A MULTIGRID METHOD FOR DETERMINING THE DEFLECTION OF LITHOSPHERIC PLATES

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Abstract

Various models are currently in existence for determining the deflection of lithospheric plates under an applied transverse load. The most popular models treat lithospheric plates as thin elastic or thin viscoelastic plates. The equations governing the deflection of such plates have been solved successfully in two dimensions using integral transform techniques. Three dimensional models have been solved using Fourier Series expansions assuming a sinusoidal variation for the load and deflection. In the engineering context, the finite element technique has also been employed. The current aim, however, is to develop an efficient solver for the three dimensional elastic and viscoelastic problems using finite difference techniques. A variety of loading functions may therefore be considered with minimum work involved in obtaining a solution for different forcing functions once the main program has been developed. The proposed method would therefore provide a valuable technique for assessing new models for the loading of lithospheric plates as well as a useful educational tool for use in geophysics laboratories.

The multigrid method, which has proved to be a fast, efficient solver for elliptic partial differential equations, is examined as the basis for a solver of both the elastic and viscoelastic problems. The viscoelastic problem, being explicitly time-dependent, is the more challenging of the two and will receive particular attention.

Multigrid proves to be a very effective method applicable to the solution of both the elastic and viscoelastic problems.
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