \]

i.e. \(C\) satisfies the wave equation (4.1). Hence, for any solution of the wave equation (4.6) there exists a linear conserved form of the wave equation (4.1) given by (4.2a,b). Since there exist infinitely many solutions of the PDE (4.6), it turns out that there are an infinite number of conserved forms of the wave equation (4.1) of the form (4.2a,b).

We can look for some particular solutions of (4.6) in order to construct some linear conserved forms given by (4.2a,b).

The simplest solution of (4.6) is \(C = 1\). From (4.5a-c) we have

\[ A = \frac{1}{c^2(x)} \quad \text{and} \quad B = D = 0. \]

The linear system (4.2a,b) then becomes

\[ v_x = \frac{u_t}{c^2(x)} , \]
\[ v_t = u_x , \]
which is the associated system (2.9a,b) considered previously.

We construct two linear conserved forms of the wave equation (4.1) by using two simple particular solutions of (4.6).

**Linear System I**

One particular solution of (4.6) is \( C = x \). In this case we have

\[
A = x/c^2(x), \quad B = 0, \quad D = -1.
\]

The system (4.2a,b) then becomes \( L_1\{x, t, u, v\} \):

\[
\begin{align*}
v_x &= \frac{xu_t}{c^2(x)}, \\
v_t &= xu_x - u.
\end{align*}
\]

(4.7a, 4.7b)

**Linear System II**

Another particular solution of (4.6) is \( C = t \). In this case we have

\[
A = t/c^2(x), \quad B = -1/c^2(x), \quad D = 0.
\]

The system (4.2a,b) then becomes \( L_2\{x, t, u, v\} \):

\[
\begin{align*}
v_x &= \frac{tu_t - u}{c^2(x)}, \\
v_t &= tu_x.
\end{align*}
\]

(4.8a, 4.8b)

In the next two sections we will give a group classification of each of the linear systems \( L_1\{x, t, u, v\} \) and \( L_2\{x, t, u, v\} \).
4.2 Group Classification of $v_x = xu_t/c^2(x)$, $v_t = xu_x - u$.

Let the linear system $L_1\{x, t, u, v\}$ given by (4.7a,b) admit an infinitesimal generator of the form

$$
X = \xi(x, t) \frac{\partial}{\partial x} + \tau(x, t) \frac{\partial}{\partial t} + (f_1(x, t)u + g_1(x, t)v) \frac{\partial}{\partial u} \\
+ (f_2(x, t)u + g_2(x, t)v) \frac{\partial}{\partial v}.
$$

(4.9)

Using Reid’s algorithm we classify the linear system $L_1\{x, t, u, v\}$ as follows:

**Case I:** $c(x) = Ax^2$ for arbitrary constant $A$. In this case the linear system $L_1\{x, t, u, v\}$ admits an infinite-parameter Lie group. (Recall that the scalar wave equation (4.1) admits an infinite-parameter Lie group if and only if $c(x) = (Ax + B)^2$.) We note that the wave speed $c(x)$ in this case is a degenerate form of the wave speed admitted by the scalar wave equation (4.1) in the infinite-group case, since the linear system $L_1\{x, t, u, v\}$ is not invariant under translations in $x$.

**Case II:**

$$
x(xH - 2)^2H''' + (5x^2(2 - xH)H' + 2x^3H^3 - 10x^2H^2 + 6xH + 12)H'' \\
+ 4x^3(H')^3 + (-4x^3H^2 + 10x^2H + 20x)(H')^2 \\
+ (-4x^2H^3 + 2xH^2 + 36H)H' - 4xH^4 + 12H^3 = 0,
$$

(4.10)

where $H = c'/c$.

Equation (4.10) is a fourth-order ODE for $c(x)$ and is invariant under a two-parameter Lie group of scalings in $x$ and scalings in $c$.

In this case $L_1\{x, t, u, v\}$ admits a four-parameter Lie group of point transformations.

The standard form of the determining equations is given by

$$
\frac{\partial \xi}{\partial x} = -\frac{x^2cc'' - x^2(c')^2 + 2c^2}{x^2cc' - 2xc^2} \xi,
$$

(4.11a)
\[
\begin{align*}
\frac{\partial \xi}{\partial t} &= -2xg_1, \\
\frac{\partial \tau}{\partial x} &= -\frac{2x}{c^2(x)}g_1, \\
\frac{\partial \tau}{\partial t} &= -\frac{x^2c'' - 2xc' + 2c}{x^2c' - 2xc} \xi, \\
\frac{\partial f_1}{\partial x} &= -\frac{c'(-x^3cc'' - 4c^2) + 2x^2c^2c'' + 2x^3c(c'')^2}{2xc(xc' - 2c^2)^2} \xi \\
&\quad - \frac{c''(-x^3c'' - 2x^2cc') + 4xc(c')^2}{2xc(xc' - 2c^2)^2} \xi, \\
\frac{\partial f_1}{\partial t} &= -\frac{x^2cc'' + x^2(c')^2 - 4xcx' + 2c^2}{xcc' - 2c^2} g_1, \\
\frac{\partial g_1}{\partial x} &= -\frac{xcc'' - x(c')^2 + cc'}{xcc' - 2c^2} g_1, \\
\frac{\partial g_1}{\partial t} &= -\frac{(-x^2c^2c' + 2xc^3)c'' + 2x^2c^2(c'')^2}{2xc(xc' - 2c)^2} \xi \\
&\quad - \frac{(-x^2c(c')^2 - 4xc^3c' + 4c^3)c'' + 2xc(c')^3 - 2c^2c'(^2)}{2xc(xc' - 2c)^2} \xi, \\
f_2 &= \frac{x^2}{c^2(x)} g_1, \\
g_2 &= \frac{-xc' + c}{xc} \xi + f_1.
\end{align*}
\]

Integrating (4.11a) we have

\[
\xi = F(x)G(t),
\]

where \(F(x) = \frac{xc}{x^2c' - 2c}\) and \(G(t)\) to be determined.

Substituting (4.12) into (4.11b) we have

\[
g_1 = -\frac{FG'}{2x}.
\]

Then by substituting (4.12) and (4.13) into (4.11h) we have

\[
\frac{(2xc^3 - x^2c^2c')c'' + 2x^2c^2(c'')^2 + (-x^2c(c')^2 - 4xcx' + 4c^3)c'' + 2xc(c')^3 - 2c^2c'^2}{(xc' - 2c)^2} = \lambda^2,
\]

\[
= \frac{G''(t)}{G(t)}
\]
where \( \lambda \) is a real or imaginary constant. One can show that (4.14) is an integral of the classification ODE (4.10). Note that (4.14) is invariant under only a one-parameter Lie group of scalings.

For \( \lambda \neq 0 \), the solution of (4.11a-j) is

\[
\begin{align*}
\xi &= \frac{xc}{xc' - 2c}(pe^{\lambda t} + qe^{-\lambda t}), \\
\tau &= -\lambda^{-1}\left(\frac{x^2cc'' - 2xcc' + 2c^2}{(xc' - 2c)^2}\right)(pe^{\lambda t} - qe^{-\lambda t}) + r, \\
f_1 &= -\lambda^{-1}\left(\frac{x^2cc'' + x^2(c')^2 - 4xcc' + 2c^2}{(xc' - 2c)^2}\right)(pe^{\lambda t} + qe^{-\lambda t}) + s, \\
g_1 &= -\frac{\lambda c}{2(xc' - 2c)}(pe^{\lambda t} - qe^{-\lambda t}), \\
f_2 &= -\frac{\lambda c^2}{2c(xc' - 2c)}(pe^{\lambda t} - qe^{-\lambda t}), \\
g_2 &= \left[\frac{-xc' + c}{xc' - 2c} - \frac{\lambda^{-1}x^2cc'' + x^2(c')^2 - 4xcc' + 2c^2}{(xc' - 2c)^2}\right](pe^{\lambda t} + qe^{-\lambda t}) + s, 
\end{align*}
\]

where \( \{p, q, r, s\} \) are arbitrary constants corresponding to the four parameters of the group.

The infinitesimal generators corresponding to (4.15a-f) are given by

\[
\begin{align*}
X_p &= e^{\lambda t}\left\{\frac{xc}{xc' - 2c}\frac{\partial}{\partial x} - \lambda^{-1}\left(\frac{x^2cc'' - 2xcc' + 2c^2}{(xc' - 2c)^2}\right)\frac{\partial}{\partial t} \\
&\quad - \left[\lambda^{-1}\left(\frac{x^2cc'' + x^2(c')^2 - 4xcc' + 2c^2}{(xc' - 2c)^2}\right)u + \frac{\lambda c}{2(xc' - 2c)}v\right]\frac{\partial}{\partial u}\right\}, \\
X_q &= e^{-\lambda t}\left\{\frac{xc}{xc' - 2c}\frac{\partial}{\partial x} + \lambda^{-1}\left(\frac{x^2cc'' - 2xcc' + 2c^2}{(xc' - 2c)^2}\right)\frac{\partial}{\partial t} \\
&\quad + \left[\lambda^{-1}\left(\frac{x^2cc'' + x^2(c')^2 - 4xcc' + 2c^2}{(xc' - 2c)^2}\right)u + \frac{\lambda c}{2(xc' - 2c)}v\right]\frac{\partial}{\partial u}\right\}, \\
X_r &= \frac{\partial}{\partial t},
\end{align*}
\]
\[ X_s = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}. \]

The nonzero commutators of the corresponding Lie algebra are

\[ [X_p, X_q] = \frac{2}{\lambda} \left\{ \frac{xc}{(x' - 2c)} \frac{d}{dx} \left( \frac{x^2cc'' - 2xcc' + 2c^2}{(x' - 2c)^2} \right) + \left( \frac{x^2cc'' - 2xcc' + 2c^2}{(x' - 2c)^2} \right)^2 \right\} X_r, \]

\[ [X_r, X_p] = \lambda X_p, \quad [X_r, X_q] = -\lambda X_q. \]

It immediately follows that

\[ \frac{xc}{(x' - 2c)} \frac{d}{dx} \left( \frac{x^2cc'' - 2xcc' + 2c^2}{(x' - 2c)^2} \right) + \left( \frac{x^2cc'' - 2xcc' + 2c^2}{(x' - 2c)^2} \right)^2 = \text{const} = \mu^2. \quad (4.16) \]

The third-order ODE (4.16) for \( c(x) \) is still invariant under a two-parameter Lie group of scalings in \( x \) and scalings in \( c \), and hence it can be reduced to a first-order ODE plus two quadratures. In particular one can let

\[ U = \frac{x'}{c}, \quad V = \frac{x^2cc'' - 2xcc' + 2c^2}{(x' - 2c)^2} \]

then (4.16) becomes

\[ \frac{dU}{dV} = \frac{UV - 2V - U + 1}{\mu^2 - V^2}. \quad (4.17) \]

Note that (4.17) is a linear first-order ODE. If \( U = \Phi(V) \) solves (4.17) then one can reduce the second-order ODE

\[ U(x, c, c') = \Phi(x, c, c', c'') \quad (4.18) \]

to two quadratures since (4.18) is invariant under a two-parameter group of scalings in \( x \) and scalings in \( c \).

One can show that (4.14) and (4.16) are two independent integrals of the classification ODE (4.10) and both (4.16) and (4.10) are invariant under the same two-parameter Lie group of scalings in \( x \) and scalings in \( c \).
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It should be emphasized that the commutator relation preserves all the symmetries admitted by the ODE (4.10), since both (4.10) and (4.16) are invariant under the same two-parameter group. Consequently, the order of ODE (4.10) can be reduced by three even though (4.10) admits only a two-parameter group!

Two of the local point symmetries, $X_p$ and $X_q$, admitted by $L_1 \{x, t, u, v\}$ are nonlocal (potential) symmetries of the scalar wave equation (4.1). Moreover, the potential symmetries of (4.1) obtained through the use of $L_1 \{x, t, u, v\}$ are all nontrivial potential symmetries of type I, since the infinitesimals $\xi$ and $\tau$ admitted by $L_1 \{x, t, u, v\}$ are essentially different from those admitted by (4.1). Most importantly, these new potential symmetries are beyond those potential symmetries obtained through the use of the associated system $S \{x, t, u, v\}$ and the cascaded system $T_2 \{x, t, u, v, w\}$.

The classification ODE (4.10) for the linear system $L_1 \{x, t, u, v\}$ is distinct from all the classification ODE’s for the wave equation (4.1) and the auxiliary systems studied previously. Hence, we can find new wave speeds such that the wave equation (4.1) admits new potential symmetries. Moreover, one can apply the method given in [8] to find a common solution set of ODE’s (1.33) and (4.10). Knowing this common solution set for $c(x)$ one can find new potential symmetries admitted by the wave equation (4.1) through the use of the linear system $L_1 \{x, t, u, v\}$.

Case III: $c(x)$ arbitrary. In this case the system $L_1 \{x, t, u, v\}$ is invariant only under translations in $t$ and scalings in $u$ and $v$.

In summary we have the following theorems:

**Theorem 4.1** The linear system $L_1 \{x, t, u, v\}$ is invariant under an infinite-parameter Lie group of point transformations if and only if the wave speed $c(x) = Ax^2$ for arbitrary constant $A$.

**Theorem 4.2** The linear system $L_1 \{x, t, u, v\}$ is invariant under a four-parameter Lie
group of point transformations if and only if the wave speed $c(x)$ satisfies the fourth-order ODE (4.10).

**Theorem 4.3** For any wave speed $c(x)$ satisfying the fourth-order ODE (4.10), two of the four point symmetries of the linear system $L_1\{x,t,u,v\}$ correspond to potential symmetries of the wave equation (4.1).

**Theorem 4.4** The potential symmetries of the wave equation (4.1) obtained through the use of the linear system $L_1\{x,t,u,v\}$ are all nontrivial potential symmetries of type I beyond those obtained by $S\{x,t,u,v\}$ and $T_2\{x,t,u,v,w\}$.

**Theorem 4.5** For any wave speed $c(x)$ simultaneously satisfies (1.33) and (4.10) the wave equation (4.1) admits new potential symmetries through $L_1\{x,t,u,v\}$ beyond those obtained by $S\{x,t,u,v\}$ and $T_2\{x,t,u,v,w\}$.

**Theorem 4.6** For any other wave speeds the linear system $L_1\{x,t,u,v\}$ is invariant under a trivial two-parameter Lie group of translations in $t$ and uniform scalings in $u$ and $v$. 
4.3 Group Classification of $v_x = (tu_t - u)/c^2(x)$, $v_t = tu_x$.

Let the linear system $L_2\{x, t, u, v\}$ given by (4.8a,b) admit an infinitesimal generator of the form

$$X = \xi(x, t) \frac{\partial}{\partial x} + \tau(x, t) \frac{\partial}{\partial t} + (f_1(x, t)u + g_1(x, t)v) \frac{\partial}{\partial u} + (f_2(x, t)u + g_2(x, t)v) \frac{\partial}{\partial v}.$$  \hspace{1cm} (4.19)

Using Reid’s algorithm we classify the linear system $L_2\{x, t, u, v\}$ as follows:

**Case I:**

$$c'' + \frac{(c')^2}{2c} = 0.$$ \hspace{1cm} (4.20)

The general solution of (4.20) is

$$c(x) = (Ax + B)^{2/3},$$ \hspace{1cm} (4.21)

for arbitrary constants $A$ and $B$. In this case the linear system $L_2\{x, t, u, v\}$ admits a four-parameter Lie group of point transformations. (We recall that for the wave speed given by (4.21) the associated wave equation $\tilde{R}\{x, t, v\}$ given by (2.10) is invariant under an infinite-parameter group.) One can show that the infinitesimals $\{\xi, \tau\}$ for the independent variables admitted by the linear system $L_2\{x, t, u, v\}$ are the same as those admitted by the associated wave equation $\tilde{R}\{x, t, v\}$ and the cascaded system $T_2\{x, t, u, v, w\}$.

**Case II:**

$$cc'c'' - 2c(c')^2 + (c')^2c'' = 0.$$ \hspace{1cm} (4.22)

The general solution of (4.22) is

$$c(x) = (Ax + B)^c,$$ \hspace{1cm} (4.23)
for any constant $C \neq 0, \frac{2}{3}$. In this case the linear system $L_2\{x, t, u, v\}$ admits a two-parameter Lie group of point transformations with the corresponding infinitesimal generators given by

$$
X_1 = (Ax + B)\frac{\partial}{\partial x} - A(C - 1)t\frac{\partial}{\partial t} - A(2C - 1)v\frac{\partial}{\partial v}, \quad (4.24a)
$$

$$
X_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \quad (4.24b)
$$

(We recall that for the wave speed given by (4.23) with $C = 2$ the wave equation (4.1) is invariant under an infinite-parameter group, and for the case where $C \neq 0, 2$ the wave equation (4.1) is invariant under a four-parameter group.) We see that none of the local symmetries (4.24a,b) admitted by the linear system $L_2\{x, t, u, v\}$ corresponds to a potential symmetry of the wave equation (4.1).

**Case III:** $c(x)$ arbitrary. In this case the linear system $L_2\{x, t, u, v\}$ admits only a one-parameter Lie group of scalings in both $u$ and $v$. (We recall that for arbitrary wave speed the wave equation (4.1) and its related systems studied previously are invariant under scalings in the dependent variables and translations in $t$.) The translational invariance in $t$ is lost since the right-hand sides of $L_2\{x, t, u, v\}$ depend explicitly on $t$.

From the above group classification of the linear system $L_2\{x, t, u, v\}$ we see that the size of the group admitted by the wave equation (4.1) is reduced when (4.1) is embedded in the linear system $L_2\{x, t, u, v\}$. Moreover, the use of $L_2\{x, t, u, v\}$ leads to a more restrictive class of wave speeds $c(x)$ admitted by the wave equation (4.1).
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4.4 Chapter Summary

In this chapter we have studied linear conserved forms of the wave equation (4.1). We have considered a general linear system given by (4.2a,b) with arbitrary coefficients satisfying a system of PDE's given by (4.4a-d). It turns out that there exist infinitely many linear conserved forms of the wave equation (4.1). Two new linear systems have been analysed in detail.

The linear system I given by $L_1\{x, t, u, v\}$ admits an infinite-parameter Lie group of point transformations if and only if the wave speed $c(x) = Ax^2$ for arbitrary constant $A$, and it admits a four-parameter local Lie group of point transformations if and only if the wave speed $c(x)$ satisfies the fourth-order ODE given by (4.10) which is distinct from the other classification ODE's found previously. For any wave speed $c(x)$ satisfying (4.10) two of the four local symmetries admitted by $L_1\{x, t, u, v\}$ correspond to nontrivial type I potential symmetries of the wave equation (4.1). Most importantly, these new potential symmetries are beyond those obtained by $S\{x, t, u, v\}$ and $T_2\{x, t, u, v, w\}$. Moreover, for any wave speed $c(x)$ simultaneously satisfying (1.33) and (4.10) there exist new potential symmetries of the wave equation (4.1) through the linear system $L_1\{x, t, u, v\}$. For any other wave speeds the linear system $L_1\{x, t, u, v\}$ admits only a trivial two-parameter Lie group of scalings in the dependent variables and translations in $t$.

The linear system II given by $L_2\{x, t, u, v\}$ admits a four-parameter local Lie group of point transformations if and only if the wave speed $c(x) = (Ax + B)^{2/3}$ for arbitrary constants $A$ and $B$, in which case one can show that the infinitesimals for the independent variables admitted by $L_2\{x, t, u, v\}$ are the same as those admitted by the associated wave equation $\bar{R}\{x, t, v\}$. For the case $c(x) = (Ax + B)^C$ where $C \neq 0, \frac{2}{3}$, the linear system $L_2\{x, t, u, v\}$ admits a two-parameter local Lie group of point transformations which correspond to local symmetries of the wave equation (4.1). For any other wave speeds
c(x) the linear system $L_2\{x, t, u, v\}$ admits only a trivial one-parameter Lie group of scalings in the dependent variables. By allowing the right-hand sides of the linear system to depend explicitly on $t$, the translational symmetry in $t$ is lost.