CHAPTER II

THE GOVERNING EQUATIONS AND THEIR NUMERICAL DISCRETIZATION

This descriptive chapter introduces the continuum equations and the proposed numerical algorithm to integrate them. In addition, we also present the basic notation used throughout the thesis, and the spaces where the solutions are sought.

1. Governing Equations

We wish to apply the proposed numerical algorithm to study the circulation of a mid-latitude baroclinic ocean. In order to facilitate the mathematical analysis we choose the simplest example of such an ocean. Thus, our ocean model consists of two layers of densities $\rho_1$ and $\rho_2$, respectively confined to a mid-latitude, $\beta$-plane, rectangular, flat bottom domain $\mathcal{D}$ rotating with angular velocity $\Omega$ and with solid boundaries. Figure 1 illustrates the arrangement of our Cartesian coordinate system, as well as the geometrical parameters of the model. The stream functions and relative vorticities of each layer are denoted by $\psi_1$ and $\omega_1$, respectively. The subindices 1 and 2 stand for the magnitudes related to the upper and lower layers, whereas $1^{\frac{1}{2}}$ stands for the magnitudes
Fig. 1 The Two Layer Model
evaluated at the interface.

The flow is driven by a variable wind stress $T(t,x)$ acting on the free surface. The coupling between layers is through the displacement of the interface. We use the QG equations in the version potential vorticity–stream function which can be found in [20]. The $z$-dependence of the variables is modelled by employing central finite differences in the vertical direction. Thus, defining the potential vorticity at each layer by

$$ q_1 = \nabla^2 \psi_1 + f - \left( \frac{f_0}{g H_1} \right) \left( \psi_1 - \psi_2 \right), \quad (1.1.1) $$

$$ q_2 = \nabla^2 \psi_2 + f + \left( \frac{f_0}{g H_2} \right) \left( \psi_1 - \psi_2 \right), \quad (1.1.2) $$

we have

$$ \frac{Dq_i}{Dt} - A_i \nabla^2 q_i + \delta_{i2} cq_i - \delta_{i1} F = 0, \text{ in } Q_T, \quad i = 1, 2, \quad (1.2.1) $$

$$ q_i(x,t) = q_{bi} \text{ on the boundary } \Gamma, \quad (1.2.2) $$

$$ q_i(x,0) = f, \quad (1.2.3) $$

where $Q_T = \Omega \times [0, T]$, 

$$ \frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla, $$

$$ f = 2|\Omega| \sin \theta_0 \approx f_0 + \beta y \text{ is the Coriolis parameter, which is approximated by projecting the earth's spherical geometry onto its tangent plane at the reference latitude } \theta = \theta_0, \text{ and } \beta = df/dy \text{ and } f_0 \text{ are parameters whose magnitudes depend on } |\Omega|, \theta_0,$$
and the radius of the earth (cf. [30]).

\[ g' = \left( \frac{\rho_1 - \rho_2}{\rho_2} \right) g, \] here \( g \) denotes the acceleration due to gravity,

the forcing \( F = \frac{\text{curl} T}{H_1} \),

\( A_H \) is the lateral eddy viscosity coefficient which is assumed constant over the whole domain \( \Omega \),

\( \varepsilon \) is a constant linear bottom friction coefficient,

\( \delta_{ij} \) is the Kronecker symbol.

For a two layer ocean, linear combinations of \( \psi_1 \) and \( \psi_2 \) lead to a set of elliptic equations which are now formulated.

Define \( \Psi(x,t) \) and \( \Phi(x,t) \) as follows.

\[ \Psi(x,t) = \psi_1(x,t) - \psi_2(x,t), \quad (1.3.1) \]

\[ \Phi(x,t) = \frac{H_1 \psi_1 + H_2 \psi_2}{H_1 + H_2}, \quad (1.3.2) \]

where \( \Psi(x,t) \) and \( \Phi(x,t) \) are the so called baroclinic and barotropic modes, respectively. According to McWilliams [26], we may further decompose the baroclinic mode \( \Psi \) in the following fashion

\[ \Psi(x,t) = \Psi_a(x,t) + C(t)\psi_s(x). \quad (1.4) \]

By considerations of mass conservation [26], \( \Psi(x,t) \) satisfies the global condition

\[ \int_{\Omega} \Psi(x,t) \, d\Omega = 0, \quad \forall t \in [0,T]. \quad (1.5) \]
In view of (1.4) and (1.5) the time dependent function \( C(t) \) is determined at any instant \( t \) by

\[
C(t) = -\frac{\int_{\Omega} \psi_{a}(x,t)d\Omega}{\int_{\Omega} \psi_{s}(x)d\Omega}, \quad \forall t \in [0,T]. \tag{1.6}
\]

By virtue of (1.1), (1.3) and (1.4) we obtain

\[
(\nabla^2 - \lambda^2)\psi_{s}(x) = 0 \quad \text{in} \quad Q_{T}, \tag{1.7.1}
\]

\[
\psi_{s}|_{\Gamma} = 1, \quad \forall t. \tag{1.7.2}
\]

\[
(\nabla^2 - \lambda^2)\psi_{a}(x,t) = q_{1} - q_{2} \quad \text{in} \quad Q_{T}, \tag{1.8.1}
\]

\[
\psi_{a}|_{\Gamma} = 0, \quad \forall t, \tag{1.8.2}
\]

where

\[
\lambda^2 = \frac{\int_{0}^{2}(H_{1} + H_{2})}{g^{1}H_{1}H_{2}}, \tag{1.8.3}
\]

\[
\nabla^2 \phi(x,t) = \frac{H_{1}q_{1} + H_{2}q_{2}}{H_{1} + H_{2}} - f \quad \text{in} \quad Q_{T}, \tag{1.9.1}
\]

\[
\phi|_{\Gamma} = 0, \quad \forall t. \tag{1.9.2}
\]

The systems (1.7)-(1.9) are closed with the initial conditions

\[
\psi_{a}(x,0) = \phi(x,0) = 0 \quad \text{in} \quad \Omega, \tag{1.10.1}
\]

or equivalently
\[ \psi_1(x,0) = \psi_2(x,0) = 0 \text{ in } \Omega. \quad (1.10.2) \]

Finally, the velocity vector \( u_i = (u_i, v_i) \) at each layer is determined at any instant \( t \) by the relation

\[ u_i = \text{curl} \psi_i, \quad i = 1, 2. \quad (1.11) \]

1.2. On Boundary Conditions.

The boundary conditions (1.7.2) and (1.8.2) state that the stream functions \( \psi_1 \) and \( \psi_2 \) are, at any instant \( t \), constant along the boundary \( \Gamma \); that is

\[ \psi_i|\Gamma = C_i(t), \quad i = 1, 2. \quad (1.12) \]

On the other hand, the physical meaning of the boundary condition (1.9.2) for the barotropic mode is that the vertically integrated normal transport is zero across a solid boundary. As is well known in the stream function-relative vorticity formulation of the N-S equations, the value of the relative vorticity is unknown on the solid boundaries of the domain; however, one can easily deduce, from the values of the velocity on the boundaries, an additional boundary condition for the stream function. Specifically, assuming that

\[ u_i|\Gamma = g_i(x,t), \quad (1.13) \]

and with \( n \) and \( t \) denoting the normal and tangent vectors on the boundary, respectively the following relations hold
\[
\frac{\partial \psi_i}{\partial n} \bigg|_{\Gamma} = -g_j \cdot t, \quad (1.14.1)
\]
\[
\frac{\partial \psi_i}{\partial t} \bigg|_{\Gamma} = g_i \cdot n = 0. \quad (1.14.2)
\]

If \( g_i \neq 0 \), then there is no shear stress on the solid wall, or in other terms, the relative vorticity is identically zero on the wall. In contrast, if \( g_i = 0 \), then there exists shear stress on the wall, and so relative vorticity is generated.

The condition \( g_i \neq 0 \) is known in fluid dynamics literature as free-slip boundary condition while the case \( g_i = 0 \) is denoted by no-slip boundary condition. Now, assuming that the value of the relative vorticity on the boundary is \( \lambda_i \), \( i = 1, 2 \) the boundary conditions for the potential vorticity are

\[
q_{bi} = \lambda_i + f - \frac{f_0}{g_i H} C(t), \quad \forall t \in [0,T]. \quad (1.15)
\]

2. Notation

We now introduce some basic notation which is used throughout the thesis. Let \( \Omega \) be an open subset of \( \mathbb{R}^2 \) with Lipschitz boundary \( \Gamma \), \( \mathbb{R}^2 \) is the two dimensional Euclidean space. Let \( C^k(\Omega) \) be the set of real valued functions on \( \Omega \) which have continuous partial derivatives of order at least \( k \), which are bounded on \( \Omega \). On \( C^k(\Omega) \) we introduce the norm
\[ |v|_{k,\omega,\Omega} = \sup_{x \in \Omega} |D^\alpha v|, \quad (2.1) \]

Boldface symbols denote either matrices, vectors or sequences; the meaning will be clear from the context. \( \alpha \) is a multi-index notation. If \( \mathbb{N} \) denotes the set of nonnegative integers

\[ \alpha := (\alpha_1, \alpha_2, \ldots, \alpha_n), \quad \alpha_i \in \mathbb{N}, \quad i = 1, 2, \ldots, n. \]

We have the following definitions

\[ |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n, \]
\[ \alpha \leq \beta \text{ iff } \alpha_i \leq \beta_i, \quad \forall i = 1, 2, \ldots, n, \]
\[ (\alpha_i + \beta_i) = \alpha_i + \beta_i, \quad \forall i = 1, 2, \ldots, n, \]
\[ (\alpha_i - \beta_i) = \alpha_i - \beta_i, \quad \forall i = 1, 2, \ldots, n, \]
\[ \alpha! = (\alpha_1!)(\alpha_2!)(\ldots)(\alpha_n!), \]
\[ (\alpha - \beta)_i = \max(\alpha_i - \beta_i; 0), \quad \forall i = 1, 2, \ldots, n, \]
\[ x^\alpha = (x_1^{\alpha_1})(x_2^{\alpha_2}) \cdots (x_n^{\alpha_n}), \]
\[ D^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}, \]

We denote by \( L^p(\Omega) \), \( 1 \leq p \leq \infty \), the space of all equivalence classes of real-valued Lebesgue measurable functions. The norm on \( L^p(\Omega) \), \( p < \infty \) is defined by

\[ \|u\|_{p,\Omega} = \left( \int_{\Omega} |u|^p d\Omega \right)^{1/p}, \quad (2.2) \]
with

\[ \|u\|_{\infty, \Omega} = \text{ess sup}|u|, \forall x \in \Omega. \]

When \( p = 2 \) we drop the subindices \( p \) and \( \Omega \) if there is no confusion.

For each integer \( m \geq 0 \) and real \( p, 1 \leq p \leq \infty \), we define the Sobolev spaces \( W^{m,p}(\Omega) \), as

\[ W^{m,p}(\Omega) = \left\{ v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega), |\alpha| \leq m, 1 \leq p \leq \infty \right\}. \]

(2.3)

The space \( W^{m,p}(\Omega) \) is a Banach space with the norm

\[ \|v\|_{m,p,\Omega} = \left( \sum_{|\alpha| \leq m} \|D^\alpha v\|_{p,\Omega} \right)^{1/p}, 1 \leq p < \infty, \]

(2.4.1)

and the seminorm

\[ |v|_{m,p,\Omega} = \left( \sum_{|\alpha| = m} |D^\alpha v|_{p,\Omega} \right)^{1/p}, \]

(2.4.2)

When \( p = \infty \)

\[ \|v\|_{m,\infty,\Omega} = \max (\text{ess sup}|D^\alpha v|), \forall x \in \Omega. \]

(2.4.3)

Also, the space \( W^{m,p}(\Omega) \) is separable for \( 1 \leq p < \infty \) and represents the completion of the space \( C^m(\Omega) \) in the norm

\[ \|v\|_{m,p,\Omega}. \]

Let \( X \) be a Banach space with norm \( \| \cdot \| \). If \( v(x,t) \)
represents a function defined on \( Q_T \), we set
\[ L^p(0,T;X) = \left\{ v: \int_0^T \|D_t v\|^p dt < \infty \right\}, \quad (2.5.1) \]

\[ \|D_t v\|^p_{L^p(0,T;X)} = \left( \int_0^T \|D_t v\|^p dt \right)^{1/p} \quad (2.5.2) \]

\[ H^m(0,T;X) = \left\{ v: \int_0^T \|D_t^\alpha v\|^2 dt < \infty, \ \alpha \leq m \right\}, \quad (2.5.3) \]

\[ \|D_t^\alpha v\|^2_{H^m(0,T;X)} = \left( \sum_{\alpha=0}^m \|D_t^\alpha v\|^2 \right)^{1/2} \quad (2.5.4) \]

Here \( D_t^\alpha \) denotes \( \frac{\partial^\alpha}{\partial t^\alpha} \).

\[ W^{m,\infty}(0,T;X) = \left\{ v: \text{ess sup}_{[0,T]} \|D_t^\alpha v\| < \infty, \ \alpha \leq m \right\}, \quad (2.6.1) \]

\[ \|D_t^\alpha v\|_{W^{m,\infty}(0,T;X)} = \max \text{ess sup}_{0 \leq \alpha \leq m, [0,T]} \|D_t^\alpha v\|, \quad m \geq 0. \quad (2.6.2) \]

In particular

\[ L^p(0,T;X) = H^0(0,T;X), \quad L^\infty(0,T;X) = W^{0,\infty}(0,T;X). \]

Sometimes we use the shorthand notation, if confusion does not arise,

\[ L^p(0,T;X) = L^p(X), \quad H^m(0,T;X) = H^m(X). \]

For \( p = 2 \), \( W^{m,p}(\Omega) \equiv H^m(\Omega) \), where \( H^m(\Omega) \) is the Hilbert space of order \( m \). The subspace \( H^m_0(\Omega) \) of \( H^m(\Omega) \) is defined by
\[ H^m_0(\Omega) = \left\{ v \in H^m(\Omega) : \left. \frac{\partial^\alpha v}{\partial \xi^\alpha} \right|_\Gamma = 0, \ 0 \leq |\alpha| \leq m-1 \right\}. \] (2.7)

3. The Weak Solution Formulation

We formulate in this section the weak solution of the problems (1.2), (1.7)-(1.10) and the velocities \( u_1 \). To begin with, we define the classes of functions which are used in the formulation of the weak form of the problems. Our basic space is \( H^1(\Omega) \); however, it is convenient to introduce some closed subsets of \( H^1(\Omega) \) which are specifically related to the solutions we are seeking. Such subsets are:

\[ S_C = \left\{ \phi \in H^1(\Omega) : \phi|_\Gamma = C(t), \ \frac{\partial \phi}{\partial n} = 0, \ \forall t \in [0,T] \right\}. \] (3.1.1)

\[ S_\zeta = \left\{ \omega \in H^1(\Omega) : \omega|_\Gamma = \zeta \right\}. \] (3.1.2)

Thus, the stream functions \( \psi_1 \in S_C \), whereas the potential and relative vorticities belong to \( S_\zeta \). The steady component \( \psi_s(x) \) of the baroclinic mode \( \psi(x,t) \) is in \( S_C \) with \( C = 1 \). On the other hand, the time dependent baroclinic component \( \psi_a(x,t) \) and the barotropic mode \( \phi(x,t) \) are in \( H^1_0(\Omega) \) for any \( t \in [0,T] \).

In order to motivate the algorithm to solve equation (1.2), we introduce here some ideas which are further developed in Chap. III. Recall the characteristic curves of the linear advection equation.
\[ \frac{\partial q}{\partial t} + u \cdot \nabla q = 0, \]
\[ q(x,0) = q_0(x), \]
\[ (3.2.1) \]

satisfy the system

\[ \frac{dX(x,s;\tau)}{d\tau} = u(X(x,s;\tau),\tau), \]
\[ X(x,s;s) = x, \]
\[ (3.2.2) \]

where \( u \) is the velocity vector with \( \text{div} u = 0 \).

The transformation \( x \to X(x,s;t) \) defines, under certain conditions to be specified below, a quasi-isometric transformation of \( \Omega \) into itself with Jacobian determinant equal to 1 almost everywhere.

To solve the potential vorticity equation (1.2) we set (cf. [31])

\[ \frac{Dq}{Dt} = \frac{\partial q}{\partial \tau}(X(x,t:\tau),\tau)|_{\tau=t}, \]
\[ (3.3) \]

where \( \frac{D}{Dt} \) denotes the total derivative.

The right hand side of (3.3) represents the time evolution of \( q \) along the characteristic curves of the flow. This consideration suggests the following two-stage inductive algorithm.

Suppose that a partition of \([0,T]\)
\[ \mathcal{P} : = \{ 0 = t_0 < t_1 < \ldots < t_m = T \} \]

is specified. We shall define the semi-discretization induced by \( \mathcal{P} \) upon (1.2) in the following two steps:

1) Given \( q^n_i : \Omega \rightarrow \mathbb{R}^2, \ i = 1,2, \ 0 \leq n \leq m, \) we assume that there is a 'reflection' map

\[ E : H^2(\Omega) \cap L^\infty(\Omega) \rightarrow H^2(\Omega') \cap L^\infty(\Omega'), \]

such that \( C^1(\Omega) \) functions are mapped onto \( C^1(\Omega') \) functions, and

\[ Eq|_\Omega = q. \]

\( E \) is linear and continuous, and \( \Omega' \) is defined by

\[ \Omega' : = \{ X(x,t_{n+1};t_n) \ \forall x \in \Omega \} , \ \Omega \subset \Omega'. \]

Thus, any function \( q^n : \Omega' \rightarrow \mathbb{R}^2 \) can be expressed as

\[ \hat{q}^n_i = Eq^n_i. \] (3.4)

Consequently, the output from step one is then given by

\[ \tilde{q}^n_i = \hat{q}^n_i(X(x,t_{n+1};t_n),t_n). \] (3.5)

The method to approximate \( \tilde{q}^n_i \) is one of the contributions of this thesis, and it will be described in detail in the next chapter. For now, it is sufficient to indicate that the function \( q^n_i(X_i(x,t_{n+1};t_n),t_n) \), where the subindex \( h \) denotes approximate values, does not belong to a finite dimensional space \( H_h \) where the solutions are sought; however, the
function $\tilde{q}^n_{hi}$ must belong to $H^n_h$. It is the fashion in which we make the function $\tilde{q}^n_{hi}$ be in $H^n_h$ that makes our method different from other Galerkin-Characteristic methods. A word on notation is now in order: We set $\tilde{q}^n_i = \tilde{q}^n_i(x^n_i)$ to remind us that $\tilde{q}^n_i$ is computed from the values $q^n_i$ takes at the points $X_i(x, t_{n+1}; t_n)$.

2) Given the output from (3.5), solve the following:

Find $q^{n+1}_i$, such that $q^{n+1}_i - q^{n+1}_{bi} \in H^1_0(\Omega)$ and $\forall t_n \in \mathcal{P}$

$$
\frac{q^{n+1}_i(x) - \tilde{q}^n_i(x^n_i)}{\Delta t}, \theta + \frac{A_H}{2}(\nabla(q^{n+1}_i + \tilde{q}^n_i(x^n_i)), \nabla \theta) - \\
\delta_{i2} \frac{c}{2}(q^{n+1}_i + \tilde{q}^n_i(x^n_i), \theta) = \delta_{i1}(F^{n+1/2}, \theta), \forall \theta \in H^1_0(\Omega), (3.6.1)
$$

$$
q^{n+1}_{bi} = \lambda^n_i + f - \frac{f_0}{g \beta_n} c^n \text{in } \Gamma, \ i = 1, 2 \quad (3.6.2)
$$

where $(u, v) = \int_{\Omega} uv \, d\Omega$, and $X_i^n = X_i(x, t_{n+1}; t_n)$.

To update the boundary values $\lambda^n_i$ of the relative vorticities $\omega^n_i$ we adopt the method of Glowinski and co-workers (cf. [11], [16]). Let $M$ be a complementary subspace of $H^1_0(\Omega)$; i.e.,

$$
H^1(\Omega) = H^1_0(\Omega) \oplus M. \quad (3.7.1)
$$

Find $\lambda^{n+1}_i \in M$, such that $\forall \mu \in M$
\[-(\lambda_i^{n+1}, \mu) = (\nabla \psi_i^{n+1}, \nabla \mu) - \int_\Gamma \frac{\partial \psi_i^{n+1}}{\partial n} \mu \, ds. \quad (3.7.2)\]

Next, we calculate the barotropic and baroclinic modes, the stream functions and the velocities.

i) Find \( \psi_s(x) \in S_c, c = 1 \), such that \( \nabla \phi \in H_0^1(\Omega) \)

\[ (\nabla \psi_s, \nabla \phi) + \lambda^2(\psi_s, \phi) = 0. \quad (3.8.1) \]

ii) Find \( \psi_a^{n+1}(x), \phi_a^{n+1}(x) \in H_0^1(\Omega) \) such that \( \nabla \phi \in H_0^1(\Omega) \)

\[ (\nabla \psi_a^{n+1}, \nabla \phi) + \lambda^2(\psi_a^{n+1}, \phi) = -(q_1^{n+1} - q_2^{n+1}). \quad (3.8.2) \]

\[ (\nabla \phi_a^{n+1}, \nabla \phi) = -(b^{n+1}, \phi). \quad (3.8.3) \]

where \( b^{n+1} = \frac{H_1 q_1^{n+1} + H_2 q_2^{n+1}}{H_1 + H_2} - f \).

iii)

\[ C^{n+1} = -\frac{\int_\Omega \psi_a^{n+1}(x) d\Omega}{\int_\Omega \psi_s(x) d\Omega}. \quad (3.9) \]

From (1.4), (3.8.1), (3.8.2) and (3.9) we are able to determine the baroclinic mode \( \psi^{n+1}(x) \) by

\[ \psi^{n+1}(x) = \psi_a^{n+1}(x) + C^{n+1} \psi_s(x). \quad (3.10) \]

Once the baroclinic and barotropic modes have been calculated, at instant \( t_{n+1} \), the stream functions \( \psi_1^{n+1} \) and \( \psi_2^{n+1} \) are
obtained in view of (1.3.1) and (1.3.2) by solving

$$
\psi^{n+1}_1 - \psi^{n+1}_2 = \psi^{n+1},
$$

$$
\frac{H_1 \psi^{n+1}_1 + H_2 \psi^{n+1}_2}{H_1 + H_2} = \phi^{n+1}.
$$

(3.11)

Finally, \( u^{n+1}(x) \) is obtained from

$$
\left\{ u^{n+1}_1, v^{n+1}_1 \right\} = \left\{ -\frac{\partial \psi^{n+1}_1}{\partial x_2}, \frac{\partial \psi^{n+1}_1}{\partial x_1} \right\}.
$$

(3.12)

Remark. Equation (3.8.1) is solved once and for all at the beginning of the computations.

Our next objective is to show that the systems (3.6) and (3.8) admit a unique solution in \( H^1(\Omega) \), for any \( t_n \in \mathcal{P} \). We shall assume that the boundary values \( q^{n}_{b1} \) and the forcing function \( F^n \) are sufficiently smooth. Also, we restrict ourselves to prove the existence and uniqueness of the solutions of (3.6) for the upper layer \( (1 = 1) \), since the proof for the lower layer \( (1 = 2) \) is essentially the same. The main tool used in the proofs is the Lax-Milgram theorem (cf. [2] and Appendix). Towards this end let us first introduce the dual spaces \( W^{-m,p'}(\Omega) \) of the spaces \( W^{m,p}(\Omega) \), and the trace spaces \( H^{m-j-1/2}(\Gamma) \), where \( \Gamma \) is the boundary of \( \Omega \).

For \( 1 \leq p < \infty \), the dual space \( W^{-m,p'}(\Omega) \) is the space of continuous linear functionals defined on \( W^{m,p}(\Omega) \) with the norm
\[ \|f\|_{m, p', \Omega} = \sup_{u \in W_0^{m, p}} \frac{|\langle f, u \rangle|}{\|u\|_{m, p, \Omega}}, \quad u \neq 0, \quad (3.13.1) \]

where \( p' \) satisfies \( p^{-1} + p'^{-1} = 1 \), and \( \langle f, u \rangle = \int_{\Omega} f u \, d\Omega \).

It is usual to denote \( \langle f, u \rangle \) as duality pairing, where \( f \) belongs to the dual space of \( u \).

The trace spaces are defined by

\[ H^{m-j-1/2}(\Gamma) = \left\{ q_b \in L^2(\Omega) : \exists u \in H^m(\Omega), \gamma_j u = q_b \text{ on } \Gamma \right\}, \]

where \( \gamma_j = \frac{\partial^j}{\partial n^j} \) is the trace operator of \( j \)-th order which can be extended to continuous linear operators mapping \( H^m(\Omega) \) onto \( H^{m-j-1/2}(\Gamma) \). In the following we use the space \( H^{m-1/2}(\Gamma) \) with norm defined by

\[ \|q_b\|_{m-1/2, 2, \Gamma} = \inf_{u \in H^m(\Omega)} \|u\|_{m, 2, \Omega} \quad \|\gamma_0 u = q_b\|. \quad (3.13.2) \]

Remark. For further properties on the trace operators and the spaces \( H^{m-j-1/2}(\Gamma) \) see Appendix of Chapter IV.

Proposition 3.1. For \( q_{b1}^{n+1} \in H^{1/2}(\Gamma) \) and \( F^{n+1/2} \in H^{-1}(\Omega) \), problem (3.6) has a unique solution in \( H^1(\Omega) \).

Proof. Since \( H^{1/2}(\Gamma) \) is the range of \( \gamma_0 \) in \( H^1(\Omega) \), then there exists a \( q_0 \in H^1(\Omega) \) such that \( q_0 = q_{b1}^{n+1} \) on \( \Gamma \), for \( i = 1, 2 \).

Next, consider the bilinear form
\[ I(u, v) = (u, v) + v(\nabla u, \nabla v), \quad u, v \in H_0^1(\Omega), \]

where \( v = \frac{\Delta t A_H}{2}. \)

It is well known (cf. [29]) that \( I(u, v) \) is continuous and \( H_0^1 \)-coercive; that is, there exist constants \( C_1, C_2 > 0 \) such that for \( u, v \) in \( H_0^1(\Omega) \)

\[
|I(u, v)| \leq C_1 \|u\|_{H_0^1} \|v\|_{H_0^1},
\]

\[
|I(u, u)| \geq C_2 \|u\|_{H_0^1}^2.
\]

Also, \( I(u, v) \) is continuous and \( H^1 \)-coercive if \( u, v \in H^1(\Omega) \). Now, the problem we have to examine for \( i = 1 \) (we drop the subindex \( i \) is:

Find \( q^{n+1}(x) \in H^1(\Omega) \) such that \( q^{n+1}(x) - q_0(x) \in H_0^1(\Omega) \) and

\[
I(q^{n+1} - q_0, \theta) = (\tilde{q}(X^n), \theta) - v(\nabla \tilde{q}(X^n), \nabla \theta) -
\]

\[
- I(q_0, \theta) + (F^{n+1/2}, \theta), \quad \forall \theta \in H_0^1(\Omega). \quad (3.14)
\]

In order to apply the Lax-Milgram theorem it remains to show that the right hand side of (3.14) defines a continuous linear functional; i.e., the mapping

\[
\theta \rightarrow (\tilde{q}(X^n), \theta) - v((\nabla \tilde{q}(X^n), \nabla \theta) - I(q_0, \theta) + (F^{n+1/2}, \theta)
\]

belongs to \( H^{-1}(\Omega) \). We first note that \( I(q_0, \theta) \) is continuous, so \( (F^{n+1/2}, \theta) - I(q_0, \theta) \in H^{-1}(\Omega) \). Next, by the Schwarz inequality
\[ \nu(\nabla\tilde{q}(x^n), \nabla \theta) \leq C \| \nabla\tilde{q}(x^n) \| \nabla \theta \| , \]

where \( C \) is a constant and \( x^n \) is in \( \Omega \)' which is the image of \( \Omega \) by the quasi-isometric homeomorphism \( x \to x^n \). By virtue of Theorem 1.1.7 of [25] and the fact

\[ \| \nabla\tilde{q}(x^n) \| = \| q^n \| , \]

since the Jacobian determinant of the quasi-isometric homeomorphism is equal to 1 almost everywhere, we have that \( \| \nabla\tilde{q} \| \) is equivalent to \( \| \nabla q^n \| \). Hence

\[ (\tilde{q}(x^n), \theta) - \nu(\nabla\tilde{q}(x^n), \nabla \theta) \in H^{-1}(\Omega) , \]

so the right hand side of (3.14) is in \( H^{-1}(\Omega) \).

With all the assumptions of the Lax-Milgram theorem satisfied, the existence and uniqueness of \( q^{n+1}(x) \in H^1(\Omega) \) follows \( \blacksquare \)

**Proposition 3.2.** Problems (3.8.2) and (3.8.3) have a unique solution in \( H^1(\Omega) \), whereas the unique solution of (3.8.1) lies in \( H^1(\Omega) \).

**Proof.** We first present the proof for the Problem (3.8.1). As before, let \( \psi_0 \in H^1(\Omega) \) such that \( \psi_0 = 1 \) on \( \Gamma \). Next, we recast the Problem (3.8.1) in the following fashion:

Find \( \psi_s - \psi_0 \in H^1_0(\Omega) \) such that \( \forall \phi \in H^1_0(\Omega) \)
\[(\nabla (\psi - \psi_0), \nabla \phi) + \lambda^2 (\psi - \psi_0, \phi) = -(\nabla \psi_0, \nabla \phi) - \lambda^2 (\psi_0, \phi) \]

(3.15)

Define now the bilinear form \( I'(u,v) \) as

\[ I'(u,v) = (\nabla u, \nabla v) + \lambda^2 (u, v). \]

\( I'(u, v) \) is continuous and \( H^1 \)- coercive, so the right hand side of (3.15) belongs to \( H^{-1}(\Omega) \). Therefore, the Lax-Milgram theorem guarantees the existence and uniqueness of the solution \( \psi_s(x) \).

Next, we proceed to prove the existence and uniqueness of the solutions of problems (3.8.2) and (3.8.3). The proof is essentially the same for both problems. First, note that \( q_{1,2}^{n+1} \) are in \( H^1(\Omega) \). Secondly, the bilinear form \( I''(u,v) \) defined by

\[ I''(u,v) = (\nabla u, \nabla v) + \mu (u, v) , u,v \in H^1_0(\Omega), \]

where \( \mu = \lambda^2 \) for Problem (3.8.2) and \( \mu = 0 \) for Problem (3.8.3), is continuous and \( H^1_0(\Omega) \)- coercive. Since all the assumptions of the Lax-Milgram theorem are satisfied, then it follows the existence and uniqueness of the solutions \( \psi_a^{n+1} \) and \( \phi^{n+1} \) in \( H^1_0(\Omega) \).

From Proposition 3.2 and (3.11) one deduces the existence and uniqueness of the stream functions \( \psi_1^{n+1} \) and \( \psi_2^{n+1} \) in the subsets \( S_{C_1} \) and \( S_{C_2} \) of \( H^1(\Omega) \), respectively, where \( C_1 \) and \( C_2 \) are defined by

\[ C_1 = \frac{-H_2 C^{n+1}}{H_1 + H_2} \quad C_2 = \frac{H_1 C^{n+1}}{H_1 + H_2} \quad \text{and} \quad C^{n+1} \text{ is given by (3.9):} \]
Remark on Regularity of the Solutions.

With reference to the regularity of the solutions the Lax-Milgram theorem states that

\[
\| q^n_{1} \|_{1,2,\Omega} \leq C_3 \| \xi_{1} f^n \|_{-1,2,\Omega} + \| q^n_{b1} \|_{1/2,2,\Gamma},
\]

\[
\| q^n \|_{1,2,\Omega} \leq C_4 \| g \|_{1/2,2,\Gamma}, \quad g = 1 \text{ on } \Gamma,
\]

\[
\| q^n_2 \|_{1,2,\Omega} \leq C_5 \| q^n_1 - q^n_2 \|_{-1,2,\Omega},
\]

\[
\| q^n \|_{1,2,\Omega} \leq C_6 \| b^n \|_{1,2,\Omega},
\]

where \( C_3, C_4, C_5 \) and \( C_6 \) are constants which depend on \( \Omega \).

However, one can assume that \( F \in L^2(0,T;L^2(\Omega)) \), because the large scale wind stress is a smooth function, and the initial condition \( q^n_1(x,0) = f \in C^\infty(\Omega) \). Since \( \Omega \) is a bounded polygon with no reentrant corners, it is reasonable to expect higher regularity of the solutions. In fact, if one assumes that the relative vorticity vanishes at the boundary, then at any instant \( t_n \) we might suppose that \( q^n_{b1} = f + C^n \in H^{3/2}(\Gamma) \) and by the regularity theory for elliptic problems (cf. [17]) we have

\[
\| q^n_1 \|_{2,2,\Omega} \leq C_7 \| F \| + \| q^n_{b1} \|_{3/2,2,\Omega}, \quad (3.16.1)
\]

where the constant \( C_7 \) depends on \( \Omega \).

When the relative vorticity does not vanish at the boundary is not clear whether \( q^n_b \in H^{3/2}(\Gamma) \); from our formulation the most we can say, in view of (3.7), is that \( q^n_b \in H^{1/2}(\Gamma) \); therefore,
by virtue of the trace theorems (cf. [29]), we have that

\[ q^n_1 \in H^1(\Omega), \forall t_n. \]  

(3.16.2)

Next, we examine the regularity of the modes. If \( \Omega \) is sufficiently smooth, then there is a real number \( k \geq 0 \) such that from the regularity theory of elliptic equations \( \psi_s(x) \in H^{k+2}(\Omega) \); but in our problem \( \Omega \) is a two dimensional polygon with no reentrant corners, so (cf. [17])

\[ \psi_s(x) \in H^2(\Omega) \]  

(3.16.3)

The regularity of \( \psi^n_a, \phi^n \), with \( \Omega \) sufficiently smooth, is given by

\[ \| \psi^n_a \|_{k+2,2,\Omega} \leq C_8(k,\Omega) \| q^n_1 \|_{k,2,\Omega}, \]

\[ \| \phi^n \|_{k+2,2,\Omega} \leq C_9(k,\Omega) \| b^n \|_{k,2,\Omega}. \]

Since \( q^n_1 \) and \( q^n_2 \) are in \( H^1(\Omega) \) then \( \psi^n_a, \phi^n \in H^3(\Omega) \); however, the constraints imposed by the geometry of the domain \( \Omega \) imply that

\[ \psi^n_a, \phi^n \in H^2(\Omega). \]  

(3.16.4)

3.2 The Finite Element Approximation

We now proceed to describe the finite element approximation of equations (3.6)-(3.12). Towards this end, we first introduce the finite element spaces. Let \( h \) denote a discretization parameter, \( 0 < h \leq h_0 < 1 \), and let \( D_h \) be a partition of \( \Omega \) into small rectangular elements \( \Omega_e \) with vertices \( x_k = (x_{1i}, x_{2j}), (i,j) \in \mathbb{Z}^2, 1 \leq i \leq I, 1 \leq j \leq J, \)

29
and sides parallel to the coordinate axes. For each vertex \( k = (i, j) \), \( i \) and \( j \) are local identifiers in the \( x_1 \) and \( x_2 \) directions, respectively. The partition \( D_h = \bigcup \Omega_e \) is regular in the sense that there exists a constant \( \sigma > 0 \) such that for \( \frac{h_e}{\rho_e} \leq \sigma, \forall \Omega_e \).

Given positive integers \( K \) and \( N_e \), where \( K \) is equal to the numbers of vertices (or grid points) in \( D_h \) and \( N_e \) is equal to the number of elements \( \Omega_e \) in \( D_h \), the finite dimensional spaces associated with the partition \( D_h \) are defined by

\[
H_h = \left\{ \nu_h \in C^0(\Omega) : \nu_h|_{\Omega_e} \in P_1, \forall \Omega_e \in D_h \right\},
\]

\[
H_{0h} = H_h \cap H^1_0(\Omega), \quad S_{C_h} = S_C \cap H_h, \quad S_{\zeta_h} = S_\zeta \cap H_h. \tag{3.17.1}
\]

In (3.17.1), \( P_1 \) is the space of polynomials in \( x_1, x_2 \) of degree \( \leq 1 \) (\( l = 1 \) in our computations). The dimension of \( H_h \) is equal to \( K \) and its canonical basis is the set \( \{ \phi_k(x_1, x_2) |_{P_1} \}^K_{k=1} \).

The interface between elements are lines of \( x_1 = \) constant and \( x_2 = \) constant, so that the restriction of \( \phi_k(x) \) along any of the interfaces is a piecewise linear polynomial in one variable. This fact permits writing \( \phi_k(x) \) as a tensor product of \( \phi_i(x_1) \) with \( \phi_j(x_2) \); i.e.,

\[
\phi_k(x_1, x_2) = \phi_i(x_1)\phi_j(x_2), \quad 1 \leq k \leq K, \quad 1 \leq i \leq I, \quad 1 \leq j \leq J.
\]
\[ \phi_i(x_{1r}) = \delta_{ir}, \quad \phi_j(x_{2s}) = \delta_{js}, \quad 1 \leq r \leq I, \quad j \leq s \leq J. \]

where \( \phi_i(x_1) \) and \( \phi_j(x_2) \) are piecewise linear polynomials in the variables \( x_1 \) and \( x_2 \), respectively. Next, according to Glowinski and Pironneau [16] we define the space \( M_h \), the finite dimensional analogue of the space \( M \) introduced in (3.7.1), by

\[ M_h = \left\{ \mu_h \in H_h : \mu_h \big|_{\Omega_e} = 0 \quad \forall \Omega_e \in D_h, \quad \Omega_e \cap \Gamma = \emptyset \right\}, \quad (3.17.2) \]

\[ H_h = H_{0h} \oplus M_h. \quad (3.17.3) \]

The dimension of \( H_h \) is equal to the number \( K_b \) of boundary points. The canonical basis \( B_h \) of \( M_h \) is defined by

\[ B_h = \left\{ w_{hi}, \quad 1 \leq i \leq K_b, \quad w_{hi} \in M_h, \quad w_{hi}(P_{1i}) = \begin{cases} 1, \text{ if } P_{1i} \in \Gamma \\ 0, \text{ otherwise} \end{cases} \right\}. \]

The support of \( M_h \) consists of the elements adjacent to the boundary.

The spaces \( H_h \) (\( H_{0h} \)) are members of the family of finite dimensional spaces \( Q_h \subset W^{1,\alpha}(\Omega) \) which is complete in \( H^1(\Omega)(H^1_{0}(\Omega)) \). We now formulate some approximation and inverse properties of \( H_h \) which are useful in the error analysis of the approximate solution (cf. [2], [12]).

A1) If \( H_h \subset H^m(\Omega), \quad 1 + 1 > m \geq 0 \), then \( \forall u \in H^r(\Omega), \quad r \geq 0, \quad 0 \leq s \leq \min(r, m) \)

\[ \inf_{\chi \in H_h} \| u - \chi \|_{s, 2, \Omega} \leq Ch^r \| u \|_{r, 2, \Omega}, \quad \forall u \in H^r(\Omega). \]
where \( \rho \leq \min(1+1-s, r-s) \).

A2) If in addition, we assume that \( u \in W^{1,\omega}(\Omega) \) then

\[
\inf_{\chi \in H_h} \| u - \chi \|_{1,\omega, \Omega} \leq C h \| u \|_{1,1, \omega, \Omega}.
\]

A3) For \( 2 \leq r \leq 1+1 \)

\[
\inf_{\chi \in H_h} \left\{ \| u - \chi \|_{1,2, \Omega} + h \| u - \chi \|_{1,1, \omega, \Omega} \right\} \leq C h^r \| u \|_{r,2, \Omega}.
\]

Let \( k_1, k_2 \) be arbitrary numbers and \( p, q \in (1, \omega] \) such that \( H_h \subset W^{k_1, p, k_2, q} \), then there exists a constant \( C = C(\omega, k_1, k_2, p, q) \) such that \( \forall v_h \in H_h \),

\[
\| v_h \|_{k_2, q, \Omega} \leq C h^{N/q-N/p+k_1-k_2} \| v_h \|_{k_1, p, \Omega},
\]

with \( N \) the dimension of the domain \( \Omega \).

\[
\| v_h \|_{1,1, \omega, \Omega} \leq C h^{-1} \| v_h \|_{1,2, \Omega},
\]

\[
\| v_h \|_{1, \omega, \Omega} \leq C h^{1-N/2} (\log h^{-1})^{1-1/N} \| v_h \|_{1,2, \Omega}.
\]

Let \( Q^n_i(x), \psi^n_{sh}(x), \psi^n_{ah}(x) \) and \( \phi^n_h(x) \) denote the approximations in \( H_h \) to \( q^n_i(x), \psi^n_s(x), \psi^n_{ah}(x) \) and \( \phi^n(x) \), respectively. To compute the finite element solutions we expand such approximations as linear combinations of the basis functions of \( H_h \). Thus
\begin{equation}
Q^n_i(x) = \sum_k Q^n_{ik} \phi_k(x) \in S_{\zeta h}
\end{equation}

\begin{equation}
\psi^n_{sh}(x) = \sum_k \psi^n_{sk} \phi_k(x) \in S_{Sh}
\end{equation}

\begin{equation}
\psi^n_{ah}(x) = \sum_k \psi^n_{ak} \phi_k(x) \in H_{Oh}
\end{equation}

\begin{equation}
\phi^n_h(x) = \sum_k \phi^n_k \phi_k(x) \in H_{Oh}.
\end{equation}

Likewise, in view of (3.13) the stream functions can also be approximated as

\begin{equation}
\psi^n_{1h}(x) = \sum_k \psi^n_{1h_k} \phi_k(x),
\end{equation}

\begin{equation}
\psi^n_{2h}(x) = \sum_k \psi^n_{2h_k} \phi_k(x).
\end{equation}

The finite element approximation of (3.6)-(3.8) is defined as follows:

Given \( Q^0_i = f, \ i = 1,2 \), \( \psi^0_{ah} = \phi^0_h = 0 \),

find \( \psi_{hs} \in S_{Sh'} \) with \( C = 1 \), such that

\begin{equation}
(\nabla \psi_{sh}', \nabla \phi_h) + \lambda^2 (\psi_{sh}', \phi_h) = 0, \ \forall \phi_h \in H_{Oh}.
\end{equation}

Then for \( n > 0 \), assuming \( Q^n_i, \lambda^n_{1h}, \psi^n_{ah}, \phi^n_{h}, \psi^n_{1h} \) and \( \psi_{sh} \) are known, solve for
$$1) \quad \left( \frac{Q_i^{n+1}(x) - \bar{Q}_i(x^n)}{\Delta t} \right) + \frac{A_H}{2} (\nabla(Q_i^{n+1}(x) + \bar{Q}_i(x^n)), \nabla \phi_h) +$$

$$+ \delta_{12} \frac{c}{2} (Q_i^{n+1}(x) + \bar{Q}_i(x^n), \phi_h) = \delta_{11}(F^{n+1/2}, \phi_h), \forall \phi_h \in H^0_h.$$ 

$$Q_i^{n+1}(x) - Q_i^{n+1} \in H^0_h. \quad (3.20)$$

$$11) \quad \psi_{ah}^{n+1}, \phi_h^{n+1} \in H^0_h \text{ such that}$$

$$(\nabla \psi_{ah}^{n+1}, \nabla \phi_h) + \lambda^2 (\psi_{ah}^{n+1}, \phi_h) = -(Q_1^{n+1} - Q_2^{n+1}, \phi_h), \forall \phi_h \in H^0_h,$$ 

$$(3.21.1)$$

$$(\nabla \psi_h^{n+1}, \nabla \phi_h) = (b_x^{n+1}, \phi_h), \forall \phi_h \in H^0_h, \quad (3.21.2)$$

$$c_{ah}^{n+1} = \frac{\int_{\Omega} \psi_{ah}^{n+1} \, d\Omega}{\int_{\Omega} \psi_{sh} \, d\Omega}, \quad (3.21.3)$$

$$\psi_h^{n+1} = \psi_{ah}^{n+1} + c_{ah}^{n+1} \psi_{sh}. \quad (3.21.4)$$

$$\psi_{1h}^{n+1} = \frac{H_2}{H_1 + H_2} \psi_h^{n+1} + \phi_h^{n+1}, \psi_{2h}^{n+1} = -\frac{H_1}{H_1 + H_2} \psi_h^{n+1} + \phi_h^{n+1}. \quad (3.22)$$

Finally, we have to compute the velocities in each layer from the values of the stream functions and (3.12). If we took the derivatives of $\psi_{1,2h}^{n+1}$ we would obtain velocity vectors which would not belong to $H_h$, but they would be in $L^2(\Omega)$; in order
to force them to be in $H_h$, we project orthogonally the
derivatives of $\psi_{i,2h}^{n+1}$ onto $H_h$. For $i = 1, 2$ define

\[ u_{ih}^{n+1} = \sum_k u_k^{n+1} \phi_k(x), \quad v_{ih}^{n+1} = \sum_k v_k^{n+1} \phi_k(x), \quad (3.23.1) \]

\[ \bar{u}_{ih}^{n+1} = -\frac{\psi_{ih}^{n+1}}{\partial x_1}, \quad \bar{v}_{ih}^{n+1} = \frac{\psi_{ih}^{n+1}}{\partial x_2}. \quad (3.23.2) \]

The orthogonal projection from $L^2(\Omega)$ onto $H_h$ is given by

\[ (u_{ih}^{n+1} - \bar{u}_{ih}^{n+1}, \phi_h) = 0, \]

\[ (v_{ih}^{n+1} - \bar{v}_{ih}^{n+1}, \phi_h) = 0, \quad \forall \phi_h \in H_{oh}. \quad (3.23.3) \]

It remains to describe the computation of the relative
vorticity on the boundary when the no-slip boundary condition
is used. This is done by constructing the discrete analogue of
(3.7.2) in the space $M_h$. Thus, by taking $\frac{\partial \psi_i}{\partial n} = 0$ in
(3.7.2), we have for $i = 1, 2$, that

\[ \int_{Be} \psi_{ih}^{n+1} \psi_{ih} ds = -\int_{Be} \omega_{ih}^{n+1} \mu_{ih} ds, \quad \forall \mu_{ih} \in M_h, \quad (3.24.1) \]

where $Be$ is the support of $M_h$. Since $Be$ is composed of the
rectangles adjacent to the boundary, then we can compute
exactly the integrals of (3.23.1) element by element of $Be$.
This yields

35
\[ \lambda_{1h}^{n+1} = \frac{3}{(|n|)^2} (\psi_{1h}^{n+1} - \psi_{1h}^{n+1}) - \frac{\omega_{1h}^{n+1}}{2}, \]  

(3.24.2)

where |n| is the distance along the normal direction from the wall point W to the first interior point I. Formula (3.24.2) is locally a second order formula known in finite difference context as Wood's formula. In a more general set up, such as triangles or curved elements as support of Be, (3.24.1) yields an algebraic linear system of equations which can be easily solved by the Cholesky method since the number of boundary points is, in general, not very large (see [16] for details).

By computing the integrals that appear in (3.19)-(3.21) one obtains the following algebraic linear systems of equations:

\[ S[\psi_{g}] = [R_{1}], \]  

(3.25.1)

\[ V_{i}[Q_{i}^{n+1}] = [R_{12}], \]  

(3.25.2)

\[ S[\psi_{a}^{n+1}] = [R_{3}], \]  

(3.25.3)

\[ T[\psi_{a}^{n+1}] = [R_{4}], \]  

(3.25.4)

where \([\ ]\) denotes a column matrix. The matrices S, V \_i and T are sparse, banded, symmetric, positive definite matrices with entries given by

\[ t_{ij} = \int_{\Omega} \nabla \phi_{hi} \nabla \phi_{hj} d\Omega, \]  

(3.26.1)
\[ S_{ij} = \int_{\Omega} (\nabla \phi_i \nabla \phi_j + \lambda^2 \phi_i \phi_j) \, d\Omega , \quad (3.26.2) \]
\[ \nu_{1ij} = \int_{\Omega} (\phi_i \phi_j + \nu \nabla \phi_i \nabla \phi_j) \, d\Omega , \quad (3.26.3) \]
\[ \nu_{2ij} = \int_{\Omega} (\mu \phi_i \phi_j + \nu \nabla \phi_i \nabla \phi_j) \, d\Omega . \quad (3.26.4) \]

where \( \nu = \left( -\frac{1}{2} \Delta t A_h \right) \) and \( \mu = (1 + \frac{1}{2} \Delta t e) \).

The calculation of the integrals is performed by using the Gauss (2,2) quadrature rule. Since all the matrices are positive definite, then the finite element solution exists and is unique. The systems (3.25) are solved by the Jacobi-Conjugate Gradient (JCG) method.