Chapter 1

Introduction

The aim of this dissertation is to provide a fast, efficient, numerical method for determining the deflection of lithospheric plates under an applied load. In particular, the multigrid method is employed in this capacity. We will examine plates having horizontal dimensions ranging from those of sedimentary basins to continental plates. We will also consider a wide range of other physical parameters. Therefore, our attention is focussed on developing a method capable of providing a solution to a variety of geophysical problems rather than on developing a good model to one particular geophysical situation. In other words, we shall concentrate mainly on the numerical aspects of the topic while still maintaining a geophysical framework within which our forcing functions and model parameters will lie.

Our discussion will begin with a justification of the treatment of lithospheric plates as thin elastic or viscoelastic plates. In elastic theory one assumes that at any given time the plate responds only to the applied load at that time so that once the load is removed, the plate instantaneously reverts to its undisturbed state. In viscoelastic theory the assumption is that the plate responds not only to the present load but also to all previous loads so that it has the capacity to remember previous states of deformation. On removal of the load, a viscoelastic plate slowly relaxes to its undisturbed state. We thus find ourselves in the regime of thin plate deflection to which much attention has been given in the engineering literature. The equations governing thin elastic or viscoelastic plate deformation have already been solved numerically, quite successfully,
using finite element packages, but such methods tend to be very expensive. In a few special cases, an analytic solution may also be obtained allowing for a more objective evaluation of the numerical results.

The application of the current work to the evolution of sedimentary basins is of particular interest since such areas tend to be rich in hydrocarbons. Sedimentary basins are generally considered to be formed by the combination of a load due to thermal contraction following a sub-surface heating event and a subsequent load due to infilling of the deflection by sedimentary layers. It is the load due to thermal contraction which is generally considered to be responsible for the initial deflection in the plate and incidentally, the thermal anomaly causing this contraction also plays an important role in the maturation of petroleum, see Nunn, Sleep and Moore [8].

Various models for the evolution of sedimentary basins have been developed for both elastic and viscoelastic plates. Beaumont [2] examined the formation of sedimentary basins due to the loading of elastic and viscoelastic lithospheres. The problem was considered in cylindrical co-ordinates with the deflection depending only on the radial component. A simple \( \delta \)-function was used to represent the spatial variation (point loading) in the forcing term for the elastic model. For the viscoelastic problem, the forcing function was augmented with the heavy-side step function to represent the time dependence. Modelling of the plate loading was therefore very simple, however, mathematical tractability was maintained in the process. Both the elastic and viscoelastic problems were solved using integral transform techniques.

Beaumont also addressed the problem of basin initiation, i.e. that mechanism by which an initial deflection is created in the lithosphere in which sediment subsequently accumulates. Six possible mechanisms were proposed including thermal contraction following a sub-surface heating event, a subsiding graben and necking due to stretching of the lithosphere. Deflection of the lithosphere due to the sedimentary infill of an
instantaneously formed graben was examined. The case of an exponentially subsiding graben was also considered where subsidence was assumed to occur over a period of 100Ma. Results from the viscoelastic, rather than elastic, model were found to be in best agreement with seismic data obtained from the North Sea basin.

Watts, Karner & Steckler [12] examined a two-dimensional model where thermal contraction was assumed to control the tectonic subsidence of the plate, i.e. that subsidence which is independent of subsequent deflection due to sedimentary infill. Fourier transform techniques were used to solve the governing equations which are simplified by the assumption of two-dimensionality. Sedimentary loading was assumed to be constant over discrete time intervals so that at the beginning of each new period a layer of sediment was assumed to instantaneously infill the cavity caused by tectonic subsidence.

Two models were considered for tectonic subsidence the latter being attributed to thermal contraction in both cases. The first assumed that the depth of the basin increases as the square root of time since basin initiation. Again, formation of the basin was assumed to occur over a period of 100Ma. The second model was based on the assumption that the lithosphere is stretched laterally at the time of basin initiation resulting in plate thinning and heating which in turn result in plate subsidence on cooling.

A variation in elastic thickness, and therefore flexural rigidity, with time was allowed for in both the elastic and viscoelastic models between time periods. Spatial variation of the flexural rigidity was also considered. The latter problem was solved using a numerical finite difference scheme rather than Fourier Transform techniques. It should be noted that this problem is simplified considerably in two-dimensions, the extension to three dimensions being non-trivial. It was concluded that the best model fit to overall basin geometry arises from an elastic model in which the flexural rigidity increases with
time after basin initiation. This increase is due to an increase in elastic thickness with age as the plate cools. It should be noted that the conclusion here is at variance with Beaumont who favoured the viscoelastic model although the latter model assumed the flexural rigidity to be constant. It was again deduced that the dominant mechanisms in the formation of sedimentary basins are thermal contraction, which controls the overall shape of the basin, and sedimentary loading.

Nunn & Sleep [7] considered the load due to both thermal contraction and sedimentary loading for an elastic and viscoelastic lithosphere. A sinusoidal spatial variation was assumed for both and solutions were obtained using Fourier Series expansions. A linear deposition of sedimentary layers over discrete time intervals was considered where a variation in sediment density from one layer to the next was incorporated in the model.

An estimate of the thickness of sedimentary layers obtained from lithostratigraphic data for the Michigan Basin was used to compute the Fourier coefficients for the expansion representing the load due to sedimentary infill. Fourier coefficients for the load due to thermal contraction were computed using the fact that the deflections due to thermal and sedimentary loading must add to yield the observed deflection at some time, \( t_{\text{max}} \), after basin initiation. These coefficients were therefore re-computed for each set of rheological parameters.

It was estimated that the load due to sedimentary infill constitutes approximately 75% of the total driving force. The best model fit to observations was obtained for a viscoelastic lithosphere with low flexural rigidity. However, it was pointed out that with the data available at present, either model yields acceptable results, and a better understanding of basin initiation, sediment budget and sediment compaction is necessary before any definitive statements can be made.

We will examine the deflection due to the loading of both elastic and viscoelastic
plates, all models being three dimensional. Simple models which account only for sedimentary loading will be considered. We therefore assume that some process such as thermal contraction is responsible for the formation of a cavity in which the sediment accumulates. In the viscoelastic case, we allow the basin to form gradually so that the sediment accumulates in a continuous fashion. The infilling sediment is assumed to have constant density.

We now present an overview of material to be found in the following chapters. In Chapter 2 we will call upon material found in the engineering literature to provide a derivation of the governing equations for the deflection of a viscoelastic plate. In doing so we will closely follow a derivation of the equation governing deflection of an elastic plate replacing the elastic constitutive equations with those for a viscoelastic solid. We will also present appropriate boundary and initial conditions along with a nondimensionalized form of the governing equations. To conclude the chapter we will consider an appropriate reformulation of the governing equations which will make their solution easier numerically.

For both the elastic and viscoelastic problems, we are faced with solving a pair of coupled, linear, elliptic equations. An efficient solver for such a system is therefore required. In Chapter 3 we present an introduction to the multigrid method and consider its application to the two problems in hand. The presentation includes a discussion of the main components of multigrid as well as some basic algorithms. The fundamental difference between the elastic and viscoelastic problems, from a numerical standpoint, is that the viscoelastic model is time-dependent whereas the elastic problem is not. Chapter 3 is therefore concluded with a discussion of multigrid for time-dependent problems. Two algorithms are considered, one of which follows immediately from the time-independent, elastic problem. The applicability of the two algorithms is discussed and some attempt is made to ascertain the conditions under which one is expected
to perform more efficiently than the other. The performance of the two algorithms is tested numerically in Chapter 5.

In Chapter 4 we examine the elastic problem on a square domain. The assumption of a square domain places some restriction on the problem geophysically, but it still maintains a definite advantage over the assumption of two-dimensionality which has been used by other authors. However, the regular domain leads to an easier treatment of the problem from a numerical standpoint since irregular domains involve added computational difficulties near the boundaries and hinder the development of fast, efficient solvers.

We begin with the presentation of an analytic solution to which later numerical results will be compared. This analytic solution is used to examine the effect of altering the elastic properties of the plate on subsequent plate deflection. A relation expressing the degree to which a load is supported by the flexural rigidity of the plate will be derived. It will be shown that the problem is potentially tougher to solve in cases where the degree of support is either very high or very low.

The rate of convergence and accuracy of the multigrid solution will be examined for various plate dimensions and elastic constants representing a wide spectrum of degrees of support. A comparison between multigrid and a simple Gauss-Seidel iteration will be carried out showing that multigrid performs relatively more efficiently except in cases where the degree of support is very low in which case the numerical solution becomes trivial. We conclude the chapter by considering a Gaussian distribution for the loading for which an analytic solution is not available.

In Chapter 5 we turn to the viscoelastic (time dependent) problem which is certainly the more interesting of the two problems in hand. We begin with the presentation of an analytic solution representing an exponentially increasing forcing function with periodic spatial dependence. This representation has some major shortcomings in the
geophysical context but it is nonetheless considered acceptable in so far as it provides a means for assessing the numerical results more objectively. Two multigrid algorithms for time-dependent problems will be contrasted and compared. After an examination of the results and a consideration of the work involved in each of the algorithms, one will be eliminated on the grounds of both efficiency and accuracy. Some analysis of the reason for the efficiency and accuracy of one algorithm over the other will also be presented.

Having chosen a suitable algorithm we then examine the performance of three different approaches to dealing with the time discretization, namely, Backward Euler, a centred difference scheme and a two-step backward difference formula. The first of these provides a first order approximation in time to the solution while the other two provide second order approximations. It will be seen that the centred difference scheme, while providing a second order approximation, involves no more work or storage than Backward Euler, the first order method, making it an extremely attractive method from the point of view of both accuracy and efficiency. We examine the effect of reducing the time step and, in consideration of the results, the centred difference scheme is selected for use in future numerical experiments.

Next we consider the effect of allowing the forcing to reach a steady state. Analytically we expect all plates, regardless of size or elastic properties, to relax until they are in hydrostatic equilibrium with the applied load. We conduct two experiments for plates of different sizes and different elastic properties and show that if integration is continued for long enough after the forcing reaches a steady state then the numerical solution indeed appears to converge to the solution representing an adjustment of the plate to hydrostatic equilibrium.

Finally, we consider a forcing function which is more reasonable geophysically. The spatial variation is represented by a Gaussian distribution and the time dependence is
such that forcing is initially zero and reaches a steady state asymptotically. Two sizes of plate with different elastic properties will be considered and again the numerical results indicate that if integration is continued for a sufficiently long period, the solution converges to one representing hydrostatic equilibrium.

Conclusions arising from the numerical experiments are presented in Chapter 6. Suggestions as to the direction of future research are also proposed.