CHAPTER 1
LINEAR STABILITY ANALYSIS

1.1 Formulation of the model

The case that we consider is that of a velocity discontinuity along a submarine ridge between two layers with different, uniformly distributed densities and no crossfrontal velocity. The front, in the basic steady laminar state, will stretch linearly from the ridge toward the heavier side in order that pressure gradient forces counteract the Coriolis forces (see fig. 1.1). The basic properties in the upper layer, referred to as layer 1, are density, \( \rho_1 \), alongridge velocity, \( U_1 \), and thickness, \( h_1(y) \), while those in the lower layer, referred to as layer 2, are density, \( \rho_2 \), alongridge velocity, \( U_2 \), and thickness, \( h_2(y) \). The crest of the submarine ridge is at a depth \( H \) under the surface and the front surfaces at a distance \( L \) from the ridge. The system is rotating at a period \( 4 \pi/f \), where \( f \) is the Coriolis parameter, assumed constant (\( f \)-plane approximation). We assume also there is no surface wave (rigid lid approximation).

In the basic state, the pressure gradient force, exerted horizontally on a water parcel, is in geostrophic balance with the crossfrontal Coriolis force in both layers:

\[
 f \rho_j U_j = - \frac{\partial P_j}{\partial y},
\]

(1.1)

where \( j=1,2 \) are indices of the layers.

The coordinate system is defined such that the origin of the crossfrontal coordinate, \( y \), is at the crest of the ridge, and that of the depth coordinate \( z \) is at the level where the front surfaces. The \( x \)-direction is into the page.
Figure 1.1. Two layers bordered by a submarine ridge, with density $\rho_1, \rho_2$, (the denser layer in region III), alongridge velocities $U_1, U_2$, and thicknesses $h_1(y), h_2(y)$, respectively. The top of the ridge is at a depth $H$. The frontal width is $L$ and depends on velocity shear and density difference. The pycnocline in the laminar situation is represented by the dashed oblique line starting from the crest.

Defining the surface pressure to be zero as a boundary condition, the integration of (1.1) yields

(1.2a) \[ P_1(y,z) = \rho_1 gz - f \rho_1 U_1 (y - L) \] in layer 1,

(1.2b) \[ P_2(y,z) = \rho_2 gz - f \rho_2 U_2 (y - L) \] in region III of layer 2.

The slope of the pycnocline depends on the density contrast and on the velocity shear, which are both constant for vertically uniform layers. Thus the depth of the pycnocline relative to the level where the front crops out is $H/(1-y/L)$. The geostrophic term, which couples the layer momentum and the Earth rotation, causes the sea surface to be slightly slanted, i.e. not quite horizontal ($z_{surface} = (y-L)U_1/f/g$). However, the order of the surface variation is negligible relative to the front and topographic feature.

The pressure equation under the pycnocline (layer 2 in region II) is

\[ P_2(y,z) = P_1(y,H(1-y/L)) + \rho_2 g(z - H(1-y/L)) \]

\[ = \rho_1 g H(1-y/L) - f \rho_1 U_1 (y - L) + \rho_2 g(z - H(1-y/L)). \]

The geostrophic balance under the pycnocline (layer 2, region II) yields
\[ f \rho_2 U_2 = -\frac{\partial P}{\partial y} = (\rho_1 - \rho_2) \frac{gH}{L} + f \rho_1 U_1 \quad \text{or} \]
\[ f(\rho_2 U_2 - \rho_1 U_1) = (\rho_1 - \rho_2) \frac{gH}{L}. \]

(1.3)

The left-hand side of (1.3) is the product of the Coriolis parameter and the momentum shear. Using the Boussinesq approximation, the density difference contributes negligibly to the momentum shear and the geostrophic balance can be rewritten

\[ f \rho \Delta U = (\rho_2 - \rho_1) \frac{gH}{L}, \]

where \( \Delta U = U_1 - U_2 \) is the velocity shear and \( \rho \) is a reference density. The representation of our model implies \( U_1 > U_2 \). The frontal length can be expressed as a function of the density difference and the velocity shear

\[ L = \frac{g' H}{f \Delta U}, \]

where \( g' = (\rho_2 - \rho_1)g/\rho \) is the reduced-gravity parameter.

The physical features vary only with the crossfrontal coordinate, \( y \). In the linear stability analysis, the perturbation fields can be treated by separation of variables, and the parts in \( x \) and \( t \) are harmonic fields.

The perturbed state is represented for each layer by four fields:

\[ \tilde{u}_j(x, y, z, t) = U_j + \varepsilon u_j(y) e^{i(kx - \sigma t)} + O(\varepsilon^2) \]
\[ \tilde{v}_j(x, y, z, t) = \varepsilon v_j(y) e^{i(kx - \sigma t)} + O(\varepsilon^2) \]
\[ \tilde{p}_j(x, y, z, t) = P_j(z) + \varepsilon p_j(y) e^{i(kx - \sigma t)} + O(\varepsilon^2) \]
\[ \tilde{h}_j(x, y, t) = h_j(y) + (-1)^j \varepsilon \eta_j(y) e^{i(kx - \sigma t)} + O(\varepsilon^2) \]

where

\( \sigma \): complex frequency (the real part is the physical frequency of a wave traveling in the \( x \)-direction and the imaginary part is the temporal growth rate)

\( k \): wave number (assumed real)

\( \varepsilon \): small parameter for the ratio of the scales of the initial disturbance and the basic state.
The variables $u_j(y)$, $v_j(y)$, $p_j(y)$ are the first-order terms of the perturbation fields of alongfrontal velocity, crossfrontal velocity and pressure, respectively. The variable $\eta(y)$ is the first-order term of the perturbation in the thickness fields. It represents an anomalous elevation of the pycnocline.

The governing equations in each layer are the nonlinear, vertically-averaged, inviscid shallow water equations without friction:

\begin{align}
\frac{\partial \tilde{u}_j}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_j}{\partial x} + \tilde{v}_j \frac{\partial \tilde{u}_j}{\partial y} - f\tilde{v}_j &= \frac{-1}{\rho} \frac{\partial \tilde{p}_j}{\partial x}, \\
\frac{\partial \tilde{v}_j}{\partial t} + \tilde{u}_j \frac{\partial \tilde{v}_j}{\partial x} + \tilde{v}_j \frac{\partial \tilde{v}_j}{\partial y} + f\tilde{u}_j &= \frac{-1}{\rho} \frac{\partial \tilde{p}_j}{\partial y}, \\
\frac{\partial \tilde{h}_j}{\partial t} + \tilde{u}_j \frac{\partial \tilde{h}_j}{\partial x} + \tilde{v}_j \frac{\partial \tilde{h}_j}{\partial y} &= -\tilde{h}_j \left( \frac{\partial \tilde{u}_j}{\partial x} + \frac{\partial \tilde{v}_j}{\partial y} \right) \tag{1.4c}
\end{align}

Eqns. 1.4a and 1.4b are the along- and crossridge momentum equations, respectively. Equation 1.4c is the continuity equation and $\tilde{h}_j$ is same as $h_j$ in regions I and III, but becomes the deformed layer thickness inside the frontal region (II). The separated fields are substituted into the system of governing equations, and the linearization is performed by removing zeroth order ($\mathcal{O}^0$) terms, that correspond to the initial balance, (1.1), and the products of perturbation elements that correspond to second order ($\mathcal{O}^2$) and higher terms. Only first-order ($\mathcal{O}^1$) terms remain. Replacing the $x$- and $t$-derivative operators by the complex factors $ik$ and $i\sigma$, the momentum equations become

\begin{align}
 i(kU_j - \sigma)u_j - f v_j &= \frac{-ik}{\rho} p_j, \\
 i(kU_j - \sigma)v_j + f u_j &= \frac{-1}{\rho} \frac{\partial p_j}{\partial y}.
\end{align}

Let $\omega_j = (kU_j - \sigma)/f$, (ratio of the Doppler-shifted frequency over the Coriolis parameter).

The solutions for the velocity fields in terms of the pressure field are

\begin{align}
 u_j = \frac{\omega_j k p_j - p_{j,xx}}{(1 - \omega_j^2) \rho f}, \tag{1.5a}
\end{align}
\[ v_j = i \frac{kp_j - \omega_j p_{j,y}}{(1 - \omega_j^2) \rho f}. \]

(1.5b)

The subscript \( y \) after the layer index represents differentiation with respect to \( y \). The perturbed velocity fields can be determined only if the system does not oscillate at the inertial frequency.

The continuity equation appears differently, according to the region. Outside of the frontal region (regions \( I \) and \( III \) respectively), the rigid lid approximation states that

\[ \frac{\partial \tilde{h}_1}{\partial t} = 0, \quad \text{thus} \quad \frac{\partial \tilde{h}_1}{\partial x} = 0. \]

In the frontal region (\( II \)),

\[ \tilde{h}_1 = H(1 - y/L) - \varepsilon \eta(x, y, t), \]
\[ \tilde{h}_2 = h_2(y) + \varepsilon \eta(x, y, t). \]

where \( \varepsilon \eta \) is the vertical displacement of the pycnocline.

The pressure of a water column is continuous at the pycnocline. The pressure under the pycnocline is then

\[ P_2(y, z) + \varepsilon p_2 = P_1(y, H(1 - y/L)) - \rho g \varepsilon \eta + \rho_2 g (z - H(1 - y/L) + \varepsilon \eta). \]

(1.6)

Removing the initial balance yields

\[ \eta = \frac{p_2 - p_1}{\rho g'}. \]

(1.7)

The substantial derivative of the layers' thickness in region \( II \) yields:

\[ \frac{\partial \tilde{h}_1}{\partial t} + U_1 \frac{\partial \tilde{h}_1}{\partial x} = -\varepsilon \left( \frac{\partial \eta}{\partial t} + U_1 \frac{\partial \eta}{\partial x} \right) = -i \omega_1 f \varepsilon \eta = -\frac{i \omega_1 f \varepsilon}{\rho g'} (p_2 - p_1) \]

and

\[ \frac{\partial \tilde{h}_2}{\partial t} + U_2 \frac{\partial \tilde{h}_2}{\partial x} = + \varepsilon \left( \frac{\partial \eta}{\partial t} + U_2 \frac{\partial \eta}{\partial x} \right) = +i \omega_2 f \varepsilon \eta = +\frac{i \omega_2 f \varepsilon}{\rho g'} (p_2 - p_1). \]

In regions \( I \) and \( III \), (1.4c) is reduced to

\[ v_j h_{j,y} = -h_j \left( i k u_j + v_{j,y} \right) \]

or, by substituting (1.5),
\[(1.8) \quad (kp_j - \omega_j p_{j,y}) \frac{\partial}{\partial y} \log(h_j(y)) = -\omega_j \left(k^2 p_j - p_{j,yy}\right).\]

In region II (the front), for the upper layer, (1.4c) becomes
\[-\frac{H}{L} v_1 - i\omega_f \eta = -H(1 - y/L)(iku_1 + v_{1,y})\]

or, using (1.5) and (1.7),
\[(1.9) \quad \frac{f^2}{g'H}(1 - \omega_1^2)(p_2 - p_1) = (1 - y/L)(k^2 p_1 - p_{1,yy}) + \frac{p_{1,y} - kp_1/\omega_1}{L}\]

and for the lower layer, (1.4c) becomes
\[v_2 h_{2,y}(y) + i\omega_2 \eta = -h_2(y)(iku_2 + v_{2,y})\]

or, using (1.5) and (1.7),
\[(1.10) \quad \frac{f^2}{g'}(1 - \omega_2^2)(p_2 - p_1) = \left(p_{2,y} - \frac{kp_2}{\omega_2}\right)h_{2,y}(y) - h_2(y)(k^2 p_2 - p_{2,yy}).\]

The crossridge direction is made nondimensional by dividing by the frontal length:
\[y \rightarrow Ly, \quad \frac{\partial}{\partial y} \rightarrow \frac{1}{L} \frac{\partial}{\partial y},\]

thus
\[u_j = \frac{\omega_j kp_j - p_{j,y}}{(1 - \omega_j^2)\rho f L}, \quad v_j = i \frac{K p_j - \omega_j p_{j,y}}{(1 - \omega_j^2)\rho f L}.\]

where \(K = kL\).

Let \(\hat{h}_1(y) = h_1(Ly)\) and \(\hat{h}_2(y) = h_2(Ly)\).

The equation for outside the frontal region, (1.8), becomes
\[(1.11) \quad \left(\frac{K}{\omega_j} p_j - p_{j,y}\right) \frac{\partial}{\partial y} \log(\hat{h}_j(y)) = -K^2 p_j + p_{j,yy}.\]

The equations for inside the frontal region, (1.9) and (1.10), become
\[(1.12a) \quad \frac{f^2 L^2}{g'H}(1 - \omega_1^2)(p_2 - p_1) = (1 - y)(K^2 p_1 - p_{1,yy}) + \frac{p_{1,y} - kp_1}{\omega_1} p_1,\]
\[(1.12b) \quad \frac{f^2 L^2}{g'H}(1 - \omega_2^2)(p_1 - p_2) = \frac{\hat{h}_1(y)}{H}(K^2 p_2 - p_{2,yy}) + \frac{\hat{h}_{2,y}(y)}{H} \left(\frac{K}{\omega_2} p_2 - p_{2,y}\right).\]

Let
\[
\beta_1 = \frac{f^2 E}{g' H} (1 - \omega_1^2), \quad \beta_2^* = \frac{f^2 E}{g' H} (1 - \omega_2^2), \quad \lambda_1 = \beta_1 - K/\omega_1,
\]

then (1.12a) becomes
\[
(1 - y) \left( K^2 p_1 - p_{1,yy} \right) + p_{1,y} + \lambda_1 p_1 = \beta_1 p_2,
\]
and (1.12b) becomes
\[
\frac{\hat{h}_2(y)}{H} \left( K^2 p_2 - p_{2,yy} \right) - \frac{\hat{h}_{2,y}(y)}{H} p_{2,y} + \left( \beta_2^* + \frac{K \hat{h}_{2,y}(y)}{\omega_2} \right) p_2 = \beta_2^* p_1.
\]

These can be rearranged as
\[
(1.13a) \quad \left[ (1 - y) \left( \frac{d^2}{dy^2} - K^2 \right) - \frac{d}{dy} - \lambda_1 \right] p_1 = -\beta_1 p_2,
\]
\[
(1.13b) \quad \left[ -\frac{\hat{h}_2(y)}{H} \left( \frac{d^2}{dy^2} - K^2 \right) - \frac{\hat{h}_{2,y}(y)}{H} \frac{d}{dy} + \beta_2^* \frac{\hat{h}_{2,y}(y)}{H} \right] p_2 = \beta_2^* p_1.
\]

This system of ordinary differential equations will use as boundary conditions the values of \(p_1, p_{1,y}\) of region I at \(y=0\), and of \(p_2, p_{2,y}\) of region III at \(y=0\).
1.2. Triangular Ridge

Consider the case of a triangular ridge (see fig.1.2) with bottom located at \( H(1 - \gamma_1 y) \) in region I, and at \( H(1 + \gamma_2 y) \) in regions II and III, where \( \gamma_1, \gamma_2 \), are nondimensionalised slopes i.e. slopes of the ridge divided by the pycnocline slope \( (H/L) \). In this section, we are going to solve for the pressure perturbation field in each layer and region by series expansions.

![Diagram of Triangular Ridge](image)

Figure 1.2. Same as Fig. 1.1, except that layer thicknesses are defined as \( \hat{h}_1(y) = H(1 - \gamma_1 y) \) in region I, as \( \hat{h}_2(y) = H(1 + \gamma_2 y) \) in region III, and, in region II, as \( \hat{h}_1(y) = H(1 - y), \hat{h}_2(y) = H(1 + \gamma_2) y \) for the upper and lower layers respectively. \( \varepsilon \eta \) is the displacement of the perturbed pycnocline.

Region I:

In region I, the differential equation for the perturbation pressure, (1.11), becomes

\[
(1 - \gamma_1 y) \left( p_{1,yy} - K^2 p_1 \right) - \gamma_1 p_{1,y} + \gamma_1 \frac{K}{\omega_1} p_1 = 0.
\]

The problem can be simplified in two steps. First, change the independent variable \( y \) to \( z = 1 - y \); so (1.14) becomes

\[
z \left[ \gamma_1^2 p_{1,zz} - K^2 p_1 \right] + \gamma_1^2 p_{1,z} + \frac{\gamma_1 K}{\omega_1} p_1 = 0.
\]

Second, far away from the front, the perturbation pressure field must decay and become smooth. The balance inside the square brackets must vanish:
\[ \gamma_1^2 p_{1,z} - K^2 p_1 \approx 0 \text{ as } z \to \infty. \]

Thus, the asymptotic behaviour of the solution is dominated by an exponential behaviour. Assuming that \( p_1 = A_z p(z, \gamma_1) \exp(-Kz/\gamma_1) \), (1.15) becomes

\[ z p_{zz} + \left(1 - \frac{2K}{\gamma_1} z\right) p_z - \frac{K}{\gamma_1} \left(1 - \frac{1}{\omega_1}\right) p = 0. \]

Making the substitution \( \alpha_i = \frac{1}{2}(1 - 1/\omega_1) \) and the change of variable \( Z_i = 2Kz/\gamma_1 \) gives

\[ Z_i p_{Z_i Z_i} + (1 - Z_i) p_{Z_i} - \alpha_i p = 0, \]

which is a confluent hypergeometric equation.

Classical solutions of (1.17) are already known (see Erdelyi [1954]). The differential equation shows a singular point at \( Z_i = 0 \) (\( y = 1/\gamma_1 \)), which is not part of the region \( I \). The global solution must behave in the far field such that it either decays smoothly or grows in some nonexponential manner:

\[ \left| p(Z_i) e^{-Z_i/2} \right| \to 0 \quad \text{and} \quad \left| \frac{d}{dZ_i} p(Z_i) e^{-Z_i/2} \right| \to 0 \quad \text{as } Z_i \to \infty. \]

This particular solution of (1.17) is known as the Tricomi solution:

\[ p(Z_i) = \Psi(\alpha_1, 1; Z_i). \]

For a development that covers all the range beyond \( Z_i = 2K/\gamma_1 \), it is numerically not practical to use a basis of the classical solutions. The solution is tediously exposed in Appendix A. It includes an asymptotic expansion from infinity, a sequence of intermediary matchings between series expansions (which can be replaced by a numerical solution of the ordinary differential equation) that extend the asymptotic solution to the point \( Z_i = 2K/\gamma_1 \). At this point, the solution provides two boundary conditions at \( y = 0 \) for \( p_1 \):

\[ p_1'(0) = A_z e^{-Z_i/2} \Psi(\alpha_1, 1; Z_i) \big|_{Z_i = 2K/\gamma_1}, \quad \text{and} \]

\[ p_1(0) = \text{constant}. \]
\[ p_{1,y}^I(0) = A_I \frac{dZ_1}{dy} e^{-Z_1/2} \left( -\frac{\Psi(\alpha_1,1;Z_1)}{2} + \frac{\partial \Psi}{\partial Z_1}(\alpha_1,1;Z_1) \right)_{Z_1=2K/\gamma_1} \]

(1.18b)

\[ = A_I K e^{-Z_1/2} \left( \Psi(\alpha_1,1;Z_1) - 2 \frac{\partial \Psi}{\partial Z_1}(\alpha_1,1;Z_1) \right)_{Z_1=2K/\gamma_1}. \]

Region III:

In a process similar to that of Region I, substitute \( z = 1 + \gamma_2 y \) into the differential equation for Region III, which becomes

(1.19)

\[ z \left[ \gamma_2^2 p_{2,z} - K^2 p_2 \right] + \gamma_2^2 p_{2,z} - \frac{\gamma_2 K}{\omega_2} p_2 = 0. \]

Assuming \( p_2 = A_{III} p(z, \gamma_2) \exp(-Kz/\gamma_2) \), the ordinary differential equation (1.19) becomes

\[ zp_z + \left( 1 - \frac{2K}{\gamma_2} \right) p_z - \frac{K}{\gamma_2} \left( 1 + \frac{1}{\omega_2} \right) p = 0. \]

The substitutions \( \alpha_2 = \frac{1}{2} \left( 1 + 1/\omega_2 \right) \) and \( Z_2 = 2K/\gamma_2 \) give another confluent hypergeometric equation

\[ Z_2 p_{z,z} + \left( 1 - Z_2 \right) p_{z,z} - \alpha_2 p = 0. \]

Its solution is found analogously to region I. It provides two boundary points at the endpoint of the front located at \( y = 1 \) \( (Z_2 = 2K(1 + \gamma_2)/\gamma_2) \):

(1.19a)

\[ p_{2}^{III}(1) = A_{III} e^{-Z_2/2} \Psi(\alpha_2,1;Z_2)_{Z_2=2K(1+\gamma_2)/\gamma_1} \quad \text{and} \]

\[ p_{2,y}^{III}(1) = A_{III} \frac{dZ_2}{dy} e^{-Z_2/2} \left( -\frac{\Psi(\alpha_2,1;Z_2)}{2} + \frac{\partial \Psi}{\partial Z_2}(\alpha_2,1;Z_2) \right)_{Z_2=2K(1+\gamma_2)/\gamma_1} \]

(1.19b)

\[ = -A_{III} K e^{-Z_2/2} \left( \Psi(\alpha_2,1;Z_2) - 2 \frac{\partial \Psi}{\partial Z_2}(\alpha_2,1;Z_2) \right)_{Z_2=2K(1+\gamma_2)/\gamma_1}. \]

Region II:

For Region II, the \( P_1, P_2 \) equations, (1.13a,b), form the coupled system

(1.20a)

\[ \left[ (1-y) \left( \frac{d^2}{dy^2} - K^2 \right) - \frac{d}{dy} - \lambda_1 \right] p_1 = -\beta_1 p_2 \]

and
\[(1 + \gamma_2) \left(1 + \gamma_2 \right) \frac{d}{dy^2} - \left(\beta_2^* + \frac{K}{\omega_2} (1 + \gamma_2) \right) \right] p_2 = -\beta_2^* p_1. \]

Let
\[\beta_2 = \frac{\beta_2^*}{1 + \gamma_2} = \frac{f^2 L^2}{g'' H} \frac{1 - \omega_2^2}{1 + \gamma_2} \]

and (1.20b) becomes
\[\left[ y \left( \frac{d^2}{dy^2} - K^2 \right) + \frac{d}{dy} - \left( \beta_2 + \frac{K}{\omega_2} \right) \right] p_2 = -\beta_2 p_1. \]

(1.21)

Then substituting \( \lambda_2 = \beta_2 + K/\omega_2 \) and combining both equations gives
\[\left[ y \left( \frac{d^2}{dy^2} - K^2 \right) + \frac{d}{dy} - \lambda_2 \right] \left(1 - y \right) \left( \frac{d^2}{dy^2} - K^2 \right) - \frac{d}{dy} - \lambda_1 \right] p_1 = \beta_1 \beta_2 p_1 \]
or, with \( D = \frac{d}{dy} \),
\[\left[ y(1 - y) D^4 + (1 - 4y) D^3 - \left\{ 2K^2 y(1 - y) + 2 + \lambda_2 + y(\lambda_1 - \lambda_2) \right\} D^2 + K^4 y(1 - y) + \left\{ (4y - 1)K^2 + \lambda_2 - \lambda_1 \right\} D \right. + \left\{ 1 + \lambda_2 + y(\lambda_1 - \lambda_2) \right\} K^2 + \lambda_1 \lambda_2 - \beta_1 \beta_2 \right] p_1 = 0. \]

(1.22)

This leads to a fourth-order, ordinary differential equation for \( p_1 \) with variable coefficients. It constitutes a boundary-value problem with the complex frequency as an eigenvalue, and two singular points at \( y=0 \) and \( y=1 \) respectively. The physical boundary conditions assume continuity of \( p_1, p_2, u_1, u_2, v_1, v_2 \) fields at the boundaries of region II:

At \( y=0, \quad p_1(0) = p_1''(0) \) and \( p_{1, y}(0) = p_{1, y}''(0) \).

At \( y=1, \quad p_2'''(1) = p_2''''(1) \) and \( p_{2, y'''}(1) = p_{2, y''''}(1) \).

It will be assumed that regular singular conditions hold at the endpoints of region II. This implies that at \( y=0, \quad 0 \)
\[\left[D^3 - (2 + \lambda_2) D^2 + (-K^2 + \lambda_2 - \lambda_1) D + (1 + \lambda_2) K^2 + \lambda_1 \lambda_2 - \beta_1 \beta_2 \right] p_1 = 0. \]

and at \( y=0, \quad 0 \)
\[\left[-3D^3 - (2 + \lambda_1) D^2 + (3K^2 + \lambda_2 - \lambda_1) D + (1 + \lambda_1) K^2 + \lambda_1 \lambda_2 - \beta_1 \beta_2 \right] p_2 = 0. \]
1.3 The dispersion relation

The Frobenius' method is used to find the series solutions for the frontal system for \( p_1 \) by expansion about \( y=0 \). Let

\[
p_1(y) = \sum_{n=0} a_n y^{n+r}.
\]

Then

\[
D p_1(y) = \sum_{n=0} a_n (n+r)y^{n+r-1},
\]

\[
D^2 p_1(y) = \sum_{n=0} a_n (n+r)(n+r-1)y^{n+r-2},
\]

\[
D^3 p_1(y) = \sum_{n=0} a_n (n+r)(n+r-1)(n+r-2)y^{n+r-3},
\]

\[
D^4 p_1(y) = \sum_{n=0} a_n (n+r)(n+r-1)(n+r-2)(n+r-3)y^{n+r-4}.
\]

We can shift powers of \( y \) to the same level by handling the indices. The various parts of (1.22) become

\[
y(1-y)D^4 p_1 + (1-4y)D^3 p_1 = \sum_{n=3} a_{n+3} (n+r+3)(n+r+2)(n+r+1)^2 y^{n+r}
\]

\[- \sum_{n=2} a_{n+2} (n+r+3)(n+r+2)(n+r+1)(n+r)y^{n+r},
\]

\[- [2K^2 y(1-y) + 2 + \lambda_1 y + \lambda_2 (1-y)]D^2 p_1 = 2K^2 \sum_{n=0} a_n (n+r)(n+r-1)y^{n+r}
\]

\[- (2K^2 + \lambda_1 - \lambda_2) \sum_{n=1} a_{n+1} (n+r+1)(n+r)y^{n+r}
\]

\[- (2 + \lambda_2) \sum_{n=2} a_{n+2} (n+r+1)(n+r+2)y^{n+r},
\]

\[[ (4y-1)K^2 - \lambda_1 + \lambda_2 ]DP_1 = 4K^2 \sum_{n=0} a_n (n+r)y^{n+r} - (K^2 + \lambda_1 - \lambda_2) \sum_{n=1} a_{n+1} (n+r+1)y^{n+r},
\]

\[[ y(1-y)K^4 + (\lambda_1 - \lambda_2)K^2 y + \Gamma ]p_1 = -K^4 \sum_{n=2} a_{n-2} y^{n+r} + \Gamma \sum_{n=0} a_n y^{n+r}
\]

\[+ K^2 (K^2 + \lambda_1 - \lambda_2) \sum_{n=1} a_{n-1} y^{n+r},
\]

where \( \Gamma = (1 + \lambda_2)K^2 + \lambda_1 \lambda_2 - \beta_1 \beta_2 \).

The overall equation (1.22) becomes
\begin{align*}
& \sum_{n=-3}^{\infty} a_{n+3}(n+r+3)(n+r+2)(n+r+1)^2 y^{n+r} \\
& - \sum_{n=-2}^{\infty} a_{n+2}[(n+r+3)(n+r)+2 + \lambda_2](n+r+2)(n+r+1)y^{n+r} \\
& - \sum_{n=-1}^{\infty} a_{n+1}[2(n+2r+1)K^2 + (\lambda_1 - \lambda_2)(n+r+1)](n+r+1)y^{n+r} \\
& + \sum_{n=0}^{\infty} a_n[2K^2(n+r)(n+r+1) + \Gamma]y^{n+r} \\
& + \sum_{n=1}^{\infty} a_{n-1}[K^4 + (\lambda_1 - \lambda_2)K^2]y^{n+r} - \sum_{n=2}^{\infty} a_{n-2}K^4y^{n+r} = 0.
\end{align*}

At each power level, the coefficients must cancel.

(1.25) \hspace{1cm} n = -3: \hspace{0.5cm} a_0r(r-1)(r-2)^2 = 0.

In order to keep \( r \) meaningful, \( a_0 \) must be arbitrary. Thus \( r = 0, 1 \) or 2. There are only three roots of (1.25), then there must be only three fundamental solutions that have a regular singular behaviour about \( y = 0 \). As the roots of (1.25) differ by an integer, their effect is to shift the power in the solution series. But it is sufficient to consider \( r = 0 \), and to build fundamental solutions by factoring the arbitrary coefficients:

\begin{align*}
& n = -2: \hspace{0.5cm} a_1(r+1)r(r-1)^2 - a_0r(r-1)[(r+1)(r-2)+2 + \lambda_2] = 0.
\end{align*}

For \( r = 0 \), \( a_1 \) remains arbitrary.

\begin{align*}
& n = -1: \hspace{0.5cm} a_2(r+2)(r+1)r^2 = a_1r(r+1)[(r-1)(r+2) + 2 + \lambda_2] \\
& \hspace{4cm} + a_0r[(2r-1)K^2 + r(\lambda_1 - \lambda_2)]
\end{align*}

(1.26)

The case \( r = 0 \) is again a trivial solution of (1.26), which leaves \( a_2 \) arbitrary. For \( n \geq 2 \), let \( k = n + 3 \) (\( k \geq 5 \)) and use \( r = 0 \) for the general indicial equation:

\begin{align*}
a_k &= a_{k-1} \frac{k(k-3) + 2 + \lambda_2}{k(k-2)} + a_{k-2} \frac{(2k-5)K^2 + (\lambda_1 - \lambda_2)(k-2)}{k(k-1)(k-2)} \\
& \hspace{1cm} - \frac{a_{k-3}[2K^2(k-2)(k-3) + \Gamma] + a_{k-4}K^2[k^2 + \lambda_1 - \lambda_2] - a_{k-5}K^4}{k(k-1)(k-2)^2}.
\end{align*}

(1.27)

The coefficients \( a_0, a_1, a_2 \) are arbitrary, so the set of fundamental solutions can be reduced to a linearly independent basis of three eigenfunctions, say:

\begin{align*}
(1.28) \hspace{1cm} \Theta_0(y) = 1 - \frac{\Gamma}{6}y^3 - \frac{(6 + \lambda_2)\Gamma}{48} + \frac{K^4 + (\lambda_1 - \lambda_2)K^2}{48}y^4 + \ldots
\end{align*}
with \( a_0 = 1, a_1 = 0, a_2 = 0 \);

\[
\Theta_1(y) = \frac{K^4 + \lambda_1 - \lambda_2}{6} y^3 - \frac{(6 + \lambda_2)(K^2 + \lambda_1 - \lambda_2)}{48} y^4 + \ldots
\]

with \( a_0 = 0, a_1 = 1, a_2 = 0 \);

\[
\Theta_2(y) = \frac{2 + \lambda_2}{3} y^3 + \frac{(2 + \lambda_2)(6 + \lambda_2) + 3K^2 + 2(\lambda_1 - \lambda_2)}{24} y^4 + \ldots
\]

with \( a_0 = 0, a_1 = 0, a_2 = 1 \).

The missing eigenfunction that would close the basis of fundamental solutions is the one that contains the singular asymptotic behaviour about the regular singular point, \( y = 0 \). It is rejected in order to respect the regular singular condition (the fourth derivative of \( p_1 \) stays finite as \( y \to 0^+ \)). The global solution expanded about \( y = 0 \) is

\[
p_1 = a_0 \Theta_0(y) + a_1 \Theta_1(y) + a_2 \Theta_2(y).
\]

The boundary conditions at \( y = 0 \) imply

\[
a_0 = p'_1(0) = A_j \Psi(\alpha_1; l; \frac{2K}{\gamma_1}) \quad \text{and}
\]

\[
a_1 = \frac{dp'_1}{dy}(0) = KA_j \left[ \Psi(\alpha_1; l; \frac{2K}{\gamma_1}) - 2\Psi'(\alpha_1; l; \frac{2K}{\gamma_1}) \right],
\]

where \( \Psi(\alpha_1; l; 2K/\gamma_1) \) is solution of the transformed differential equation in the region \( l \) at the boundary, and \( \Psi'' \) is the derivative with respect to the element after the semicolon, i.e., \( 2K/\gamma_1 \) in this case. The coefficient \( a_2 \) can not be determined on this side. Therefore, the global solution can be split into two solutions: one for which the slope and the magnitude at the origin depend upon conditions in the region \( l \), and that has no inflection at the origin; the other solution behaves freely, without magnitude, nor slope, but curved at the origin.

For \( p_2 \), we get a similar fourth-order ordinary differential equation:

\[
\left[ y(1-y)D^4 + (3-4y)D^3 - \{2K^2y(1-y) + 2 + \lambda_2 + y(\lambda_1 - \lambda_2)\}D^2 \\
+ \{(4y-3)K^2 + \lambda_2 - \lambda_1\}D + K^4y(1-y) + \\
K^2\{1 + \lambda_2 + y(\lambda_1 - \lambda_2)\} + \lambda_1\lambda_2 - \beta_1\beta_2 \right]p_2 = 0.
\]
Let
\[ p_2(y) = \sum_{n=0} b_n y^{n+s}. \]
(1.35)

The lowest power level in (1.34) is provided by the two first terms:
\[
[y(1-y)D^4 + (3-4y)D^3]p_2 = b_0 s(s-1)(s-2)^2 y^{s-2} + O(y^{s-1}).
\]

So \( s = 0, 1, \) or \( 2. \) Similarly with \( p_1, \) after rejection of the singular solution, the series solutions can be treated as analytic and the three first constants \( \{b_1, b_2, b_3\} \) remain arbitrary. Instead of creating another general indicial equation with five terms for the \( b_n \)'s, we can use the fact that \( p_1 \) and \( p_2 \) are coupled in order to shorten the algebra. Then, with the substitution of (1.24) and (1.35) in (1.20b), we get
\[
\sum_{n=0} \left\{ a_{n+2}(n+2)(n+1) - a_{n+1}(n+1)^2 - (\lambda_1 + K^2)a_n \right\} y^n + K^2 \sum_{n=1} a_{n-1} y^n = -\beta_1 \sum_{n=0} b_n y^n.
\]
(1.44)

Then, it follows from the regular singular condition
\[
b_0 = \frac{(\lambda_1 + K^2)a_0 + a_1 - 2a_2}{\beta_1}
\]
and the first general indicial equation is
\[
a_n = a_{n-1} \frac{n-1}{n} + \frac{(\lambda_1 + K^2)a_{n-2} - K^2a_{n-3} - \beta_1 b_{n-2}}{n(n-1)}.
\]
(1.45)

With the substitution of (1.24) and (1.35) in (1.21), we get
\[
\sum_{n=0} \left\{ (n+1)^2 b_{n+1} - \lambda_2 b_n \right\} y^n - K^2 \sum_{n=1} b_{n-1} y^n = -\beta_2 \sum_{n=0} a_n y^n.
\]
Then
\[ b_1 = \lambda_2 b_0 - \beta_2 a_0 \]
and the second general indicial equation is
\[
b_n = \frac{\lambda_2 b_{n-1} + K^2 b_{n-2} - \beta_2 a_{n-1}}{n^2}.
\]
(1.46)

All \( b_n \)'s can be determined with respect to \( a_0, a_1 \) and \( a_2. \) Setting \( a_2 = 0 \) and prescribing \( a_0 \) and \( a_1 \) by the boundary conditions on the left side of the front, we get the parts of \( p_1 \) and \( p_2 \)
that depend on the arbitrary magnitude of $p_1$ in region $I$ ($A_I$). Setting $a_0 = 0$, $a_1 = 0$, and leaving $a_2$ arbitrary, we get the other parts of $p_1$ and $p_2$. Thus,

$$p_1(y) = A_I p_1(a_0, a_1, 0, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y) + a_2 p_1(0, 0, 1, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y)$$

$$p_2(y) = A_I p_2(a_0, a_1, 0, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y) + a_2 p_2(0, 0, 1, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y),$$

where $a_0 = p_1^I(0)/A_I$ and $a_1 = p_{1,y}^I(0)/A_I$.

The radius of convergence of the series (1.24) is

$$R_c = \lim_{n \to \infty} \frac{a_n}{a_{n-1}} = \lim_{n \to \infty} \left[ \frac{n-1}{n} + O(n^{-2}) \right] = 1,$$

provided the nondimensional wavenumber $K$ and the $\lambda_1$, $\beta_1$ parameters stay bounded. And series (1.35) converges for any $y$ if $K$ and the parameters $\lambda_2$, $\beta_2$ stay bounded. Together, the series solutions do not converge at $y=0$. We need another point from which the expansions can describe the neighborhood of $y=1$. Use the boundary point with region $III$ and let $z=1-y$. Thus (1.20b) and (1.21) become

$$\left[ (1-z) \left( \frac{d^2}{dz^2} - K^2 \right) - \frac{d}{dz} - \lambda_2 \right] p_2 = -\beta_2 p_1,$$

$$(1.47)$$

$$\left[ z \left( \frac{d^2}{dz^2} - K^2 \right) + \frac{d}{dz} - \lambda_1 \right] p_1 = -\beta_1 p_2.$$  

$$(1.48)$$

The set of equations (1.47,1.48) with the boundary conditions from the region $III$ stay similar to the original set, thus series solutions can be found similarly by swapping the indices:

$$p_2(z) = A_{III} P_1(a_0^*, a_1^*, 0, K^2, \lambda_2, \beta_1, \lambda_2, \beta_1; z) + a_2^* P_1(0, 0, 1, K^2, \lambda_2, \beta_1, \beta_2, \beta_1; z),$$

$$p_1(z) = A_{III} P_2(a_0^*, a_1^*, 0, K^2, \lambda_2, \beta_1, \lambda_2, \beta_1; z) + a_2^* P_2(0, 0, 1, K^2, \lambda_2, \beta_1, \beta_2, \beta_1; z),$$

where $a_0^* = p_{1,I}^{III}(1)/A_{III}$, $a_1^* = -p_{1,y}^{III}(1)/A_{III}$ and

$$A_{III}^* = A_I e^{2K_1(z)}.$$  

The solutions for $p_1$ and $p_2$ do not converge at $z=1$ ($y=0$), but both sets can be connected at $y=z=\frac{1}{2}$:

$$p_1(p_2(a_0, a_1, 0, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y) + a_2 P_2(0, 0, 1, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y) =$$

$$A_{III} P_1(a_0^*, a_1^*, 0, K^2, \lambda_2, \lambda_1, \beta_1, \beta_2; z) + a_2^* P_1(0, 0, 1, K^2, \lambda_2, \lambda_1, \beta_2, \beta_1; z),$$  

$$(1.49)$$
\[ A_1 P_1(a_0, a_1, 0, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y) + a_2 P_2(0, 0, 1, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y) =
\]
\[ A'_1 P_2(a_0', a_1', 0, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y) + a'_2 P_2(0, 0, 1, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y) =
\]
(1.50)
\[ A'_m P_2(a_0', a_1', 0, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; z) + a'_2 P_2(0, 0, 1, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; z).
\]

The arbitrary constants of integration are \( A_1, A'_m, a_2 \) and \( a'_2 \). Two other equations are needed for the matching. They are provided by differentiation of the solutions evaluated at \( y = z = 1/2 \).

\[ A_1 P_2''(a_0, a_1, 0, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y) + a_2 P_2''(0, 0, 1, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y) =
\]
\[ -A'_m P_2''(a_0', a_1', 0, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y) - a'_2 P_2''(0, 0, 1, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y),
\]
(1.51)
\[ A_1 P_2(a_0, a_1, 0, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y) + a_2 P_2(0, 0, 1, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y) =
\]
\[ -A'_m P_2(a_0', a_1', 0, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y) - a'_2 P_2(0, 0, 1, K^2, \lambda_1, \lambda_2, \beta_1, \beta_2; y).
\]
(1.52)

Primes denote differentiation with respect to \( y \) and \( z \), according to the origins from which functions are expanded. The conditions that solve for the arbitrary constants form a linear system:

\[
\begin{bmatrix}
P_1(a_0, a_1, 0) & P_1(0, 0, 1) & -\overline{P}_2(a_0', a_1', 0) & -\overline{P}_2(0, 0, 1) & A_1 \\
\overline{P}_1'(a_0, a_1, 0) & \overline{P}_1'(0, 0, 1) & \overline{P}_2'(a_0', a_1', 0) & \overline{P}_2'(0, 0, 1) & a_2 \\
P_2(a_0, a_1, 0) & P_2(0, 0, 1) & -\overline{P}_1(a_0', a_1', 0) & -\overline{P}_1(0, 0, 1) & A'_m \\
\overline{P}_2'(a_0, a_1, 0) & \overline{P}_2'(0, 0, 1) & \overline{P}_1'(a_0', a_1', 0) & \overline{P}_1'(0, 0, 1) & a'_2
\end{bmatrix} = 0,
\]
(1.53)
1.4. Initial parameters

The perturbation of the frontal system is described by a fourth-order differential equation with four boundary conditions; therefore the eigenvalue problem has four degrees of freedom. The set of parameters defined in this chapter must be constructed upon these degrees of freedom and will be used as original inputs.

The first input parameter to consider is the nondimensional wavenumber $K$ of the disturbance. Considering that the system is frictionless, we have the freedom to choose the frame of inertia to match the centre of motion between the water masses.

Let

$$\omega = k \frac{U_1 + U_2}{2f} - \frac{\sigma}{f}$$

be the eigenvalue of the system. The next parameters will not require further knowledge of the momenta other than the momentum difference or nondimensional Doppler shift. From the eigenvalue, $\omega_1$ and $\omega_2$ are readily defined:

$$\omega_1 = \omega + \frac{K(U_1 - U_2)}{2fL} \quad \text{and} \quad \omega_2 = \omega - \frac{K(U_1 - U_2)}{2fL}.$$  

But

$$\frac{U_1 - U_2}{fL} = \frac{\Delta U}{fL} = \frac{(\Delta U)^2}{g'H}.$$  

Let

$$Ri = \frac{f^2L^2}{g'H} = \frac{g'H}{(\Delta U)^2}.$$  

This new input parameter is analogous to a Richardson number and is a measure of the ratio of the available potential energy over the relative kinetic energy. Its reciprocal can be also seen as the squared ratio of the baroclinic Rossby radius over the frontal length, i.e. as a squared Burger number. It is the parameter $F$ in the work of Gawarkiewicz [1991]. An alternative selection of input parameter would have been a Rossby number equal to the nondimensional Doppler shift, $Ro = K/(2Ri)$, which was preferred by Kotschin [1932], Orlanski [1968] and Flagg and Beardsley [1978] instead of the nondimensional wavenumber.
Thus
\[
\omega_1 = \omega + \frac{K}{2Ri} \quad \text{and} \quad \omega_2 = \omega - \frac{K}{2Ri}
\]

Four other parameters result from the above inputs and the eigenvalue.

From the boundary conditions:
\[
\alpha_1 = \frac{1}{2} \left( 1 - \frac{1}{\omega_1} \right),
\]
\[
\alpha_2 = \frac{1}{2} \left( 1 + \frac{1}{\omega_2} \right).
\]

From the pycnocline region:
\[
\beta_1 = Ri \left( 1 - \omega_1^2 \right),
\]
\[
\lambda_1 = \beta_1 - \frac{K}{\omega_1}.
\]

The straight slope under the pycnocline calls for a third input, \( \gamma_2 \), and two other parameters:
\[
\beta_2 = Ri \frac{1 - \omega_2^2}{1 + \gamma_2},
\]
\[
\lambda_2 = \beta_2 + \frac{K}{\omega_2}.
\]

In his treatment, Gawarkiewicz [1991] assumes perturbations close to a geostrophic balance, such that \( |\omega_1|^2, |\omega_2|^2 \ll 1 \). From that approximation, \( \beta_1 \) and \( (1 + \gamma_2)\beta_2 \) are equivalent to his Richardson's parameter \( (g'\mathcal{H}/(\Delta U)^2) \). So, \( \beta_1 \) and \( \beta_2 \) are denoted modified Richardson's numbers. The last input to define is \( \gamma_1 \), and comes with the left boundary conditions. It was implicitly set to zero in the shelf-break model of Gawarkiewicz [1991].

The set of inputs, \( \{\gamma_1, \gamma_2, K, Ri\} \), is made from real-valued parameters. Thus for any solution of the dispersion relation, the complex conjugate of the eigenvalue is also solution. This implies the system is either unstable or marginally stable (no growth rate, no decay rate) for a set of original inputs, but can not be stable (in the sense of allowing only a decaying mode).