CHAPTER III

A GALERKIN-CHARACTERISTIC ALGORITHM FOR THE HYPERBOLIC STAGE

This Chapter is devoted to the formulation and analysis of the first stage of the algorithm. For this purpose, we take as prototype equation the linear advection (transport) equation, the solution of which is constant along the characteristic curves of the flow. An interesting property of this equation is that under very weak regularity assumptions the weak solution coincides almost everywhere with the classical solution. Our main concern here is the approximation of the weak solution by a combination of Galerkin and Characteristic methods. Towards this end we cover the computational domain \( \Omega, \Omega \subset \Omega_1 \subset \mathbb{R}^2 \) (\( \Omega_1 \) is supposed to be a rectangular domain), with a rectangular grid and assign one particle to each node of the grid. Then we trace back the positions of the particles along the characteristic curves during a time interval \( \tau \) and assume that the magnitude of the dependent variable is transported by the particles, so the values of the dependent variable at the grid points \( x_k \) at instant \( t + \tau \) are the values possessed by the particles at the points \( X(x_k, t + \tau; t) \). Here and in the sequel, \( X(x_k, t + \tau; t) \) denotes the position at instant \( t \) of the particle that at instant \( t + \tau \) is at \( x_k \).

The mathematical representation of this physical idealization would be a linear combination of Dirac delta
functions centered at \( X(x_k^*, t + \tau; t) \). But this representation is not very convenient for numerical computations, so we must regularize it mathematically. This is achieved if one substitutes smoother piecewise functions for the Dirac delta functions. From a physical point of view, this amounts to replacing point particles by parcels of limited extension which are advected with their centroid velocities. We choose as smoother piecewise functions the basis functions of the finite element space centered at the departure points \( X(x_k^*, t + \tau; t) \). Therefore, the particle solution at instant \( s = t + \tau \) is given by

\[
\omega(x, s) = \sum_k \alpha_k(t) \phi_k(x - X(x_k^*, s; s)),
\]

where \( \alpha_k(t) \) denote the values of \( \omega(x, t) \) at the points \( X(x_k^*, s; t) \). Since \( \phi_k(x - X(x_k^*, s; s)) = \phi_k(x) \), then \( \omega(x, s) \) belongs to the finite element space \( H_h \). To determine the weights \( \alpha_k(t) \) we project the particle solution onto \( H_h \) and show that this is equivalent to computing the values of \( \alpha_k(t) \) by cubic spline interpolation of the grid point values of a given functional of \( \omega(x, t) \). We prove that this algorithm is conservative, unconditionally stable in the \( L^2 \)-norm and convergent. Moreover, for sufficiently smooth functions the solution is superconvergent at the foot of the characteristic curves.

1. Preliminaries

Let the computational domain \( \Omega \) be an open subset of \( \mathbb{R}^2 \) with Lipschitz boundary \( \Gamma \), \( \mathbb{R}^2 \) is the two-dimensional Euclidean
space. We now introduce the definitions of splines and B-splines as well as some of their properties. Let \( I_1 = [a, b] \), \( I_2 = [c, d] \) and assume that the domain \( \Omega = I_1 \times I_2 \). Let \( I, J \) be positive integers such that \( \Delta_1 = \{ x_{1i} \}_{i=1}^{I}, \Delta_2 = \{ x_{2j} \}_{j=1}^{J} \) are partitions of \( I_1 \) and \( I_2 \) respectively, which satisfy

\[
\begin{align*}
& a = x_{11} < x_{12} < \ldots \ldots < x_{1I} = b, \\
& c = x_{21} < x_{22} < \ldots \ldots < x_{2J} = d.
\end{align*}
\]

For \( r, s \) positive integers, we define the linear space of spline functions of order \( r \) over \( I_1 \) as

\[
S_{r, \Delta_1}(I_1) = \{ S(x_1) \in H^r_p(I_1) : D^r S(x_1) = 0 \text{ for } x_1 \text{ in } [x_{1i}, x_{1i+1}] \},
\]

for \( i = 1, \ldots, I \). Here

\[
H^r_p(I_1) = \{ f \in C^{r-2}(I_1) : D^{r-2}f \text{ is absolutely continuous}, \\
D^{r-1}f \in L^p(I_1) \}
\]

and \( H^1_p(I_1) = L^p(I_1) \).

Specifically, a spline \( S(x_1) \in S_{r, \Delta_1}(I_1) \) is a polynomial of degree \( r - 2 \) which is continuous and with continuous derivatives up to order \( r - 2 \).

The linear space of spline functions of order \( s \) over \( I_2 \) is defined similarly. We are interested in the class of tensor product splines \( S_{r, \Delta_1} \circ S_{s, \Delta_2} \) defined as follows.
\[ S_{r, \Delta_1} \otimes_{S_{s, \Delta_2}} = \{ S(x_1, x_2) = S_1(x_1)S_2(x_2) : S_1 \in S_{r, \Delta_1}, S_2 \in S_{s, \Delta_2} \} \]

We note that

\[ S_{r, \Delta_1} \otimes_{S_{s, \Delta_2}} \subset W^k, p(\Omega), \]

where \( k = \min(r, s) \).

In order to simplify the notation, we denote by \( S_{k, h}(\Omega) \) the restriction to \( \Omega \) of the tensor product spline of \( S_{r, \Delta_1} \) with \( S_{s, \Delta_2} \). We now formulate a lemma which is a special case of a result due to Widlund [36] for the spline approximation procedure \( S \). To this end let \( S(\Omega) \) be a linear space of functions in which \( S \) is defined. Typically \( L^p(\Omega) \subset S(\Omega) \) for \( 1 \leq p \leq \infty, m = 0, 1, \ldots, k \). Furthermore, assume that \( S \) satisfies the following properties.

1) **Uniqueness.** For any \( f \in S(\Omega) \) there exists one and only one \( f_h \in S_{k, h}(\Omega) \) such that \( Sf = f_h \).

2) **Stability.** There is a constant \( C \), such that

\[ \| f_h \|_p \leq C \| f \|_p, f \in L^p (\Omega) \cap S(\Omega). \]

3) **Quasi-linearity in \( L^p (\Omega) \).**

\[ \| (f_1 + f_2)_h \|_p \leq \| f_1 \|_p + \| f_2 \|_p, f_1, f_2 \in S(\Omega) \]

4) **Optimal accuracy.** For all sufficiently smooth functions \( f \)

\[ \| f_h - f \|_p \leq C h^k |f|_{p, k}, \]

where \( k \) denotes the order of the spline.
Lemma 1.1 [36]. Let the approximation procedure \( S \) satisfy i)-iv), then

\[
\| f_n - f \|_p \leq Ch^r |f|_{p,r}.
\]  

(1.1)

for \( 0 < r < k \).

An alternate and constructive definition of \( S_{r,A_1} \) and \( S_{s,A_2} \) involves \( B \)-splines as follows [7]. Enlarge the sequences \( \{x_{1i}\}_{i=1}^I \) and \( \{x_{2j}\}_{j=1}^J \) to non-decreasing sequences \( \{x_{1i+r}\}_{i=1}^{I+r} \) and \( \{x_{2j+s}\}_{j=1}^{J+s} \), the additional points being otherwise arbitrary. For \( l = 1, \ldots, I+r \) and \( j = 1, \ldots, J+s \), the \( i \)-th \( B \)-spline (\( j \)-th \( B \)-spline) of order \( r \) (\( s \)) for these sequences is

\[
B_{i,r}(x) = (x_{1i+r-1} - x_{1i})[x_{1i}, \ldots, x_{1i+r}](x_{1i} - x)^{r-1},
\]

(1.2)

for all \( x \) in \( I_1 \). Similarly for \( B_{j,s}(y) \), for all \( y \) in \( I_2 \).

Here \( [x_{1i}, \ldots, x_{1i+r}] \) is the \( r \)-th divided difference on the function \( (x_{1i} - x)^{r-1} = \max(0, x_{1i} - x) \) keeping \( x \) fixed. From (1.2), the following recurrence relation is obtained.

\[
B_{i,1}(x) = \begin{cases} 1, & x_{1i} \leq x \leq x_{1i+1} \\ 0, & \text{otherwise.} \end{cases}
\]

(1.3.1)

\[
B_{i,r}(x) = \frac{x - x_{1i}}{x_{1i+r-1} - x_{1i}} B_{i,r-1}(x) + \frac{x_{1i+r} - x}{x_{1i+r} - x_{1i+1}} B_{i+1,r-1}(x).
\]

(1.3.2)

We recall some properties of the \( B \)-splines which are needed in the thesis (cf. [7]).
B1) \( B_{i,r}(x) = 0 \) for \( x \) not in \([x_{1i}, x_{1i+r}]\).

B2) For any \((x_1, x_2)\) in \(\Omega\)

\[
\sum_{i=1}^{1} \sum_{j=1}^{J} B_{i,r}(x_1)B_{j,s}(x_2) = 1 .
\] (1.4)

B3)

\[
S_{k,h}(\Omega) = \text{span} \{B_{i,r}(x_1)B_{j,s}(x_2)\}_{i,j=1}^{I,J} .
\]

B4) Let the non-decreasing sequence \(\{x_{1i}\}_{i=1}^{I+r} \times \{x_{2j}\}_{j=1}^{J+s}\) form the interior knots of the corresponding sequence of B-splines \(\{B_{i,r}(x_1)B_{j,s}(x_2)\}_{i,j=1}^{I,J}\). For the strictly increasing sequence \(\chi = \{x_{1i}\}_{i=1}^{I} \times \{x_{2j}\}_{j=1}^{J}\) of data points, the spline

\[
S(x_1, x_2) = \sum_{i,j} c_{ij} B_{i,r}(x_1)B_{j,s}(x_2) .
\] (1.5)

will agree with the given function \(f\) at \(\chi\) if and only if

\[
S(x_{1i}, x_{2j}) = f(x_{1i}, x_{2j}) , \ \forall \ i,j .
\]

Schoenberg-Whitney theorem (cf.[7]) guarantees the unique solution of (1.5) if and only if

\[
B_{i,r}(x_{1i})B_{j,s}(x_{2j}) \neq 0 , \ \forall \ i,j , \quad (1.6.1)
\]
or, equivalently, if and only if

\[
x_{1i} < x_{1i} < x_{1i+r} , \ \forall \ i , \quad (1.6.2)
\]

\[
x_{2j} < x_{2j} < x_{2j+s} , \ \forall j .
\]
2. Description of the Algorithm

2.1. The Continuous Problem

Consider the Cauchy problem for the scalar advection equation for \( \omega(x, t) \) in the cylinder \( Q_T \),

\[
\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \frac{D \omega}{D t} = 0 ,
\]

(2.1.1)

\[
\omega(x, 0) = \omega_0(x) \quad \text{in} \quad \Omega ,
\]

(2.1.2)

where \( \frac{D}{D t} \) denotes the material derivative of \( \omega \) in the flow \( u \). We assume that the velocity field is incompressible, i.e.,

\[
\nabla \cdot u = 0 \quad \text{in} \quad Q_T ,
\]

(2.2)

and tangent to the boundary \( \Gamma \) of \( \Omega \), i.e.,

\[
u \cdot n |_\Gamma = 0 .
\]

(2.3)

We also require

\[
u(x, t) \in L^\infty(0, T; W^{1,\infty}(\Omega)) .
\]

(2.4)

To compute the solution of (2.1) we introduce the characteristic curves \( X(x, s; t) \) of (2.1) which satisfy the system of differential equations
\[
\frac{dX(x,s;t)}{dt} = u(X(x,s;t), t), \quad (2.5.1)
\]
\[
X(x,s;s) = x. \quad (2.5.2)
\]

We state without proof the existence and uniqueness of the solution of (2.5) and summarize some regularity results.

**Proposition 2.1.** Under condition (2.4) there exists a unique solution \( t \rightarrow X(x,s;t) \) of the system (2.5) such that \( X(x,s;t) \) is in \( C^0(0,T;W^{1,\infty}(\Omega)) \). Furthermore, assume that for some integer \( k \geq 1 \) is in \( L^\infty(0,T;W^{k,\infty}(\Omega)) \), then for all \( \alpha \in \mathbb{N}^n \), \( 1 \leq |\alpha| \leq k \), \( t \rightarrow D^\alpha X(x,s;t) \in C^0(0,T;L^\infty(Q)) \).

The solution of (2.5) can be expressed as
\[
X(x,s;t) - x = \int_s^t u(X(x,s;\tau), \tau) d\tau, \quad (2.6)
\]
where \( X(x,s;t) \) denotes the position at time \( t \) of the particle of fluid which is at point \( x \) at instant \( s \).

We need further results of the solution of (2.5) which we now formulate.

**Lemma 2.1.** Under assumptions (2.2), (2.3) and (2.4) and for \( |s-t| \) sufficiently small, \( x \rightarrow X(x,s;t) \) is a quasi-isometric homeomorphism of \( \Omega \) into itself with Jacobian determinant equal to 1 a.e. Moreover,
\[
L^{-1} |x - y| \leq |X(x,s;t) - X(y,s;t)| \leq L |x - y|, \quad (2.7)
\]
where \( L = \exp(|s-t| \cdot |u|_{1,\infty,Q}) \).
Proof. For all $s,t$ in $(0,T)$ we have from Proposition 2.1 (cf. [18]) that there is a neighborhood $U(x)$ for which $X(x,s;t)$ exists and is unique. By fixing $s$, $X(x,s;t)$ can be considered as a mapping $\phi^t_s(x)$ from $U(x)$ into a neighborhood of $X(x,s;t)$. Proposition 2.1 guaranties this mapping is one to one and continuous, as well as its inverse $\phi^s_t = X(X(x,s;t),t;s) = x$. On the other hand, since the velocity satisfies (2.3), then it follows that $X(x,s;t)$ maps $\Omega$ into itself. It is easy to obtain the inequality (2.7) by applying Gronwall’s inequality to (2.5) and (2.6). Hence, $x \rightarrow X(x,s;t)$ is a quasi-isometric homeomorphism. The Jacobian determinant of the above transformation is given as $J(x,s;t) = \det(DX(x,s;t))$. Proposition 2.1 again guaranties that $t \rightarrow J(x,s;t)$ is continuous. Liouville's formula and condition (2.2) yield $J(x,s;t) = 1$ a.e.

Remark. It can be proved (see [25], Th. 1.1.7) that $x \rightarrow X(x,s;t)$ being a quasi-isometric mapping of class $C^{m-1,1}(\Omega)$, $m \geq 1$, which maps $\Omega$ into $\Omega^*$, the norms $\| \cdot \|_{m,p,\Omega}$ and $\| \cdot \|_{m,p,\Omega^*}$ are equivalent. It is a simple matter to check that if the Jacobian determinant of the transformation is 1 a.e. then $\| \cdot \|_{2,\Omega} = \| \cdot \|_{2,\Omega^*}$.

The unique classical solution of (2.1) is given by

$$\omega(x,t) = \omega_0(X(x,t;0)) \quad \text{(2.8)}$$

This condition holds under very weak regularity assumptions. It is well known [29] that for some integer $m \geq 1$, $u \in$
\[ L^\infty(0,T;W^m,\Omega) \] and \( \omega_0 \in W^m,\Omega \) the weak solution of the problem (2.1) belongs to \( L^\infty(0,T;W^m,\Omega) \) and is given by (2.8). Now, let \( \phi(\cdot) \in D(R^2) \), where \( D(R^2) = \{ \phi \in C^\infty(R^2) : \phi \) has compact support \( \} \), then

\[ \int_{R^2} \omega(x,t)\phi(x)dx = \int_{R^2} \omega(0)\phi(X(y,0;t))dy , \quad (2.9) \]

since the Jacobian determinant of the transformation \( x \to X(x,t;s) \) is equal to 1. a.e. In (2.9) \( y = X(x,t;0) \).

2.2. The Discrete Problem

Let \( M \) be a positive integer, \( \Delta t = T/M \) and \( t_m = m\Delta t \) for \( 0 \leq m \leq M-1 \). Let \( u_h(x,t) \) be an approximation to \( u(x,t) \). We assume that \( u_h(x,t) \) satisfies conditions(2.2)-(2.4). For \( s = t_m \), the approximate trajectories of the points \( x \) in the time interval \( [t_m,s] \) are given, according to (2.6), by

\[ X_h(x,s;s-\tau) = x - \int_0^\tau u_h(X_h(x,s;s-\varepsilon),s-\varepsilon)d\varepsilon , \quad (2.10) \]

\[ 0 \leq \tau \leq \Delta t \]

For \( \tau = \Delta t \), \( X_h(x,s;t_m) \) denotes the departure point of the trajectory which reaches the point \( x \) at instant \( s=t_{m+1} \).

The error

\[ e(\tau) = X(x,s;s-\tau) - X_h(x,s;s-\tau) \]
committed in approximating the trajectories by (2.10) satisfies the following lemma.

Lemma 2.2. Assume that \( u_h(x,t) \) and \( u(x,t) \) fulfill conditions (2.2)-(2.4), then for any integer \( m \) in the interval \([0, M-1]\) and for any real \( \tau \) such that \( 0 \leq \tau \leq \Delta t \) the error \( e(\tau) \) is bounded by

\[
|e(\tau)| \leq \frac{|u_h - u|_{\omega, Q}}{|\nabla u|_{\omega, Q}} \left( \exp(\tau |\nabla u|_{\omega, Q}) - 1 \right). \tag{2.11}
\]

Proof. Subtracting (2.6) from (2.10) it follows that

\[
|e(\tau)| \leq \int_0^\tau |u_h(X_h(x,s;\varepsilon),s-\varepsilon) - u(X(x,s;\varepsilon),s-\varepsilon)| \, ds
\]

\[
\leq \int_0^\tau |u_h(X_h(x,s;\varepsilon),s-\varepsilon) - u(X_h(x,s;\varepsilon),s-\varepsilon)| \, ds
\]

\[
+ \int_0^\tau |\nabla u|_{\omega, Q} |e(\varepsilon)| \, ds.
\]

Applying Gronwall's inequality yields the bound (2.11). \( \square \)

At any instant \( t_{m+1} \) we approximate the weak solution \( \omega(x,t) \) by a two step process. First, a particle approximation of (2.8) is set up, then this approximation is projected onto the finite element space \( H_h \). Towards this end we introduce the set \( \bar{H} \) of cut-off functions \( \bar{\phi}_k(x) \) defined as follows. For \( k = 1, 2, \ldots, K \) and \( 1 \leq i \leq I, 1 \leq j \leq J \)

a) \( \bar{\phi}_k(x) \) is bilinear in \( x_1 \) and \( x_2 \)

b) \( \text{supp}\bar{\phi}_k(x) = [-h_{1i-1}, h_{1i}] \times [-h_{2j-1}, h_{2j}], \forall i,j, \)

with \( -h_{1i-1} \leq x_1 \leq h_{1i}, -h_{2j-1} \leq x_2 \leq h_{2j} \).

c) \( \bar{\phi}_k(x-x_k) = \phi_k(x), \phi_k(x) \in \{\phi_k(x)\}_{k=1}^K, \) canonical basis of
Thus

\[ \overline{H} = \{ \phi_k(x) \}_{k=1}^K. \]

Notice that \( \text{supp} \phi_k(x) \) is the shifted support of \( \phi_k(x) \) to the intervals \([-h_{1i-1}, h_{1i}] \times [-h_{2j-1}, h_{2j}], \forall i, j \). It is clear that \( \phi_k(x) \) have compact support in \( \mathbb{R}^2 \) and satisfy

1) \( \phi_k(x) \in W^{1,2}(\mathbb{R}^2), \quad (2.12.1) \)

2) \( (4/\mu_k) \int_{\mathbb{R}^2} \phi_k(x) dx = 1, \quad (2.12.2) \)

where \( \mu_k = \text{meas}(\text{supp} \phi_k(x)). \)

Let \( X_h(x_k, t_{m+1}; t_m) \) be the departure point of the trajectory which reaches the grid knot \( x_k \) at instant \( t_{m+1} \). Let us consider \( \phi_k(x - X_h(x_k, t_{m+1}; t_m)) = \delta(x - X_h(x_k, t_{m+1}; t_m)) * \phi_k(x), \) where \( \delta(\cdot) \) is the Dirac measure. Another property of \( \phi_k(x - X_h(x_k, t_{m+1}; t_m)) \) which is required for our algorithm to make sense is the following:

3) If for some \( k \) and \( j \) \( \text{supp} \phi_k(x) \cap \text{supp} \phi_j(x) \neq \emptyset \), then \( \text{supp} \phi_k(x - X_h(x_k, t_{m+1}; t_m)) \cap \text{supp} \phi_j(x - X_h(x_j, t_{m+1}; t_m)) \neq \emptyset \) \( (2.13) \)

Let us check that iii) is satisfied.

Assume \( \Delta t = O(h), \) by virtue of (2.7), (2.11) and the regularity of the finite element partition one has
\[ |X_h(x_j, t_{m+1}; t_m) - X_h(x_k, t_{m+1}; t_m)| \leq |X(x_j, t_{m+1}; t_m) - X_h(x_j, t_{m+1}; t_m)| + |X(x_k', t_{m+1}; t_m) - X_h(x_k, t_{m+1}; t_m)| + |X(x_j, t_{m+1}; t_m) - X(x_k', t_{m+1}; t_m)| \leq |x_j - x_k| + O(h \Delta t + |u - u_h|_{\infty, Q} \Delta t). \] (2.14)

Since \( \text{supp}\phi(x) \cap \text{supp}\phi_j(x) \neq \emptyset \) is of order \( O(h) \), then it is clear from (2.14) that (iii) is satisfied. For reasonably structured grids, not necessarily uniform, this condition also ensures that for any \( k \) in \( (1, \ldots, K) \) there exists at least an index \( j \) in \( (1, \ldots, K) \) such that \( x_k \in \text{supp}\phi_j(x - X_h(x_j, t_{m+1}; t_m)) \). Next, assume that \( y \in \text{supp}\phi_k(x) \), then if one takes \( y \) for \( x_j \) in (3.14) it follows that \( \text{supp}\phi_k(x - X_h(x_k, t_{m+1}; t_m)) \) approximates up to \( O(h \Delta t + |u - u_h|_{\infty, Q} \Delta t) \) the element whose points are \( X_h(y, t_{m+1}; t_m) \) for any \( y \) in \( \text{supp}\phi_k(x) \).

We are now in conditions to state the main result of this section and provide a description of the proposed algorithm to compute the weak solution at the points \( X_h(x_k, t_{m+1}; t_m) \). Before doing so let us introduce some notational simplification. Hereafter, we set \( X_h(x_k, t_{m+1}; t_m) = X_{hk}^m \) with Cartesian components \( (X_{hk1}^m, X_{hk2}^m) \), and denote the matrices \( B_{4,1}(x_{1p}) \) and \( B_{4,2}(x_{2q}) \), generated by the B-splines \( B_{4,1}(x) \) and \( B_{4,2}(x) \) at the grid points \( \{x_{1p}\} \times \{x_{2q}\}, 1 \leq p \leq I, 1 \leq q \leq J, \) by \( A^*_1 \) and \( A^*_2 \), respectively. Such matrices are invertible.

Let \( \{\omega_k^{m+1}\}_{k=1}^K \) be the values of the approximate solution at the grid points at instant \( t_{m+1} \) and \( A \) a symmetric positive
definite matrix with entries \( a_{rs} = (\phi_r, \phi_s) \). The product (see below the proof of the Theorem 2.1)

\[
A[\omega^{m+1}] = [B]
\]  

(2.15)

satisfies the theorem that follows. In (2.15) the entries \( b_k \) of \([B]\) represent the values at instant \( t_m \) of the projection of the approximate solution \( \omega_h(x, t_m) \) at the departure points \( x^m_{hk} \).

Theorem 2.1. Consider the set \( \{B_{i,4}(x_1)B_{j,4}(x_2)\}^I,J_{i,j=1} \) of B-spline basis for the space \( S_{4,h}(\Omega) \). Then at any instant \( t_m \), \( 0 \leq m \leq M-1 \), the entries of the matrix \([B]\) are given by

\[
b_k = \sum_{i=1}^I \sum_{j=1}^J c_{ij}^{m} B_{i,4}(x^m_{hk1})B_{j,4}(x^m_{hk2}), \forall k \text{ in } 1,2,...,K.
\]  

(2.16.1)

Moreover, the coefficients \( c_{ij}^{m} \) satisfy the relation

\[
A^{\star}_1 A^{\star}_2 (c^{m}) = \tilde{A}[\omega^{m}].
\]  

(2.16.2)

For the definition of the matrix \( \tilde{A} \) see below the proof of the theorem. In general, for well structured grids it is advantageous to take \( \tilde{A} = A \) for reasons to be explained below.

Theorem 2.1 simply states that at any instant \( t_m \) the approximate solution \( \omega_h(x, t_{m+1}) \) is updated at the grid points by performing cubic spline interpolation of \( \omega(x, t_m) \) at the departure points \( x^m_{hk} \).

Thus, the algorithm to compute the first hyperbolic stage consists of the following steps:
Given \( [\omega^0] \):

i) \( \forall m \in [0, M-1] \), compute \([c^m]\) by solving (2.16.2). The most efficient way of performing this step is using the properties of the tensor product of matrices. Let \( \tilde{A}[\omega^m] = \tilde{A} \). The column vector \( \tilde{A} \) can be arranged as an \( I \times J \) matrix \( \tilde{A} = (r_{ij}) \), \( 1 \leq i \leq I \) and \( 1 \leq j \leq J \). Likewise, the column vector \([c^m]\) is arranged as an \( I \times J \) matrix \( C \) such that

\[
CA^* = Y.
\]

Then, for \( 1 \leq j \leq J \), solve

\[
A^*_1[y_{ij}] = [r_{ij}] \quad 1 \leq i \leq I.
\]

Finally, for \( 1 \leq i \leq I \), solve

\[
A^*_2[c_{ij}] = [y_{ij}] \quad 1 \leq j \leq J.
\]

Since the matrices \( A^*_1 \) and \( A^*_2 \) are diagonally dominant, they can be efficiently inverted by Gauss elimination without pivoting (cf.\([7]\)). The number of long operations at this step is \( O(I \times J) \).

ii) At any instant \( t_m \), determine by sequential or binary search the intervals where the points \( (X^m_{hk}) \) lie.

iii) For each pair \( (X^m_{hp1}, X^m_{hq2}) \), \( x_{1p-1} \leq X^m_{hp1} \leq x_{1p} \), \( x_{2q-1} \leq X^m_{hq2} \leq x_{2q} \), evaluate the B-spline basis

\[
v_i = B_{4,i}(X^m_{hp1}), \quad \text{for } i = p, p+1, p+2, p+3
\]

using the relations (1.3.1) and (1.3.2).

iv) For \( j = q, q+1, q+2, q+3 \) form

\[
d_j = \sum_{i=p}^{p+3} v_i c^m_{ij}.
\]

v) Evaluate
\[
\sum_{j=1}^{J} \int_{J,j,J}^{} \frac{C_{j}^{1}(x^{m}_{hq2})}{J}
\]

Finally, solve the system (2.15) in \(O(I \times J)\) operations to obtain \(\omega_{m+1}^{k}\).

The number of long operations (multiplications and divisions) taken by steps iii) - v) is \(4r^{2} + o(r)\) per point. Here, \(r\) denotes the order of the spline. Hence, the total number of operations to carry out the computations at \(K\) departure points is \(64K + O(K)\). This number is slightly larger than that of the standard bicubic spline procedure which is \(49K + O(K)\). Thus, from a computational point of view the use of B-splines is more expensive than the standard bicubic spline interpolation procedure; however, the local nature of the B-splines may offer some advantages when the domain is not rectangular or the solution is not globally smooth.

**Proof of Theorem 2.1**

According to Raviart [32] and Mas-Gallic and Raviart [24] we may define a particle approximation of the weak solution \(\omega(x,t)\) as follows. We approximate the initial condition \(\omega_{0}(y)\) by a linear combination of Dirac measures

\[
\omega_{0p}(y) = \sum_{k=1}^{K} \rho_{k}^{0} \omega_{hk}^{0}(y - y_{k})
\]

(2.17)

for some set \(\{y_{k}, \rho_{k}^{0} \omega_{hk}^{0}\}\) of points \(y_{k} \in \Omega\) and \(\rho_{k}^{0} \omega_{hk}^{0} \in R\). By substituting (2.17) into (2.9) a direct calculation yields
\[ \omega_p(x, t) = \sum_{k=1}^{K} p_k \omega^0_{hk}(x - X_h(x_k, t; t)) . \tag{2.18} \]

Here \( X_h(x_k, t; 0) = y_k \).

In [32, Th.4.1.] is proved the convergence of (2.18) to the weak solution (2.9) for any \( \phi \in C_0^0(R^2) \). From a practical point of view it is convenient to regularize (2.18) by a convolution product with a cut-off function. Several types of cut-off functions have been proposed in the literature (cf.[32]); however, we are interested in using cut-off functions of low degree of smoothness and which are suitable to work with \( C^0 \) - conforming finite elements. Next, we define particle approximation to \( \omega(x, t) \) at instant \( t_{m+1} \) as

\[ \omega_p(x, t_{m+1}) = \sum_{k=1}^{K} p_k \omega_{hk}(t_m) \delta(x - X_h(x_k, t_{m+1}; t_{m+1})) , \tag{2.19} \]

where \( \omega_{hk}(t_m) \) is to be understood as an approximate value to \( \omega(X_h(x_k, t_{m+1}; t_m), t_m) \), \( p_k \) denotes the value of a weight function at \( X_{hk}^m \) and \( x_k \) is a grid point. The regularization process of (2.19) amounts to replacing point particles, which move along the characteristic curves passing through the vertices \( \{x_k\} \), by parcels of limited extension moving with their centroids. We denote by \( \omega_{ph}(x, t_{m+1}) \) the regularized form of \( \omega_p(x, t_{m+1}) \) which is constructed as follows.

\[ \omega_{ph}(x, t_{m+1}) = \sum_{k=1}^{K} p_k \omega_{hk}(t_m) \delta(x - X_h(x_k, t_{m+1}; t_{m+1})) * (p_{-1} \phi_k(x)) \]
\[ \sum_{k=1}^{K} \omega_{hk}(t_m \phi_k(x - \bar{X}_h(x, t_{m+1}; t_{m+1})) = (2.20) \]

We note right away that \( \omega_{ph}(x, t_{m+1}) \) is in \( L^2(\Omega) \) since by Young's theorem each convolution operator \( \rho_k \omega_{hk}(t_m \delta(\cdot, \cdot; \cdot)) \) is in \( L^2(\Omega) \). By the definition of \( \bar{\phi}_k \) and given that the points \( \bar{X}_h(x, s; t) \) are in \( \Omega \) as one deduces from Lemma 2.1, it is clear that \( \omega_{ph}(x, t_{m+1}) \in W^{1,2}(\Omega) \) for any \( t_{m+1} \). Fig.2 depicts a graphical representation of the regularization process at point \( \bar{X}_{hk} \) which arrives at vertex \( x_k \) at instant \( t_{m+1} \).

Now, we look for approximating the solution (2.20) in the space \( H_h \). Since \( \bar{\phi}_k(x - \bar{X}_h(x, t_{m+1}; t_{m+1})) \) coincides with \( \phi_k(x) \), then we can set \( \omega_{ph}(x, t_{m+1}) = \omega_{h}(x, t_{m+1}) \) where

\[ \omega_h(x, t_{m+1}) = \sum_{k=1}^{K} \omega_{hk} \phi_k(x), \ x \in \Omega, \quad (2.21) \]

and such that

\[ (\omega_h(x, t_{m+1}), \phi_k) = (\omega_h(y, t_m), \bar{\phi}_k(y - \bar{X}_h(x, t_{m+1}; t_m)), \ \forall k, m, \quad (2.22) \]

where \( (u,v) = \int_{\Omega} uv d\Omega \).

From (2.22) one obtains that the weights \( \omega_{hk} \) satisfy the algebraic linear system

\[ A[\omega] = [B]. \quad (2.15) \]

Our next step in the proof is to show that for any \( k, 1 \leq k \leq K \), the entries \( b_k \) of \( [B] \) are the interpolant values of a
bicubic spline at the points \(X^m_{hk}\). By virtue of the properties of the tensor product of functions on rectangular grids we can prove this assertion for the one dimensional case without loss of generality. Let \(a, b\) be the end points of the interval \(I_1 := a = x_1 < \ldots < x_1 = b\) and consider

\[
K(X_k) = \sum_r \omega_r \int_a^b \tilde{\varphi}_k(x - X_k)\phi_r(x)dx, \quad \forall \, k, r, \tag{2.23}
\]

where

\[
\tilde{\varphi}_k(x - X_k) = \begin{cases} 
\frac{(x+h_k) - X_k}{h_k}, & x \in [X_k, X_k + h_k], \\
-\frac{(x - h_{k+1}) - X_k}{h_{k+1}}, & x \in [X_k, X_k + h_{k+1}], \\
0, & \text{otherwise}.
\end{cases}
\]

Dropping the symbol \(\Sigma\), \(K(X_k)\) can now be written as

\[
K(X_k) = -\int_a^{X_k} \left( \frac{X_k - x}{h_k} \right)\phi_r(x)dx + \int_{X_k}^b \left( \frac{X_k - x}{h_{k+1}} \right)\phi_r(x)dx
\]

\[
+ \int_a^b \phi_r(x)dx.
\]

Clearly, \(K(X_k)\) is a piecewise cubic polynomial in \(X_k\). Now, if the grid is either uniform in each coordinate direction or progressively smoothly variable, i.e. a well structured grid, then by virtue of (2.14) one has that for any \(k\) there is an \(r\) such that \(K(X_k)\) possesses first and second order continuous derivatives in the interval \([x_r, x_{r+1}]\), and with breaking points \(\{x_r\}\). Such collection of piecewise cubic polynomials generate the linear space \(\mathcal{P}_{4,x_r,3}\), of piecewise cubic polynomials with continuous derivatives up to second order at
the breaking points \( \{ x_r \} \). According to Curry-Schoenberg theorem (cf. [7]) \( P_4, x_r, 3 \), coincides in the two dimensional case with the space \( S_{4, h}^4(\Omega) \). Hence, there exists a cubic spline \( S(x) \) in \( S_{4, h}^4(\Omega) \) such that \( S(x^m_{hk}) = K(x^m_{hk}) \). In order to characterize \( S(x) \) we have to ascertain the values \( S(x_r) \). It is obvious that if one takes \( x^m_{hk} = x_r \) in the two dimensional analog of (2.23), then it follows that

\[
S(x_r) = \tilde{A}[\omega^m],
\]

where the entries of \( \tilde{A} \) are given by the inner products \( \langle \phi_k(x - x_r), \phi_r(x) \rangle \). In the case of grids which are uniform in each direction \( \tilde{A} = A \). Since any \( S(x) \in S_{4, h}^4(\Omega) \) can be expressed as a linear combination of tensor products of cubic B-splines, then for any \( k \),

\[
b_k = S(x^m_{hk}) = \sum_{i=1}^{I} \sum_{j=1}^{J} c^m_{ij} B_{i,4}(x^m_{1hk}) B_{j,4}(x^m_{2hk}).
\]

Taking \( (x^m_{1hk}, x^m_{2hk}) = (x_{1k}, x_{2k}) \) above and using (2.23) yields

\[
A_1 \otimes A_2^* [c^m] = \tilde{A}[\omega^m].
\]

**Remark.** If a nonuniform grid in each direction is used, then \( \tilde{A} \neq A \). This implies that the matrix \( \tilde{A} \) has to be recomputed every time step; however, from a practical point of view, and since the algorithm requires well structured grids (this may be a limitation) one might be tempted to take \( \tilde{A} = A \). There are several reasons for doing so:
i) Savings in computational time and core memory. The matrix $\tilde{A}$ is calculated once and for all at the beginning of the computations.

ii) The conservation of $\omega(x,t)$, as we shall see in the next section, is better achieved. In fact, if $\tilde{A}$ were used, $\omega(x,t)$ would be conserved up to $O(\Delta t)$. Other properties as the stability and the convergence do not suffer.

iii) If the CFL condition is less than 1, then $\tilde{A} = A$ whether the grid is uniform in each direction or not.

3. Properties

In this section we study some important features of the algorithm. Specifically, we show that it is conservative and unconditionally stable in the $L^2$-norm.

Conservation.

Theorem 3.1. Let $\mu_k = \text{meas}(\text{supp}\phi_k)$. For $1 \leq k \leq K$ and $m \in [0, M-1]$ algorithm (2.15)-(2.16) conserves $\sum_k \mu_k \omega^m_k$.

Proof. Let

$$\omega_h(x, t_{m+1}) = \sum_k \omega^{m+1}_k \phi_k(x),$$

$$[\tilde{\omega}^{m+1}] = A[\omega^{m+1}].$$

Then

$$\int_{\Omega} \omega_h(x, t_{m+1}) \, dx = \sum_k \mu_k \omega^{m+1}_k = \sum_k \tilde{w}^{m+1}_k.$$
Now, recalling (1.4), (2.15) and (2.16) and with $\tilde{A} = A$ it follows immediately

$$\sum_k \mu_k \omega_{mk}^{m+1} = \sum_k \mu_k \omega_{mk}^m.$$  \hspace{1cm} (3.1)

Stability.

**Theorem 3.2.** Algorithm (2.15)-(2.16) is unconditionally stable in the $L^2$-norm.

**Proof.** By virtue of (2.22) we can write

$$(\omega_h(x, t_{m+1}), \phi_k) = (\omega_h(y, t_m), \tilde{\phi}_k(y - x_{hk}^m)), \forall k.$$ \hspace{1cm} \text{(3.2)}

Multiplying by $\omega_k^{m+1}$ and summing over $k$ indices gives

$$\|\omega_h(x, t_{m+1})\|_2^2 = \|\omega_h(y, t_m)\|_2 \|\sum_k \omega_k^{m+1} \phi_k(y - x_{hk}^m)\|.$$ \hspace{1cm} \text{(3.3)}

Now, it remains to estimate the second term on the right hand side of the inequality. To begin with, we recall that $\text{supp}\phi_k(y - X_{hk}^m)$ is the shifted support of $\phi_k(x)$. Let $E_k$ and $SE_k$ denote the supports of $\phi_k(x)$ and $\tilde{\phi}_k(y - X_{hk}^m)$ respectively. Then we may write

$$\|\sum_k \omega_k^{m+1} \phi_k(y - x_{hk}^m)\|_2^2 = \sum_{j \in SE_k} (\sum_k \omega_k^{m+1} \phi_j(y - x_{hk}^m) \tilde{\phi}_j(y - x_{hj}^m))d\Omega_j.$$ \hspace{1cm} \text{(3.4)}

From (2.14) we have that if $\text{supp}\phi_j \cap \text{supp}\phi_k$ is not empty, then $\text{supp}\phi_j \cap \text{supp}\phi_k = \text{supp}\phi_j(y - x_{hj}^m) \cap \text{supp}\phi_k(y - x_{hk}^m) + O(h\Delta t)$, where we have assumed that $|u - u_h|_{Q, Q} \leq O(h)$. With this
information and taking into account that the amount of the overlapping is finite one easily gets

\[ \| \sum_k w_k \varphi_k (y - x_h^m) \| \leq (1 + C\Delta t) \| \omega_h(x, t_{m+1}) \|. \]

Hence

\[ \| \omega_h(x, t_{m+1}) \| \leq (1 + C\Delta t) \| \omega_h(x, t_m) \| , \quad (3.2) \]

and so stability.

4. Error Analysis

Let \( \omega^{m+1}(x) = \omega(x, t_{m+1}) \), which according to (2.8) satisfies

\[ \omega^{m+1}(x) = \omega^m(X_h(x, t_{m+1}; t_m)) = \omega^m(x_h^m) , \quad (4.1) \]

where \( X_h^m \) is a shorthand notation for \( X_h(x, t_{m+1}; t_m) \). For any \( x \) in \( E_k \), where \( E_k \) is the support of \( \varphi_k \), that is, the macroelement composed of those elements which meet at the vertex \( x_k \), consider the transformation

\[ x \to x - x_k + x_h^m = x + \alpha^m_{hk} . \quad (4.2) \]

By this transformation the element \( E_k \) is shifted by \( \alpha^m_{hk} \). As in the proof of stability, we denote the shifted element by \( SE_k \). Let us introduce the function \( \omega^{m+1}(x) \) which is defined by
\[ \omega^{m+1}(x) = \omega^m(x + \alpha^m_{hk}). \] (4.3)

We also consider the approximation \( \omega^{m+1}_h(x) \) to the solution \( \omega^{m+1}(x) \) in the finite element space \( H_h \). \( \omega^{m+1}_h(x) \) is defined by (cf. [28])

\[ (\omega^{m+1}_h(x), \phi_k) = (\omega^m_h(x), \phi_k), \forall k. \] (4.4)

Let \( \Pi_h \) be the orthogonal projection from \( L^2(\Omega) \) onto \( H_h \) with respect to the inner product \( (u, v) \), i.e.,

\[ (\Pi_h u, \chi) = (u, \chi), \forall \chi \in H_h. \] (4.5)

Then \( \omega^{m+1}_h(x) \) may be expressed as

\[ \omega^{m+1}_h(x) = \Pi_h \omega^m_h(x). \] (4.6)

Next, for any \( x \) in \( E_k \), we consider the particle approximation to \( \omega^{m+1}_h(x) \) in \( H_h \), which we define as

\[ \omega^{m+1}_h(x) = \sum \omega_{hk}(t_m)\phi_k(x - x_{hk}^{m+1}). \] (4.7)

where the values \( \omega_{hk}(t_m) \) are obtained by the relation

\[ (\omega^{m+1}_h, \phi_k) = (\omega^m_h, \phi_k(y - x_{hk}^m)). \] (4.8)

The inner products on the right hand side of (4.8) are given by integrals of the form

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\[
\int_{SE} \bar{\omega}_h^m(y) \hat{\phi}_k(y - \chi_{hk}^m) dy . \tag{4.9}
\]

By virtue of (4.2) we may write

\[y = x + \alpha_{hk}^m\]

and consequently the integrals of (4.9) become

\[
\int_{E} \bar{\omega}_h^m(x + \alpha_{hk}^m) \phi_k(x) dx , \forall k . \tag{4.10}
\]

The formulation (4.10) can be viewed as an area-weighting formulation according to [27]. By virtue of (4.10) we may write (4.8) as

\[
(\bar{\omega}_h^{m+1}, \phi_k) = (\bar{\omega}_h^m(x + \alpha_{hk}^m), \phi_k) , \forall k . \tag{4.11}
\]

Finally, let us define

\[
\nu^{m+1}(x) = \omega^{m+1}(x) - \bar{\omega}_h^{m+1}(x) . \tag{4.12}
\]

The formulation (4.4) is analysed in [31] and a method to perform the computations of the inner products on the right hand side is described in [19]. Morton et al [27] have recently shown that the approximation of the right hand side inner products of (4.4) by several quadrature rules might render the formulation conditionally unstable. We wish to ascertain, in the \(L^2\)-norm, the error incurred by replacing \(\omega^{m+1}(x)\) by \(\omega_h^{m+1}(x)\) at any instant \(t_{m+1}\). Towards this end we
\[ \omega^{m+1}(x) - \omega_h^{m+1}(x) = \omega^{m+1}(x) - \omega_h^{m+1}(x) + \omega_h^{m+1}(x) - \omega_h^{m+1}(x). \quad (4.13) \]

By the triangle inequality

\[ \| \omega^{m+1}(x) - \omega_h^{m+1}(x) \| \leq \| \omega^{m+1}(x) - \omega_h^{m+1}(x) \| + \| \omega_h^{m+1}(x) - \omega_h^{m+1}(x) \|. \quad (4.14) \]

The estimate of the first term on the right side of (4.14) is given by the following lemma.

**Lemma 4.1.** For \( \omega(x,t) \) in \( L^\infty(0,T;W^{k+1,2}(\Omega)) \), \( u(x,t) \) in \( L^\infty(0,T;W^{1,\infty}(\Omega)) \) and using \( C^0 \) finite elements of degree \( k, k \leq 1 \)

\[ \| \omega^{m+1}(x) - \omega_h^{m+1}(x) \| \leq \| \omega^{m}(x) - \omega_h^{m}(x) \| + \| \omega_h^{m}(x) - \omega_h^{m}(x) \| + |\omega|_{1,\omega,Q} \|u - u_h\|_{\omega, Q} \Delta t + C h^{k+1} \| \omega_m \|_{k+1, p, \Omega}. \quad (4.15) \]

**Proof.** See [31]. \[ \]

To estimate the second term on the right hand side of (4.14) we can adopt by virtue of (4.10) the method used in [27]. We observe that by the triangle inequality

\[ \| \omega_h^{m+1} - \omega_h^{m+1} \| \leq \| \omega_h^{m+1} - \omega_h^{m+1} \| + \| \omega_h^{m+1} - \omega_h^{m+1} \|. \quad (4.16) \]

Let us study the first term on the right side of the above inequality. From the relationship

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\[
(\tilde{\omega}^{m+1}_h(x) - \hat{\omega}^{m+1}_h(x), \phi_k) = (\tilde{\omega}^m_h(x_h^m) - \hat{\omega}^m_h(x + a_{h_h}^m), \phi_k), \quad \forall \ k,
\]

it follows that

\[
\| \tilde{\omega}^{m+1}_h(x) - \hat{\omega}^{m+1}_h(x) \| \leq \| \tilde{\omega}^m_h(x_h^m) - \hat{\omega}^m_h(x + a_{h_h}^m) \|, \quad (4.17)
\]

so that we have to estimate the right hand side of (4.17).

Using (4.12) and the triangle inequality we obtain

\[
\| \tilde{\omega}^m_h(x_h^m) - \hat{\omega}^m_h(x + a_{h_h}^m) \| \leq \| \omega^m(x_h^m) - \omega^m_h(x + a_{h_h}^m) \|
+ \| v^m(x_h^m) - v^m(x + a_{h_h}^m) \|. \quad (4.18)
\]

The first term on the right side of the above inequality is bounded as

\[
\| \omega^m(x_h^m) - \omega^m_h(x + a_{h_h}^m) \| \leq C|\chi_h^m - (x + a_{h_h}^m)|_\omega, Q \|v\|_\omega, Q. \quad (4.19)
\]

From (2.14) and (4.2)

\[
|\chi_h^m - (x + a_{h_h}^m)| = O(h\Delta t + |u - u_h|\Delta t). \quad (4.20)
\]

Hereafter we assume that \(|u - u_h| = O(h)\), then (4.19) becomes

\[
\| \omega^m(x_h^m) - \omega^m_h(x + a_{h_h}^m) \| \leq C\|v\|_\omega, Q \ h\Delta t. \quad (4.21)
\]

The second term on the right hand side of (4.18) yields

\[
v^m(x_h^m) - v^m(x + a_{h_h}^m)
= \int_0^1 (\chi_h^m - (x + a_{h_h}^m)) \cdot Dv^m(F_{\theta}(x))d\theta, \quad (4.22)
\]

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where \( F_\theta(x) = x + \alpha_{hk}^m + \theta(X_h^m - (x + \alpha_{hk}^m)) \). \( (4.23) \)

Thus

\[
\| v^m(x_h^m) - v^m(x + \alpha_{hk}^m) \|^2 \leq \sum_k \int_{E_k} |X_h^m - x - \alpha_{hk}^m|^2 d\nu^m(F_\theta(x)) d\theta d\Omega.
\]

Since \( F_\theta(x) \) is a quasi-isometry of \( E_k \) onto \( \hat{E}_k \), \( \hat{E}_k \) being the image of \( E_k \) by the transformation \( x \rightarrow X_h(x, t_{m+1}; t_m) \), then we may change the variables in the second integral to obtain, employing \( (4.20) \),

\[
\| v^m(x_h^m) - v^m(x + \alpha_{hk}^m) \| \leq C h \Delta t \| v \|,
\]

By virtue of the approximation properties of \( H_h \) (cf. Chap. II, Sect. 3.2)

\[
\| v^m(x_h^m) - v^m(x + \alpha_{hk}^m) \| \leq C h \Delta t \| v \|_{2,2,\Omega}, \ (4.24)
\]

Finally, we come to estimate the second term on the right hand side of the inequality \( (4.16) \).

\[
(\hat{\omega}^{m+1}_h(x) - \omega^{m+1}_h(x), \phi_k) = (\hat{\omega}^m_h(y) - \omega^m_h(y), \hat{\phi}_k(y - x_h^m)), \forall k.
\]

Using the same argument from the stability proof yields

\[
\| \hat{\omega}^{m+1}_h - \omega^{m+1}_h \| \leq (1 + C \Delta t) \| \hat{\omega}^m_h - \omega^m_h \|. \ (4.25)
\]

Collecting the estimates \( (4.21) \), \( (4.24) \) and \( (4.25) \) we obtain
\[ \| \tilde{\omega}_{h}^{m+1} - \omega_{h}^{m+1} \| \leq (1 + C_{1} \Delta t) \| \tilde{\omega}_{h}^{m} - \omega_{h}^{m} \| \\
+ C_{2} \Delta t |\nabla \omega|_{\omega, Q} + C_{3} \Delta t h^{2} \| \omega^{m} \|_{2, 2, \Omega}. \quad (4.26) \]

Now, from (4.14), (4.15) and (4.26) we find that

\[ \| \omega_{h}^{m+1} - \omega_{h}^{m+1} \| \leq K_{1} (\Delta t) h^{2} \| \omega \|_{2, 2, \Omega} + K_{2} \Delta t |\nabla \omega|_{\omega, Q} \\
+ (1 + C \Delta t) \| \omega^{m} - \tilde{\omega}_{h}^{m} \| + \| \tilde{\omega}_{h}^{m} - \omega_{h}^{m} \|. \quad (4.27) \]

Assuming that \( \omega_{h}^{0} = \tilde{\omega}_{h}^{0} = \omega_{0} \), then successive substitutions into (4.27) and Gronwall's inequality give

\[ \| \omega_{h}^{m+1} - \omega_{h}^{m+1} \| \leq K \frac{T}{\Delta t} \| h^{2} \| \| \omega \|_{2, 2, \Omega} + Q \\
+ \Delta t |\nabla \omega|_{\omega, Q} \}, \quad (4.28) \]

where \( K(h, \Delta t) = \exp(C \Delta t) \). This result brings us to the following theorem.

**Theorem 4.1.** For \( \omega(x, t) \) in \( L^{\infty}(0, T; W^{2, 2}(\Omega)) \land W^{1, \infty}(\Omega) \), \( u \) in \( L^{\infty}(0, T; W^{1, \infty}(\Omega)) \) and \( \Delta t = O(h) \), algorithm (2.15)-(2.16) with \( C^{0} \)
finite elements of degree \( k = 1 \) converges in the \( L^{\infty}(0, T; L^{2}(\Omega)) \)

norm with error \( O(h) \).

Clearly, the estimate (4.28) is suboptimal. Numerical computations mentioned in [27] show that for sufficiently smooth functions the error committed may be \( O(h^{3}) \). On the other hand, in the previous analysis no consideration has been paid to the fact that we are using cubic spline to interpolate the values at the departure points, and it is well known that
this might yield a pointwise error as high as $O(h^4)$ at those points. In order to ascertain the role played by the cubic spline interpolation in our algorithm we now estimate the error in the maximum norm. Towards this end, we assume that $\omega(x,t) \in L^r(0,T;W^s,\omega(\Omega)) \cap W^s,\omega(\Omega))$ and $u(x,t) \in L^r(0,T;W^s,\omega(\Omega))$, $r$ and $s$ integers satisfying, $1 \leq r \leq \omega$, $1 \leq s \leq 4$. At any instant $t_{n+1}$ the approximate solution $\omega_{n+1}^m(x)$ is simply

$$\omega_{n+1}^m(x) = \sum_j (S\omega_h^m(x_{hj}))\phi_j(x) = pS\omega_h^m(x_{hj})_1,$$  \hspace{1cm} (4.29)

where $S$ is the cubic spline interpolant of the approximate solution $\omega_h^m(x)$ at the points $x_{hj}$ and $p$ is identified as the prolongation operator from the space $W^h = \{ \omega_{hj}, 1 \leq j \leq K \}$ into the finite dimensional space $H_h$. Then

$$\omega_{n+1}^m(x) - \omega_{n+1}^m(x) = \omega_{n+1}^m(x) - I\omega_{n+1}^m(x)$$

$$+ I\omega_{n+1}^m(x) - pS\omega_h^m(x_{hj})_1,$$ \hspace{1cm} (4.30)

where $I\omega$ denotes the interpolant of $\omega$. The first term on the right hand side of (4.30) represents the approximation error whereas the second term is the evolutionary error. Its estimation will concern us because it accumulates at each time step and controls important aspects of the method as, for instance, the truncation error and the numerical dissipation. A further decomposition yields

$$I\omega_{n+1}^m - pS\omega_h^m(x_{hj})_1 = p(\omega_h^m(x_{hj}) - \omega_h^m(x_{hj})_1) + p(\omega_h^m(x_{hj}) - S\omega_h^m(x_{hj})_1)$$

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\[ + p(\omega_h^m(x_{hj}^m) - \omega_h^m(x_{hj}^m)) . \]

Hence (dropping the subindex $\Omega$)

\[ |\omega^{m+1}(x) - \omega_h^{m+1}(x)|_\infty \leq |\omega^{m+1}(x) - I\omega^{m+1}(x)|_\infty \]

\[ + |\omega^m(x_j^m) - \omega_h^m(x_{hj}^m)|_\infty \]

\[ + |\omega^m(x_{hj}^m) - S\omega_h^m(x_{hj}^m)|_\infty \]

\[ + |S\omega_h^m(x_{hj}^m) - S\omega_h^m(x_{hj}^m)|_\infty . \quad (B1-B4), \quad (4.31) \]

The term $B1$ gives with bilinear finite elements

\[ |\omega^{m+1}(x) - I\omega^{m+1}(x)|_\infty \leq C h^2 , \quad (4.32) \]

by interpolation theory. In regard to $B2$, from Lemma 2.2 we find (see also [2],[28])

\[ |\omega^m(x_j^m) - \omega^m(x_{hj}^m)|_\infty \leq C|\nabla \omega|_\infty, Q_T(|u_h - u|_\infty, Q_T) . \quad (4.33) \]

A bound for $B3$ is obtained by using Lemma 1.1, i.e.,

\[ |\omega^m(x_{hj}^m) - S\omega^m(x_{hj}^m)|_\infty \leq C h^s |\omega|_{S, \infty} . \quad (4.34) \]

Finally, it remains to bound $B4$. From the stability property of the splines one gets

\[ |S\omega_h^m(x_{hj}^m) - S\omega_h^m(x_{hj}^m)|_\infty \leq C |\omega^m(x) - \omega_h^m(x)|_\infty . \quad (4.35) \]

From (4.30)-(4.35) and assuming that $\omega^0 = \omega_h^0$, it follows on application of Gronwall's inequality
\[ |\omega^{m+1} - \omega_h^{m+1}|_{\omega Q} \leq K \frac{T}{\Delta t} (h^\sigma + |\nabla \omega|_{\infty, Q} h \Delta t), \quad (4.36) \]

where \( \sigma = \min(2, s) \) and \( K \) a positive constant. The result (4.36) is formulated as a theorem.

**Theorem 4.2.** For \( \omega(x, t) \in L^\infty(0, T; \mathcal{W}^2, \infty(\Omega) \cap \mathcal{W}^s, \infty(\Omega)), u(x, t) \in L^\infty(0, T; \mathcal{W}^1, \infty(\Omega)), s \) an integer such that \( 1 \leq s \leq 4 \) and \( \sigma = \min(2, s) \); algorithm (2.15)-(2.16) with \( C^0 \) bilinear finite elements on a rectangular grid and \( \Delta t = O(h) \) converges in the maximum norm as given by (4.36).

Note that if \( \omega \) is sufficiently smooth, say \( s=4 \), (4.34) gives a super convergent result, in the finite element framework, at the departure points \( x_{hj}^m \). Moreover, since the estimate of Lemma 1.1 is local, we expect that in those regions where \( \omega(x, t) \) is smooth enough the interpolation at \( x_{hj}^m \) will be of high order.

**Remark.** Theorem 4.2 may explain some of the numerical results mentioned by Morton et al. [27] using Lagrange-Galerkin methods with \( C^0 \) finite elements to integrate the one dimensional linear advection problem. In this simple case they were able to compute exactly the integrals of the inner products; but as Theorem 2.1 shows this is equivalent to interpolation by cubic splines at the foot of the characteristic curves. If one takes \( u_h = u \), then (4.34) yields the error estimate in the \( L^\infty(0, T; \ell^2) \)-norm of \( O(h^4/\Delta t) \).
5. Numerical Experiments

To illustrate the conservation and stability properties of our scheme as well as to verify the validity of the error analysis we consider two model problems. The first one is the advection of a cone in a fixed rotating velocity field \( \hat{\Omega} \). The parameters of this example are

\[
\omega(x,0) = \begin{cases} 
H(1- \frac{\bar{x}}{r}) , & (\bar{x} \leq r) \\
0 , & \text{otherwise} 
\end{cases} \tag{5.1}
\]

where \( r \) is a positive parameter, \( H = 100 \), and

\[
\bar{x} = (x_1 - x_0)^2 + x_2^2 .
\]

\[
u = (-\hat{\Omega}x_2 , \hat{\Omega}x_1) . \tag{5.2}
\]

We use an uniform square grid on the domain \([-1,1] \times [-1,1]\), with \( h = (1/63) \), \( x_0 = -15h \), \( r = 8h \) and the CFL condition equal to 2.9 at the point of maximum velocity. The number of time steps to complete a revolution is equal to 96. Fig. 3a shows the initial condition whereas Fig. 3b displays the cone after 6 revolutions. The height at the vertex is now 87 and some wiggle activity is observed at the base of the cone. Table 1 shows the time evolution of the cone every 48 time steps (1/2 of a revolution) in terms of mass \( \int \omega \), square of mass \( \int \omega^2 \), and maximum and minimum values. The mass and square of mass values have been divided by their initial values, while the maximum and minimum values at different instants have been divided by \( H \). We observe that there is mass conservation. As for the dissipative effects, they are strong at the very beginning to decrease as time passes. For example, after 1/4
of a revolution the rate of dissipation of square of mass per
time step is $6 \times 10^{-5}$ decreasing to $3 \times 10^{-5}$ at the end of the
6-th revolution. We also observe that the error committed at
the vertex of the cone is $O(10^{-2})$ per time step. This conforms
to our error analysis which is $O(h/\Delta t)$ in this case since
$\omega(x,t) \in W^{1,\infty}$.

Our second test is a more severe one. We consider the
'slotted' cylinder problem which was proposed by Zalesak in 1979
to test high order flux corrected transport algorithms. The
idea behind this experiment is to compare the performance of
our algorithm with the results reported by Zalesak [37] and
Munz [28] using high order finite difference schemes. The
parameters of this experiment are those used in [28]. Fig. 4a
shows the initial condition, where $h = 0.01$, $CFL = 4.2$ and the
height of the cylinder $H = 4$. The number of time steps to
complete a revolution is now equal to 96. We emphasize the
fact that this initial condition is not in $W^{1,p}$, but does
belong to $Lip^{1,\infty}$, where $Lip^{m,p}$ is the $(m,p)$ Lipschitz space.
However, we still can apply Lemma 1.1 (See[33]). Figs. 4b, 4c, 4d
show the cylinder after 1/8, 1 and 6 revolutions respectively.
We observe in these figures the outstanding ability of the
scheme to maintain the shape of the initial condition and keep
under a strict control the wiggles generated at the
discontinuities. This is a remarkable achievement of a scheme
which is not specifically designed to handle strong
discontinuities. The ability of the scheme to keep the wiggles
under control is due to its $L^2$-stability. Table 2 shows the
time evolution of the cylinder every 24 time steps of a 6 revolution experiment. The conclusions drawn from this results are, in some respects, similar to those of the cone experiment. There exists mass conservation. The dissipation of square of mass is strong at the initial stages to decrease progressively as time passes. For instance, the rate of dissipation is $10^{-4}$ after the first revolution, to decrease to $10^{-5}$ after the second revolution. The trend is towards a further decrease. Large wiggles appear concentrated around the external, upper and lower, rings of the cylinder and the slot; but after few time steps they are damped out and the results indicate that their maximum and minimum values tend asymptotically to $\pm 10\%$. 
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<th>Conservation mass</th>
<th>Conservation square mass</th>
<th>Maximum</th>
<th>Minimum</th>
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TABLE 1. Time evolution of the cone

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<tr>
<th>N° of time steps</th>
<th>Conservation mass</th>
<th>Conservation square mass</th>
<th>Maximum</th>
<th>Minimum</th>
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TABLE 2. Time evolution of the slotted cylinder.
Fig. 2 Regularization of the Particle Points
Fig. 3a Initial Condition for the Cone
Fig. 3b The Cone after 6 Revolutions
Fig. 4a Initial Condition for the Cylinder
Fig. 4b The Cylinder after 1/8 of a Revolution
Fig. 4c The Cylinder after 1 Revolution
Fig. 4d The Cylinder after 6 Revolutions