CHAPTER V

CONCLUSIONS

We have formulated and analysed an algorithm of the type Galerkin–Characteristic methods to integrate convection-dominated diffusion problems. The proposed algorithm has been combined with a mixed finite element method, similar to the one given in [11], to produce a formulation of the quasi-geostrophic equations (potential vorticity – stream function equations) for a mid-latitude baroclinic ocean.

The integration of the transport-diffusion equations by our algorithm is essentially a two stage process. The first stage corresponds to the integration of the advection operator along the characteristic curves of the flow in combination with Galerkin method. For this stage, we show that a reinterpretation of the usual Galerkin–Characteristic method in terms of particle methodology, with rectangular grids, yields a computationally efficient scheme, which consists of interpolating by cubic splines the grid point values of a functional of the dependent variable at the departure points of the particles. The scheme thus devised is conservative, unconditionally stable and convergent. Our error analysis in the maximum norm for this stage proves that for sufficiently smooth functions the approximate solution is super convergent at the foot of the characteristic curves. To assess the performance of our algorithm for the hyperbolic stage we have
carried out two types of advective experiments. The first one is a fairly hard problem which consists of advecting a cone in a rotating flow field. Munz [28] has reported the results of this problem obtained by high resolution finite difference schemes of the type total variation diminishing (TVD) such as MUSCL, UNO, etc., which are considered to be the best finite difference schemes for hyperbolic problems. A visual comparison of our results with those portrayed in figures 6 to 16 of [28] shows that:

1) Our scheme is able to keep the shape of the cone better than any high resolution scheme.

2) The 'numerical diffusion' of our scheme is lower than that of the high resolution finite difference schemes, with the possible exception of the superbee scheme.

3) Our scheme exhibits small wiggle activity at the base of the cone, whereas the high resolution finite difference schemes are wiggle free because they possess the TVD property.

4) Our scheme is able to use much larger time steps than any high resolution finite difference scheme because ours is unconditionally stable with respect to the $L^2$-norm. This property is responsible for keeping the wiggles under strict control.

Our second numerical experiment consists of advecting a 'slotted' cylinder in a rotating flow field. This is a very hard problem because of the discontinuities on the lateral surface and the 'slotted' region, respectively. A visual
comparison of our results with those reported in [37], using
SHASTA and modified SHASTA schemes, and [28], specifically
figures 6 to 16, using the aforementioned high resolution
schemes, yields the same conclusion as in the cone experiment.

The second stage corresponds to the time progression of
the algorithm via the diffusion mechanism starting with the
output of the previous stage. The time progression is carried
out in our analysis by the Crank–Nicholson scheme. The error
analysis with respect to $L^2$-norm, based on techniques
developed in [10] and [35], reveals for $\Delta t = O(h)$ the presence
of a term of order $O(h)$ in the evolutionary component of the
error. This term is due to the particle approximation of the
first stage and is suspected to be inherited by those
Galerkin–Characteristic methods which approximate the inner
products by alternative techniques, such as the ones proposed
in [27] and [5]. In this sense, the Crank–Nicholson scheme is
suboptimal when used in conjunction with our method; however,
for larger time steps it leads to more accurate results than
the backward Euler scheme, which appears to be optimal for $\Delta t
= O(h)$. Our error analysis technique is more general than the
ones used in [13], [31] and [33], which cannot be used for the
Crank–Nicholson scheme.

The constants $C_{01}$ and $C_{02}$ in Chap.IV (2.11), which
multiply the terms of the evolutionary error, are the product
of the exponential constant of the Gronwall’s inequality with
the norm of the derivatives of the variable along the
characteristic curves. For convection dominated flows the
speed of variation along the characteristics is less rapid than in the $t$ direction, so the algorithm permits larger time steps without loss of accuracy. On the other hand, since the algorithm in the second stage is also unconditionally stable, then the exponential constant of the Gronwall's inequality is less than one; so that as time progresses the influence of the evolutionary error on the global accuracy of the method decreases, and eventually for $T \to \infty$ the error of the approximate solution is the approximation error. The latter type of error is $O(h^2)$ with free-slip boundary conditions, and $O(h)$ with no-slip boundary conditions. These estimates are one order higher than previous estimates given in the literature. Finally, we prove that the approximation error with respect to $L^2$-norm for the stream functions is $O(h^2)$, which is optimal, while the $L^2$-norm approximation error for the velocities is $O(h)$. Numerical experiments with no-slip boundary conditions shows the validity of our error estimates.

As a final remark, we should mention that our analysis can be extended to approximate the solution of Navier-Stokes equations by the proposed algorithm.