Chapter 2

Derivation of the Governing Equation

2.1 Lithospheric Plates

In considering the deflection of lithospheric plates, we will be encountering horizontal dimensions ranging from 250km to 1000km. This covers anything from the formation of sedimentary basins to the deflection of continental plates due to topographic loading. In order to derive an equation governing plate deflection we make the following observations. First, that the horizontal extent of the plate is not so large as to make the effects of the earth's curvature of importance. We therefore assume that in their undisturbed state, the plates are completely flat. Second, the thickness of the plate is very small in comparison to the horizontal dimension. In general, lithospheric plates have an elastic or mechanical thickness of between 25 and 50km depending on whether the plate is oceanic or continental. This is considerably less than the seismic thickness which is of the order of 100km, see Nunn & Sleep [7, p.590]. It is the mechanical thickness which is of importance in an elastic or viscoelastic model so we therefore assume that lithospheric plates may be treated as thin elastic or viscoelastic plates.

In elastic theory, we assume that at any given time the plate responds only to the applied load at that time so that on removal of the load, the plate instantaneously returns to its undisturbed state, having no memory of the previously applied load. Should we wish to examine the deflection of the lithosphere under an applied load at a given point in time, the present day for example, then elastic theory provides us
Chapter 2. Derivation of the Governing Equation

with an adequate model for achieving this. However, if we wish to study the evolution of the plate's deflection then a viscoelastic model is more appropriate. In viscoelastic theory the elastic properties are assumed to be a function of time so that materials have the capacity to remember, and therefore respond to, previously applied loads as well as the current one. If the load is removed at a given time, a viscoelastic plate does not instantaneously revert to its undisturbed state but rather relaxes, possibly to a steady state, as time goes on. Obviously, from a geophysical standpoint, we will not be considering the complete removal of the load but, in the case where the load reaches a steady state, the subsequent evolution of the plate's deflection is of interest.

2.2 Classical Elasticity and Viscoelastic Theory

Having justified the treatment of lithospheric plates as thin elastic or viscoelastic plates we now derive an expression governing the deflection of thin viscoelastic plates under the action of transverse loading.

We begin with an overview of classical elasticity introducing the ideas of stress, strain and subsequent particle displacement. Consider a volume element of an elastic material as shown in Figure 2.1 with Cartesian axes in the given orientation. This volume element is subject to normal and shear stresses due to its interaction with neighbouring volume elements. Figure 2.1 indicates the direction in which these stresses are taken as positive. Note that stresses acting on surfaces normal to the $x_2$-direction have been omitted for clarity. Subscripts on the stresses are in the following convention: (i) the first represents the direction normal to the surface on which the stress is acting, (ii) the second indicates the direction parallel to which the stress is acting. We adopt a suffix notation with respect to Cartesian axes $(x_1, x_2, x_3)$ and assume the Einstein Summation Convention to be in operation over repeated suffices. The notation $(.)_j$
represents differentiation with respect to $x_j$.

Following Vinson [11], we now present the governing equations for the deformation of an elastic body. A more in-depth discussion than that given in Vinson may be found in Hunter [6]. Consideration of conservation of momentum for a static, elastic body in the absence of body forces (such as gravity) yields the following equilibrium equations

$$\sigma_{ij,j} = 0$$

(2.2.1)

Conservation of angular momentum requires that

$$\sigma_{ij} = \sigma_{ji}$$

(2.2.2)

in other words that the stress tensor be symmetric.

A deformed elastic body has associated with it not only stresses but also strains and a displacement field. Assuming small displacements, it can be shown that the strains $\varepsilon_{ij}$ (suffix convention being the same as for the stresses $\sigma_{ij}$) are related to the
displacement field $u_i$ in the following way

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$ (2.2.3)

Finally we have the constitutive or strain stress relations. Assuming the material to be isothermal, isotropic and homogeneous we have:

$$\varepsilon_{ij}^e = \frac{1}{E} \left[(1 + \nu) \sigma_{ij} - \nu \sigma_{kk} \delta_{ij}\right]$$ (2.2.4)

where the superscript $e$ refers to an elastic solid, $E =$ Young’s Modulus , $\nu =$ Poisson’s Ratio and

$$\delta_{ij} = \text{Kronecker Delta} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Drucker [3, p.267] shows that the strain-stress relations for a linear elastic solid lead directly to those for a linear viscous solid with the replacement of the strains $\varepsilon_{ij}^e$ by the rate of strain $\dot{\varepsilon}_{ij}^v$ where the dot denotes differentiation with respect to time and the superscript $v$ refers to a viscous solid. For an isotropic material we replace Young’s Modulus $E$, by the coefficient of viscosity, $\mu$. Hence, for a linear viscous solid we have:

$$\dot{\varepsilon}_{ij}^v = \frac{1}{\mu} \left[(1 + \nu) \sigma_{ij} - \nu \sigma_{kk} \delta_{ij}\right]$$ (2.2.5)

where the assumption of isotropy and homogeneity has been maintained.

The linear viscoelastic model consists of a combination of the above linear and viscous models. Two possible formulations exist, the Maxwell model and the Kelvin model. The Maxwell model assumes that the strain for a viscoelastic body is given by the sum of the strains arising from the elastic and viscous models. The Kelvin model, however, assumes that the stress for a viscoelastic body is given by the sum of the stresses arising from these models. As pointed out by Turcotte [9, p.70], the usual approach, at least in the geophysical context, is to deal with the problem using the Maxwell model.
Chapter 2. Derivation of the Governing Equation

We therefore assume that the rate of strain, $\dot{\varepsilon}_{ij}$, is given by

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^v_{ij},$$

hence, using equations (2.2.4) and (2.2.5) we have:

$$\dot{\varepsilon}_{ij} = (1 + \nu) \left[ \frac{\sigma_{ij}}{E} + \frac{\sigma_{ij}}{\mu} \right] - \nu \left[ \frac{\sigma_{kk}}{E} + \frac{\sigma_{kk}}{\mu} \right] \delta_{ij} \tag{2.2.6}$$

As will be seen shortly, for our purposes it will be of use to have a form of the constitutive equations where summation from only 1 to 2 and not to 3 is implied over repeated suffices. Inverting equation (2.2.6) with this in mind we have:

$$\dot{\sigma}_{ij} + \frac{\sigma_{ij}}{\tau} = \frac{E}{(1 + \nu)} \left[ \dot{\varepsilon}_{ij} + \frac{\nu}{(1 - \nu)} \varepsilon_{kk} \delta_{ij} \right] \tag{2.2.7}$$

where

$$\tau = \frac{\mu}{E} \tag{2.2.8}$$

is known as the relaxation constant.

2.3 Thin Elastic Plates

Vinson [11] derives an equation for the deflection of a thin elastic plate. We shall follow along the same lines using the constitutive equations for a viscoelastic solid in place of those for an elastic solid, hence deriving an equation governing the deflection of a thin viscoelastic plate. Consider a thin, rectangular, isotropic plate with horizontal dimensions $x_1^L$, $x_2^L$ and thickness $h$. We assume that the conditions $h \ll x_1^L$ and $h \ll x_2^L$ define a thin plate. We now position a set of Cartesian axes at the centroid of the plate as shown in Figure 2.2 so that the mid-plane of the plate is the plane $x_3 = 0$.

Consider a cuboid element of the plate extending through the plate thickness as shown. We assume that on the application of a transverse load this element remains
normal to the deformed mid-plane, in other words, it undergoes at most a translation and rotation with respect to the co-ordinate system. The element therefore remains straight upon load application. This means that sides of the cuboid initially lying in the $x_1$ and $x_2$-directions remain perpendicular to sides lying in the $x_3$-direction after deformation. Hunter [6, p.111] interprets the strains $\epsilon_{ij} (i \neq j)$ as representing a measure of the deviation from perpendicularity, after deformation, of sides initially pointing in the $i$ and $j$-directions. Since in the current problem sides in the $x_1$ and $x_2$-directions remain perpendicular to sides in the $x_3$-direction, we may deduce that $\epsilon_{13} = \epsilon_{23} = 0$.

We also assume that an applied transverse load results in plate bending but does not lead to compression or tension in the plate thickness direction. This is in contrast to the behaviour of a sponge where a transverse load is absorbed by compression in the thickness direction and not by bending. Again referring to Hunter [6, p.111], we see that the strains $\epsilon_{ij} (i = j)$ may be interpreted as representing some measure of
the elongation or compression of an element in the \( i \)-direction. Since in the current application there is no elongation or compression in the \( x_3 \)-direction we therefore deduce that \( \varepsilon_{33} = 0 \). Equation (2.2.3) then implies that \( u_3 \) (the displacement in the \( x_3 \)-direction) is independent of \( x_3 \).

Hence, the admissible form of the subsequent displacement field for a thin elastic plate is:

\[
\begin{align*}
    u_1 &= u_1(x_1, x_2, x_3, t) = u_1^0(x_1, x_2, t) + x_3 \alpha_1(x_1, x_2, t) \\
    u_2 &= u_2(x_1, x_2, x_3, t) = u_2^0(x_1, x_2, t) + x_3 \alpha_2(x_1, x_2, t) \\
    u_3 &= u_3(x_1, x_2, t)
\end{align*}
\]

(2.3.1)

where \( u_i^0(x_1, x_2, t) = u_i(x_1, x_2, 0, t) \) for \( i=1,2 \) and \( \alpha_i(x_1, x_2, t) \) represent rotations to be defined later.

From here we shall adopt modified forms of the suffix notation and summation convention which have been used to this point. We shall assume that all suffices run only from 1 to 2 and not to 3. Any variable with a suffix 3 will be written explicitly.

We now derive expressions for the net moments and forces acting on an element across the plate thickness. The moments, \( m_{ij} \), per unit area about the mid-plane are given by:

\[
m_{ij} = \sigma_{ij} x_3
\]

Integrating these across the plate thickness we have:

\[
M_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij} x_3 \, dx_3
\]

(2.3.2)

The transverse shears, \( q_i \), in the \( x_3 \)-direction are given by:

\[
q_i = \sigma_{i3}
\]

and so integrating over the plate thickness we have:

\[
Q_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{i3} \, dx_3
\]

(2.3.3)
We now derive an integrated form of the equilibrium equations (2.2.1). Multiplying (2.2.1) by \(x_3\) and integrating over the plate thickness we have

\[
\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij,j} x_3 \, dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{i3,3} x_3 \, dx_3 = 0
\]

\[
\Rightarrow \quad M_{ij,j} + \sigma_{i3} x_3 \bigg|_{-\frac{h}{2}}^{\frac{h}{2}} - \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{i3} \, dx_3 = 0
\]

\[
\Rightarrow \quad M_{ij,j} - Q_i = 0 \quad (2.3.4)
\]

assuming there are no shear stresses acting on the surface of the plate. Integrating (2.2.1) for \(i = 3\) we have

\[
\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{3j,j} \, dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{33,3} \, dx_3 = 0
\]

\[
\Rightarrow \quad Q_{j,j} + \sigma_{33} \bigg|_{-\frac{h}{2}}^{\frac{h}{2}} = 0
\]

\[
\Rightarrow \quad Q_{j,j} + p = 0 \quad (2.3.5)
\]

where

\[
p = p(x_1, x_2, t) = \sigma_{33} \bigg|_{-\frac{h}{2}}^{\frac{h}{2}}
\]

represents the net force per unit area acting normal to the plate surface. Combining equations (2.3.4) and (2.3.5) we have

\[
M_{ij,j} + p = 0 \quad (2.3.6)
\]

We now relate the moments \(M_{ij}\) to the vertical displacement of the plate. Combining equations (2.2.3) and (2.2.7) we have

\[
\dot{\sigma}_{ij} + \frac{\sigma_{ij}}{\tau} = \frac{E}{(1 + \nu)} \left[ \frac{1}{2} (\ddot{u}_{i,j} + \ddot{u}_{j,i}) + \frac{\nu}{(1 - \nu)} \dddot{u}_{kk} \delta_{ij} \right]
\]
where the zero subscript has been dropped from $E$ and $\tau$. Multiplying through by $x_3$, integrating and making use of (2.3.1) we have:

$$
\int_{-b}^{b} \left( \dot{\sigma}_{ij} + \frac{\sigma_{ij}}{\tau} \right) x_3 \, dx_3 = 
$$

$$
\frac{E}{1 + \nu} \int_{-b}^{b} \left[ \frac{1}{2} (\dot{u}_{i,j}^0 + \dot{u}_{j,i}^0) x_3 + \frac{\nu}{1 - \nu} \dot{u}_{k,k}^0 x_3 \delta_{ij} \right] \, dx_3
$$

$$
+ \frac{E}{1 + \nu} \int_{-b}^{b} \left[ \frac{1}{2} (\dot{\alpha}_{i,j} + \dot{\alpha}_{j,i}) x_3^2 + \nu \delta_{k,k} x_3^2 \delta_{ij} \right] \, dx_3 \quad (2.3.7)
$$

We now relate the rotations $\alpha_i(x_1, x_2, t)$ to the vertical deflection $u_3(x_1, x_2, t)$ which from here will be denoted by $w = w(x_1, x_2, t)$. Equations (2.2.3) and the assumption that $\varepsilon_{13} = \varepsilon_{23} = 0$ imply:

$$
\varepsilon_{13} = \frac{1}{2} \left( \alpha_1 + \frac{\partial w}{\partial x_1} \right) = 0 \quad \Rightarrow \quad \alpha_1 = -\frac{\partial w}{\partial x_1}
$$

$$
\varepsilon_{23} = \frac{1}{2} \left( \alpha_2 + \frac{\partial w}{\partial x_2} \right) = 0 \quad \Rightarrow \quad \alpha_2 = -\frac{\partial w}{\partial x_2}
$$

Notice we have dropped the comma notation for derivatives and will continue to do so for the remainder of this discussion. Equation (2.3.7) may now be written

$$
\dot{M}_{ij} + \frac{M_{ij}}{\tau} = -D \left[ (1 - \nu) \frac{\partial^2 \dot{w}}{\partial x_i \partial x_j} + \nu \frac{\partial^2 \dot{w}}{\partial x_k \partial x_k} \delta_{ij} \right] \quad (2.3.8)
$$

where $D$, the flexural rigidity of the plate, is given by

$$
D = \frac{E h^3}{12(1 - \nu^2)}
$$

Equations (2.3.6) and (2.3.8) represent a closed system of equations for the vertical deflection $w$, in terms of the applied transverse load $p$. Eliminating $M_{ij}$ from these equations we have

$$
\frac{\partial^2}{\partial x_1^2} \left\{ D \left[ \frac{\partial^2 \dot{w}}{\partial x_1^2} + \nu \frac{\partial^2 \dot{w}}{\partial x_2^2} \right] \right\} + 2 \frac{\partial^2}{\partial x_1 \partial x_2} \left\{ D(1 - \nu) \frac{\partial^2 \dot{w}}{\partial x_1 \partial x_2} \right\}
$$
\[ + \frac{\partial^2}{\partial x_2^2} \left\{ D \left[ \frac{\partial^2 \tilde{w}}{\partial x_1^2} + \nu \frac{\partial^2 \tilde{w}}{\partial x_1^2} \right] \right\} = \frac{p}{\tau} + \dot{p} \]

which may be written

\[ \nabla^2(D(\nabla^2 \tilde{w})) - (1 - \nu)\nabla^4(D, \tilde{w}) = \frac{p}{\tau} + \dot{p} \quad (2.3.9) \]

where

\[ \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \]

and

\[ \nabla^4(D, \tilde{w}) = \frac{\partial^2 D \partial^2 \tilde{w}}{\partial x_1^2 \partial x_2^2} - 2 \frac{\partial^2 D}{\partial x_1 \partial x_2} \frac{\partial^2 \tilde{w}}{\partial x_1 \partial x_2} + \frac{\partial^2 D}{\partial x_2^2} \frac{\partial^2 \tilde{w}}{\partial x_1^2} \]

Recall that we have assumed the plate to be homogeneous so that \(E\) and \(\nu\) are constant.

If we further make the assumption that the plate thickness is uniform we may deduce that \(D\) is independent of position (although it may still maintain some time dependence if \(h\) changes with time) and so we have:

\[ D \nabla^4 \tilde{w} = \frac{p}{\tau} + \dot{p} \quad (2.3.10) \]

where

\[ \nabla^4 \tilde{w} = \nabla^2(\nabla^2 \tilde{w}) \]

Inverting equations (2.2.4) and using the resulting equations in place of (2.2.7) in the above derivation leads to an equation relating the vertical deflection of an elastic, rather than viscoelastic, plate to transverse loading. Vinson [11, p.16] shows that the equation thus obtained is

\[ D \nabla^4 w = p \quad (2.3.11) \]

instead of (2.3.10).
Chapter 2. Derivation of the Governing Equation

Figure 2.3: Hydrostatic Restoring Force

2.4 Transverse Loading

When considering the deflection of a lithospheric plate due to a transverse load, \( p \), we must take into account the hydrostatic forces which result from such deflections. If a lithospheric plate having an overlying layer of sediment is deflected downward, the sediment which is usually less dense than the mantle, will infill the cavity thus formed, resulting in a net hydrostatic restoring force. Following Turcotte & Schubert [10, p.121] we consider the case shown in figure 2.3 where the continental lithosphere of density \( \rho_m \) and thickness \( h_m \) rides on the fluid mantle, also of density \( \rho_m \). Overlying the lithosphere is a layer of sediment of thickness \( h_c \) and density \( \rho_c \). An applied transverse load \( p_a \) causes a deflection \( w \) in the lithospheric plate. We shall now calculate the hydrostatic restoring force and hence the net transverse load. The pressure acting at the point A in Figure 2.3 below the deflected plate is given by

\[
\rho_c g (h_c + w) + \rho_m g h_m
\]  \( (2.4.1) \)
Chapter 2. *Derivation of the Governing Equation*

whereas the pressure at the point B under an undisturbed portion of the plate is given by:

\[ \rho_c g h_c + \rho_m g (h_m + w) \]  \hspace{1cm} (2.4.2)

Subtracting (2.4.2) from (2.4.1) we have the hydrostatic force acting on the deflected plate:

\[ (\rho_c - \rho_m) g w \]  \hspace{1cm} (2.4.3)

Hence, the net transverse load is given by

\[ p = p_a + (\rho_c - \rho_m) g w = p_a - \gamma w \]  \hspace{1cm} (2.4.4)

where \( \gamma = (\rho_m - \rho_c) g \). Hence, the governing equation for deflection of a viscoelastic plate under an applied load \( p_a \) may now be written:

\[ D \nabla^4 \dot{w} + \gamma \left( \frac{w}{\tau} + \dot{w} \right) = \frac{p_a}{\tau} + \dot{p}_a \]  \hspace{1cm} (2.4.5)

and for an elastic plate we have:

\[ D \nabla^4 w + \gamma w = p_a \]  \hspace{1cm} (2.4.6)

From hereon, the subscript 'a' on the loading, \( p \), will be dropped so that \( p \) will be understood to mean the applied load.

### 2.5 Nondimensionalization

For the purposes of numerical computation we will now express equations (2.4.5) and (2.4.6) in nondimensional form. As our solution domains will be square, we scale all lengths by a factor \( L \). All times will be scaled by a factor \( T \). Using primes to denote nondimensional parameters, we have:

\[ w = w'L, \quad p = p'/L, \quad D = D'L^2 \]
\[ \gamma = \gamma' / L^2, \quad \tau = \tau' T \]

\[ \frac{\partial}{\partial x_1} = \frac{1}{L} \frac{\partial}{\partial x'_1}, \quad \frac{\partial}{\partial x_2} = \frac{1}{L} \frac{\partial}{\partial x'_2}, \quad \frac{\partial}{\partial t} = \frac{1}{T} \frac{\partial}{\partial t'} \]

Hence, equation (2.2.5) becomes

\[ (L^2 D') \frac{1}{L^4} \nabla^4 \left( \frac{L \dot{w}'}{T} \right) + \left( \frac{\gamma'}{L^2} \right) \left( \frac{L}{T} \right) \left( \frac{w'}{\tau'} + \dot{w}' \right) = \frac{1}{LT} \left( \frac{p'}{\tau'} + \dot{p}' \right) \]

Cancelling the common factor \(1/LT\) from each of the terms and dropping the primes we have:

\[ D \nabla^4 \dot{w} + \gamma \left( \frac{w}{\tau} + \dot{w} \right) = \frac{p}{\tau} + \dot{p} \]

(2.5.1)

where now, all variables and parameters are nondimensionalized with respect to a length scale \(L\) and a time scale \(T\). Similarly, equation (2.4.6) remains essentially unaltered in nondimensional form.

### 2.6 Boundary and Initial Conditions

Consider a solution domain \(\Omega\), with boundary \(\partial \Omega\). Following Nunn & Sleep [7, p.597] we assume the plate to be simply supported at the edges so that there is zero deflection and zero curvature at the boundary, \(\partial \Omega\). Hence,

\[ w = 0, \quad \nabla^2 w = 0 \quad \text{on} \quad \partial \Omega \]  

(2.6.1)

For the time-dependent (viscoelastic) problem, we also need some initial condition. Any condition consistent with (2.6.1) at \(t = 0\) will suffice. In cases where the exact solution, \(w_e\), is known we will take:

\[ w = w_e(t = 0) \quad \text{on} \quad \Omega \]  

(2.6.2)

as our initial condition where it is assumed that \(w_e\) satisfies (2.6.1). In cases where the exact solution is not known, we follow Nunn & Sleep [7, p.630] in assuming that the
deflection is initially zero so we have:

\[ w(t = 0) = 0 \quad \text{on } \Omega \] (2.6.3)

At this point we shall draw attention to an alternative formulation of (2.6.1) which will be of use to us later. As the boundary conditions (2.6.1) are assumed to hold for all time they may be written equivalently as:

\[ \dot{w} = 0, \quad \nabla^2 \dot{w} = 0, \quad \text{on } \partial \Omega \] (2.6.4)

Note that while the b.c.'s do not depend on time, the forcing function does. In the following chapter we shall consider possible solution methods for equations (2.4.5) and (2.4.6) with the above boundary and initial conditions.

### 2.7 Reformulation of the governing equations

Particular analytic solutions are readily available for certain choices of the forcing function. Examples of these will be encountered later where an analytic solution will be used to assess the accuracy of a numerical solution. Nunn & Sleep [7] provide an in-depth discussion of the solution for periodic loading using the elastic and viscoelastic models.

The current aim however, is to provide a means for solving the elastic or viscoelastic problem numerically in as efficient a manner as possible. We propose to solve the equations using finite difference discretizations, however, as both problems have 4th order derivatives in space, we are presented with several computational difficulties if they are left in their current formulation. We will alleviate some of the difficulties by re-writing each equation as a pair of coupled, second order, p.d.e.'s as follows. Equation
Chapter 2. Derivation of the Governing Equation

(2.4.5) for the viscoelastic problem becomes:

\[
\begin{align*}
D \nabla^2 v + \gamma \left( \frac{w}{\tau} + u \right) &= \frac{p}{\tau} + \dot{p} \\
\nabla^2 u &= v \\
\dot{u} &= \dot{w}
\end{align*}
\]  

(2.7.1)

with boundary conditions:

\[ u = 0, \; v = 0 \text{ on } \partial \Omega \]  

(2.7.2)

and initial condition:

\[ w(t = 0) = 0 \text{ on } \Omega \]  

(2.7.3)

assuming the exact solution is unknown. Equation (2.4.6) for the elastic problem may be written:

\[
\begin{align*}
D \nabla^2 u + \gamma w &= p \\
\nabla^2 w &= v
\end{align*}
\]  

(2.7.4)

with boundary conditions:

\[ w = 0, \; v = 0 \text{ on } \partial \Omega \]  

(2.7.5)

Notice that the boundary and initial conditions arise from a reformulation of those presented in section 2.6.

It is clear that the elastic, rather than viscoelastic, problem is the easier of the two to solve. The solution of the elastic problem requires an efficient solver for coupled elliptic p.d.e.'s of the form:

\[
\begin{align*}
\mathcal{L}^1(u^1, u^2) &= f^1 \quad \text{on } \Omega \\
\mathcal{L}^2(u^1, u^2) &= 0
\end{align*}
\]  

(2.7.6)

where

\[ u^1 = u^2 = 0 \text{ on } \partial \Omega \]  

(2.7.7)

Later it will be seen that having chosen a discretization for the time derivative, the viscoelastic problem may, in part, be written in the same form. So, once an efficient
solver has been developed for the elastic problem, it may also be used to solve part of the viscoelastic problem which, on the grounds of efficiency and modularity, is desirable.