OPTIMAL FISHERIES INVESTMENT

by

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Abstract

This thesis explores problems of optimal investment arising in fisheries and other renewable resource industries. In such industries, two simultaneous investment problems must be addressed: investment in the resource stock (the biomass) and investment in the capital stock (harvesting capacity). Each of these investment problems faces a key complication; investment in the resource is constrained by the natural population dynamics, while in many cases investment in the physical capital stock suffers from irreversibility, since capital used in natural resource industries is often non-malleable. In addition, all investment decisions must be made within an uncertain environment; full information is never available.

Building upon the work of Clark, Clarke and Munro (Econometrica, 1979), we develop a two-state two-control model which incorporates investment delays and stochastic resource fluctuations within a seasonal (discrete-time) framework. A dynamic programming approach is used to analyse the model heuristically and to obtain numerical results, beginning with a study of the ideal deterministic case and proceeding to a full stochastic analysis. The key assumption of irreversible investment is maintained throughout the thesis.

We have examined the qualitative and quantitative effects on optimal investment strategies of several economic and ecological factors: (i) delays in investment, (ii) population dynamics parameters, (iii) selling price, (iv) capital cost and operating cost, (v) depreciation rate, (vi) discount rate, and
(vii) the level of uncertainty in the resource stock. We have found that the key cost parameter for the investment problem is the ratio of unit capital costs to unit operating costs. Depreciation can play a rather counter-intuitive role; in some circumstances optimal investment levels can increase with the depreciation rate, contrary to the usual treatment of depreciation as an additional cost of capital.

The introduction of uncertainty in the form of stochastic resource fluctuations can substantially change the optimal investment policy, but this tends to have little effect on the value of the fishery. We analyse the factors which determine the role of randomness in optimal fisheries investment, and discuss in some detail the implications for management.

Solution of the stochastic optimization problem studied here requires the use of rather complicated numerical methods, which are described in detail in the thesis. These methods are quite general, and should prove useful in analysing other related stochastic models.
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Chapter I. Introduction

The problem of determining optimal investment levels and optimal capital stocks is a pervasive one in the economics literature. Indeed investment decisions together with decisions regarding the optimal levels of production are key to the efficient operation of an economy. [See Gordon (1961) for a discussion on the importance of capital investment in resource industries, and Freidenfelds (1981) for a comprehensive treatment of simple capacity expansion models.]

In recent years, considerable research has been undertaken in expanding the traditional analysis of investment, both at the level of the firm and of the society. Classical theory, including such standard topics as marginal efficiency of capital, the investment multiplier and IS-LM analysis, suffers in particular from its static deterministic nature. Led by K.J. Arrow and his co-workers (Arrow and Lind, 1970; Arrow and Kurz, 1970), modern Capital Theory has adopted optimal control theory as a tool to study, as Arrow puts it, "investment as optimization over time".

From the outset it has been realized that perhaps the two major complications in determining optimal investment strategies are the lack of malleability of capital and the uncertain environment in which investment decisions must be made. On the one hand, investment decisions tend to be irreversible, while on the other hand such decisions can never be based on full
information, since the appropriate level of investment for long-lived capital depends on the state of the world several years from the present.

Uncertainty in investment has been addressed by many researchers, including Arrow and Lind (1970), who discussed the general question of risk and risk-sharing in public investments, Baumol (1968), Brock and Mirman (1978) and Hirshleifer (1966). Recently, McDonald and Siegel (1982) have undertaken an interesting application of option theory to the problem of investment under uncertainty. Their model allows for the possibility that firms will shut down if production is uneconomical; this approach seems particularly useful in considering natural resource industries.

The non-malleability of capital, and the corresponding problem of irreversible investment, was discussed in general terms by Arrow (1968). Campbell (1980) and Lasserre (1982) examined irreversible investment models of exhaustible resource production, while Margolick, Charles and Helliwell (1981) studied the problem of optimal investment in hydro-electric capacity under demand uncertainty.

Perhaps nowhere is the optimal irreversible investment problem more difficult to analyse than in the area of renewable resource management, and fisheries management in particular, where one is faced with a strongly fluctuating resource, for which the underlying dynamics are known only approximately, together with volatile prices and a complicated industrial structure of fishermen and processors. This may in part explain
why most models of fisheries management problems to date have concentrated on the dynamics of the natural capital, the fish stock, rather than the manmade capital, in the form of boats and gear. The prototypical control problem (see for example Clark and Munro, 1975) tends to involve a differential or difference equation governing the size of the fish stock, an objective function to be maximized, and a control (usually instantaneous harvesting effort or end-of-year escapement). It is often assumed that the fishing effort, \( E(t) \), is subject to a constraint \( 0 \leq E(T) \leq K \) at any time \( t \), where \( K \) is a fixed constant representing a given capital stock, or "capacity" of the fishery.

Certainly, many authors have noted that the capital stock \( K \) is unlikely to remain constant over time. For example, Smith (1968) considered the case where capital follows an uncontrolled differential equation, with the level of capital responding to fishery rents. In a control-theoretic context, the common assumption that escapement can be set at any value less than or equal to the recruitment implicitly assumes that unlimited amounts of capital are available, if needed, to harvest the stock. In both these cases, capital is considered to be malleable, so that both investment and disinvestment are possible. While this is not unreasonable if the fishery can utilize the capital stock of a larger neighbouring fishery (as is the case with the British Columbia herring fishery in relation to the salmon fishery) it is unlikely to hold in general.
Given that a vessel entering a particular fishery may have essentially no alternative uses, one must return to the question of optimal irreversible investment. For the particular case of the fishery, Clark and Kirkwood (1979), Dudley and Waugh (1980) and Silvert (1977) have addressed the question of how many vessels (or in general how much harvesting capacity) should be optimally allocated to a fishery by its sole owner or social manager. In each of these papers, however, the models being considered allow the investment problem to be reduced to a once-and-for-all decision at the outset. A more general analysis has been undertaken by Clark, Clarke and Munro (1979), who obtained a full analytic solution to the continuous-time deterministic investment problem with 2 state variables (biomass and fleet capacity) and 2 controls (fishing effort and investment). Recently McKelvey (1982) has studied a similar model, involving the optimal mix of "specialist" and "generalist" vessels in a fishing fleet.

This thesis attempts to extend the study of optimal capital investment in renewable resource industries by expanding the work of Clark, Clarke and Munro (1979) in a number of directions. In Chapter II, we analyze a model similar to theirs but more realistic in a number of respects: (i) time is considered to be discrete between fishing seasons although continuous within each season, (ii) the decision variables are end-of-season escapement and yearly investment (as opposed to instantaneous fishing effort and investment in the CCM continuous time case), and (iii) delays are allowed between the
time at which investment decisions are made and the time at which these investments come on-line. A dynamic programming approach is utilized, allowing us to study arbitrary stock-recruitment functions, including the Beverton-Holt and Ricker forms, and to obtain comparative dynamics results.

In particular we describe the effect on optimal investment/escapement policies of the following factors: (i) discrete-time analysis, (ii) investment delays, (iii) fecundity and carrying capacity of the stock, (iv) fish price, (v) capital cost, (vi) discount rate, and (vii) depreciation rate.

In Chapter III, this model is modified to incorporate stochastic fluctuations in the resource stock, and a study is made of optimal investment under uncertainty. We examine the appearance of an optimally-managed stochastic fishery, the role of several model parameters in determining how uncertainty affects optimal policies, and the relative performance of deterministic vs. stochastic strategies.

Finally, Chapter IV highlights the more important results, discusses the implementation of optimal investment programs, and offers some concluding remarks regarding the effects of economic and ecological parameters, and the role of uncertainty, in determining optimal investment strategies.

Our method of analysis involves analytic, heuristic and numerical approaches. Dynamic programming is the primary tool of both Chapters II and III: useful insights can be obtained from an analytic treatment of a simplified model and from heuristic analysis of the general case, but it is the numerical method and
results which occupy most of our attention.

A word on notation: this thesis was produced using the documentation program FMT on the University of British Columbia computer system, together with a preprocessor developed by Donald Ludwig of U.B.C. This program was capable of producing all symbols required for this thesis apart from three: the symbol for infinity (denoted instead as "inf"), the square root sign ("sqrt"), and the integral sign, which is denoted "\[\int\]" herein.
Chapter II. Optimal Fisheries Investment: The Deterministic Case

We begin our examination of optimal fisheries investment by considering a deterministic model which, in its simplest form, represents the discrete-time analogue of the Clark, Clarke and Munro (1979) model (referred to here as "CCM"). Throughout the chapter, our model and results are compared to those of CCM; we shall see that while the qualitative behaviour is similar, some important differences do arise. In addition, the dynamic programming approach used here allows several additional features to be considered. We will be able to describe in detail the role of (i) discrete-time analysis, (ii) investment delays, (iii) fecundity and carrying capacity of the stock, (iv) fish price, (v) capital cost, (vi) discount rate, and (vii) depreciation rate in determining optimal investment/escapement policies.

The outline of the chapter is as follows: Section A describes the model, section B presents simple analytic results, while in section C a heuristic analysis of the model is undertaken. Section D discusses the numerical method and in section E the results are presented. Finally, in section F, a summary of the results together with a discussion completes the chapter. (The reader wishing to arrive more quickly at the main results in section E will find that sections B and D can be omitted without loss of continuity.)
A. The Model

(i) The fish stock is assumed to be represented by a single quantity, the biomass. This common assumption ignores multi-species and multi-cohort problems, but is a useful simplification so as to place primary emphasis on the role of investment and fleet capacity in our analysis.

The biomass is governed by a stock-recruitment relationship \( R = F(S^n) \), which in our applications will be either pure compensatory \( (F' > 0, F'' < 0) \) or overcompensatory, using the Beverton-Holt or Ricker forms respectively (Beverton & Holt, 1957; Ricker, 1954). Intraseasonal growth of individuals in the population is neglected, so that the natural population dynamics (reproduction and natural mortality) occurs at just one instant in each year. However, fishing mortality is assumed to occur continuously during the season, so that during the course of a single season, the biomass follows the common differential equation \( \frac{dx}{dt} = -h(t) = -qE(t)x(t) \) where \( h(t) \) is the harvest rate, \( E(t) \) is the instantaneous aggregated fishing effort, \( q \) is a constant catchability coefficient and initially \( x(0) = R \). The

\[
S = R \exp \left\{ -q \int_0^T E(t) \, dt \right\}
\]

escapement is then \( S = R \exp \left\{ -q \int_0^T E(t) \, dt \right\} \) where \( T \) is the arbitrary maximum season length.

(ii) The capital stock, or fleet capacity, \( K \), is represented by the maximum instantaneous fishing effort: at any point \( t \) in the
season \( n \), we must have \( 0 \leq E(t) \leq K \). Hence \( K \) depicts the catching power of the fleet. Clearly this is a highly aggregated measure, including such elements as actual vessels, nets, machinery, engines and training of the fishermen. In addition, it ignores questions of gear type; the type of vessel, or mix of vessel types, chosen for a fishery can affect operating costs, the proportions of factor inputs (labour, fuel, etc.) and even the health of the fish stock. (In the British Columbia fishery, gillnetters, seiners and trollers each claim that the other two gear types are more damaging to the resource.) On the other hand, use of catching power as the measure of capacity is superior to measuring capital simply by the fleet size; it is often observed that control of the number of boats in a fishery (through licensing) tends to have little effect on the overall harvesting potential. Restricting entry merely inspires existing fishermen to upgrade their own catching power.

The cost per unit of new fleet capacity is assumed to be a constant, \( \delta \), irrespective of the current level of capacity. Furthermore it is assumed that this cost must be paid in full at the time the new capacity is ordered.

Depreciation is assumed to occur at the end of each season, with a constant fraction \( \gamma \) (the depreciation rate) of the current capital stock wearing out or otherwise being removed from the fishery at that time.

Perhaps the most important assumption in our model, as in the CCM model, is the irreversibility of investment. We assume here that the fleet capacity cannot be decreased at will, but
only through the process of depreciation. Hence the dynamics of the capital stock, K, can be expressed as follows:

\[ K_{n+1} = (1-\gamma)K_n + I_n \quad ; \quad I_n \geq 0 \]

where the investment \( I_n \) becomes available in year \( n+1 \). [This key assumption could be relaxed somewhat if we were to allow a positive scrap value for fishing capital (see CCM, pp35-37, and the discussion in (iii) below) or the possibility of bringing outside vessels (either domestic or foreign) into the fishery on a temporary basis (see, for example, McKelvey, 1981).]

A further consideration in dealing with investment policies is the possibility of a delay existing between the time an investment decision is made and the time the corresponding new capacity becomes available. Such delays may arise due to the time necessary to construct new vessels and/or transport them to the fishing grounds. In a stochastic fishery, the existence of delays will increase the level of uncertainty in planning fisheries investment. On the other hand, investment delays in a deterministic world are of minor importance to the analysis, increasing the effective capital cost but not affecting the substance of the management problem. We have chosen to undertake most of our analysis using the more realistic delayed investment assumption, both for the sake of realism and to compare our deterministic results with those obtained using the stochastic model of Chapter III. To simplify the structure of the model while incorporating a reasonable delay, we shall assume that, in any given year, the decision regarding next season's optimal
capacity must be made before the end of the current season and full payment for any new investment must be made in the current season.

(iii) The fishery management problem involves yearly escapement and investment decisions. The timing of the two decisions depends on the assumption regarding delays in bringing investment on-line. The following applies to the delayed investment case. Given the recruitment $R_n$, the optimal escapement $S^*$ is chosen, subject to the constraint

$$R_n \exp\{-qTK\} \leq S^* \leq R$$

where the lower limit is reached by fishing at maximum effort throughout the season. Then the optimal investment for next year, $I_{n+1} \geq 0$, is chosen, based on the value of $S^*$: payment is made in year $n$ for this new capacity.

To this point, we have not discussed the industrial structure of the fishery. There are two primary components of a fishing industry, the harvesting and the processing sectors; incorporation of the latter in a fisheries model can be quite complicated (Clark and Munro, 1980), particularly if processors have considerable monopsonistic power. A full analysis of the optimal capacity problem would have to include both fleet capacity and processing capacity as separate variables, with their corresponding investment rates as controls. In addition, variable processing costs must be dependent on actual harvest rather than on effort per se. We avoid these complications here.
by essentially omitting the processing sector from the model. On the other hand, one may think of the model as applying to the entire fishing industry (for a particular species), with an appropriate fraction of the processors' capital costs included in the unit capacity cost, \( \delta \). This may be reasonable, at least for a given fish stock, since presumably one unit of harvesting capacity will require a certain amount of processing capacity to handle its output.

In any case, we shall make the additional assumption that the fishery faces a given constant selling price, \( p \), and a known cost-of-effort function \( c(E) \), with convex (U-shaped) marginal and average cost curves. In particular, this implies perfect elasticity of demand for fish, an assumption which simplifies the economics in our model, but is likely to be reasonable only if the fishery does not enjoy too large a market share. The interesting and realistic problem of price uncertainty will be considered in future work.

Our model allows for the possibility of an "alternative fishery", which provides a backup for the principal fishery. For simplicity we assume this alternative fishery is capable of producing a constant net revenue \( \rho \) per unit effort per unit time. In addition we assume that \( \rho < \{1-\alpha(1-\gamma)\}^\delta / \alpha T \), so the alternative fishery is not worth developing on its own (capital costs \( \delta \) cannot be covered by the total present value of rents, namely \( \alpha \rho T [1+\alpha(1-\gamma)+\alpha^2(1-\gamma)^2+...]=\alpha \rho T / \{1-\alpha(1-\gamma)\} \) ). The overall effect of the alternative fishery is to provide a use for excess capital (for which the capital cost is a by-gone), while also
increasing intraseasonal opportunity costs. (The role of an alternative fishery is somewhat related to the existence of a market for scrap, since both provide a use for excess capacity arising in the principal fishery. However, scrapping of capacity is in itself irreversible, whereas an alternative fishery plays the part of a "sideline", to be used whenever desirable.)

Left out from the model are such complications to the fishery management problem as: (i) international (transboundary) problems, (ii) the possibility of set-up costs and fixed (non-capital) costs , and (iii) differentiation between labour and capital as inputs to the fishery, together with questions of multispecies effects, gear type, processing, demand elasticity, nonlinear costs and mixed fleets discussed above. Despite these drawbacks, the model hopefully contains at least the essence of the optimal fisheries investment problem.

The yearly rents accruing to the fleet, as a function of recruitment, capacity, escapement and investment, are given by:

\[
\pi(R,K,S,I) = \int_0^T \left[ pqEx - c(E) \right] dt + \int_0^T \rho(K-E) dt - 6I
\]

\[
= p(R-S) - \int_0^T c(E(t)) dt - (\rho/q) log(R/S) + \rho TK - 6I
\]

using \(dx/dt = -qEx\), \(x(0) = R\), \(x(T) = S\).

This rents function is determined once we have found an expression for the cost-minimizing effort path and season length which reduce the biomass from \(R\) to \(S\) within the maximum season length \(T\). Consider the problem:

\[
\text{Min} \left\{ \int_0^T c(E(t)) dt \right\} \text{ subject to } 0 \leq E(t) \leq K \text{ and } \int_0^T E(t) dt = q^{-1} \log(R/S).
\]
Note first that whatever the optimal season length, effort can be considered to be constant within the season: this follows from our assumption of a convex average cost curve. If effort varies during the season, costs could be reduced by using an averaged effort level. Hence the cost-minimization problem can be re-stated:

$$\text{Min} \{ \tau \cdot c(E) \} \text{ subject to } 0 \leq E \leq K \text{ and } \tau = (qE)^{-1} \log(R/S) \text{ where } 0 \leq \tau \leq T.$$  

This in turn can be simplified to the problem of minimizing

$$c(E)/E \text{ subject to } (qT)^{-1} \log(R/S) \leq E \leq K.$$  

Define $E_0$ to be the minimum average cost effort level, so that $c(E_0)/E_0 = c'(E_0)$. If $E_0$ lies in the interval $[(qT)^{-1} \log(R/S), K]$ then the optimal effort is $E^* = E_0$. If $E_0 < (qT)^{-1} \log(R/S)$ then $E^* = (qT)^{-1} \log(R/S)$ and if $E_0 > K$ then $E^* = K$.

Using this result together with the definition of $\tau$ above, we can now write the minimum yearly variable costs as a function of $R$, $K$, and $S$:

$$\tau \cdot c(E) = \begin{cases} 
\text{Tc}([qT]^{-1} \log[R/S]) & K > E_0, R \leq S < R \\
\log(R/S) \cdot c(E_0)/qE_0 & K > E_0, R \leq S < R \\
\log(R/S) \cdot c(K)/qK & K < E_0 \end{cases}$$

The yearly rents function is then given explicitly when this expression is inserted in the equation:

$$\pi(R, K, S, I) = p(R - S) - \tau \cdot c(E) - (\rho/q) \log(R/S) + \rho TK - \delta I$$

The "objective" of the fishery manager in reality is likely to be a complicated composite of such considerations as rent maximization, harvest maximization, harvest and employment stability, conservation of the fish stock and distribution of

There is, of course, no reason to exclude any of these objectives and indeed the appropriate balance between conflicting objectives is an ethical and political decision. To simplify the analysis here we shall adopt the common economic objective of present value rent maximization. (In Chapter III, we briefly consider the case of risk aversion, replacing yearly rents, \( \pi \), with \( U(\pi) \), where the utility function \( U \) is increasing and concave \([U'>0, \ U''<0]\).) Hence our problem can be stated as follows:

\[
\text{Max} \sum_{i=1}^{n-1} \alpha^i \pi(R,K,S,I) \quad \text{subject to:} \\
\{S_1;I_2;S_2;\ldots\} \quad n \quad n \quad n \quad n+1 \\
R = F(S), \quad K = (1-\gamma)K + I, \quad R \exp(-qTK) \leq S \leq R, \quad I \leq 0, \\
R^{n+1} \quad n \quad n+1 \quad n \quad n \quad n \quad n+1
\]

where \( \alpha \) is the annual discount factor.

It is worthwhile noting at this point that we have posed here an overall optimization problem, without addressing the difficult task of determining how to reach this optimum. Regulatory measures to achieve optimal effort or optimal harvests within the fishing season have been the subject of much discussion [see for example Pearse (1979), Scott and Neher (1981), and Clark (1980)]. The problem of achieving the optimal fleet capacity through regulation has not been as widely discussed, but is perhaps not as difficult a problem conceptually; we shall return to this question in Chapter IV.
(iv) The dynamic programming equation for the value of the fishery in state \((R_n, K_n)\) at the start of a season \(n\) is given by:

\[
V(R_n, K_n) = \max_{n} \max_{S_n} \left\{ \pi(R_n, K_n, S_n, I_n) + e^n V(R_{n+1}, K_{n+1}) \right\}
\]

where \(R_n = F(S_n)\), \(K_{n+1} = (1-\gamma)K_n + I_n\), the outer maximization is subject to \(R_n \leq S_n \leq K_n\) and the inner maximization is over the range \(I_n \geq 0\). This is simply a statement, using \(n+1\) Bellman's (1957) principle of optimality, that the value of the fishery is given by the maximum value of the sum of current rents plus the discounted future value of the fishery, where the escapement and investment levels are chosen from the set of all feasible values. Removing the subscripts on the variables, this can be rewritten:

\[
V(R, K) = \max_{R_n} \max_{S_n} \left\{ \pi(R, K, S, I) + e^n V(F(S_n), (1-\gamma)K_n + I_n) \right\}
\]

where \(\pi(R, K, S, I) = p(R-S) - r \cdot c(E) - (\rho/q) \log(R/S) + \rho TK - \delta I\), with \(r \cdot c(E)\) being determined as above. This equation (1) will form the basis for most of the analysis and results presented in this chapter. The following three sections study equation (1) from analytic, heuristic and numerical perspectives, respectively.
B. Analytic Results

To obtain some analytic insights into the optimal investment problem, we first consider a simplification of the model described above. The stock-recruitment function $F(S)$ is given by $F(S) = aS/(1+aS/b)$ in the Beverton-Holt case. If the maximum intrinsic growth rate, $a$, is very large, recruitment is practically independent of the previous year's escapement. Indeed as $a$ tends to infinity, we obtain the approximation $F(S) = b = \text{constant}$. This rather trivial stock-recruitment function allows us to concentrate on the capital investment problem, and in particular to obtain an analytic upper bound on the true optimal fleet capacity (since the assumption $F(S) = b$ is as optimistic as one can be regarding the productivity of the fishery). The assumption that recruitment is independent of escapements may be considered a reasonable approximation in some circumstances; Australian prawn stocks appear to fit this assumption, a factor which has enabled fairly detailed models of these fisheries to be analyzed (Clark & Kirkwood, 1979; Dudley & Waugh, 1980). On the other hand, Walters & Ludwig (1981) have shown that errors in measuring fish stocks can lead to the appearance of independence when in fact there may be considerable compensation in the stock-recruitment relationship. If bionomic equilibrium occurs at a sufficiently high stock size one need not be overly concerned with over-harvesting and the threat of extinction. However, to allow for the possible risk of overly low escapements we shall arbitrarily include in the objective function a positive, increasing function of
escapement, \( b(S) \), representing a measure of the economic or social benefits of an escapement \( S \). In a sense, this function corrects for the extreme assumption of constant recruitment.

Thus the model we consider here is identical to that of section A, except that we specify \( R = F(S) = b = \text{constant} \) (which we shall write simply as \( R \)) and add a benefit function \( b(S) \) to the rents \( \pi(R,K,S,I) \). The latter function becomes:

\[
\pi(R,K,S,I) = p(R-S) - r \cdot c(E) - (\rho/q) \log(R/S) + \rho TK - \delta I + b(S)
\]

The dynamic programming equation (1) can now be written:

\[
(2) \quad V(R,K) = \max \quad \max_{I \geq 0} \quad R \cdot \exp\{-qTK\} \leq S \leq R \quad [\pi(R,K,S,I) + \sigma V(R,(1-r)K+I)]
\]

where \( R \) is constant. Letting \( v(K) = V(R,K) \) and differentiating with respect to \( I \), we see that the optimal capacity for next season is given by \( v'(K^*) = \delta / \sigma \) where we have used the equation

\[
\pi(R,K,S,I) = -\delta.
\]

Assuming that \( v(\cdot) \) is an increasing function of \( I \) capacity, \( K \), it is clear that there is a unique optimal level of fleet capacity, \( K^* \). Hence whatever the initial values of \( R \) and \( K \), the optimal policy will be to invest up to the capacity level \( K^* \), which can be chosen as a one-time irreversible decision at the outset. (If initially \( K > K^* \), one must allow depreciation to take place until \( K \) has dropped below \( K^* \) and then invest up to \( K^* \)). Whatever the level of capacity in a given year, the optimal escapement \( S^* \) is chosen subject to \( Re^{-qTK} \leq S \leq R \) and based on cost minimization in harvesting the stock from \( R \) to \( S \). Since the level of escapement has no effect on the fishery in future years, the optimal escapement can be found simply by maximizing
the yearly rents. Each year's decision is identical, so that the
problem reduces to optimizing over one season, with capital
costs suitably amortized. Assuming risk neutrality, \( U(\pi) = \pi \), the
problem can be stated as follows:

\[
\begin{align*}
\text{(3) } \max & \quad \max \quad \left[ p(R-S) - \min \int_{K \geq 0} c(E) dt + b(S) \right. \\
& \quad \left. \exp{-qTK} \leq S \leq R \quad \tau; E \rightarrow 0 \right] \\
& \quad -\left( \frac{\rho}{q} \right) \log\left( \frac{R}{S} \right) - \left( \kappa - \rho T \right) K
\end{align*}
\]

where the total variable cost is to be minimized subject to
\( 0 \leq \tau \leq T, \quad 0 \leq E(\cdot) \leq K, \) and \( q \int_{0}^{T} E(t) dt = \log\left( \frac{R}{S} \right) \). The constant \( \kappa \) is found
by adding the annual "rental" cost of capital [that is, the
amortized capital cost \( = \delta/\Sigma a_{n} \)] to the annual depreciation
\( \tau \delta \). Hence \( \kappa = \left[ (1-\alpha)/\alpha + \tau \right] \delta \). The cost minimization, given \( R \)
and \( S \), was performed in section A, where we obtained the result:

\[
\begin{align*}
\min & \quad \int_{\tau; E \rightarrow 0} c(E) dt = \left\{ \begin{array}{ll}
\log\left( \frac{R}{S} \right) \cdot c(E_{0})/qE_{0} & ; K > E_{0}, Re \leq S \leq R \\
\log\left( \frac{R}{S} \right) \cdot c(K)/qK & ; K < E_{0}
\end{array} \right.
\end{align*}
\]

We wish now to determine the optimal \( S^{*} \) for any combination of \( R \)
and \( K \). Let us assume initially that \( K > E_{0} \). Then, defining
\( -qT_{E_{0}} \), we have:

\[
\begin{align*}
S' &= Re \\
\text{Re} & \leq S \leq S'; \quad \pi = p(R-S) - \text{Tc}\left( \frac{1}{qT} \log\left( \frac{R}{S} \right) \right) - \left( \frac{\rho}{q} \right) \log\left( \frac{R}{S} \right) + b(S) \\
S' \leq S \leq R; \quad \pi = p(R-S) - q^{-1} \log\left( \frac{R}{S} \right) \cdot \left[ c(E_{0})/E_{0} \right] + b(S)
\end{align*}
\]

Differentiating each of these equations and setting \( d\pi/dS = 0 \) we
obtain:
\[-q_{TK}\]
\[
Re \leq S \leq S'; \quad [p-b'(S)]qS = c'([1/qT]\log[R/S]) + \rho
\]
\[
S' \leq S \leq R ; \quad [p-b'(S)]qS = c'(E_0) + \rho
\]
Define \( S = s(R) \) to be the solution of the first of these equations and \( S_0 \) to be the solution of the second, so that \([p-b'(S_0)]qS_0 = c'(E_0) + \rho\). Define \( R_*(K) \) as the solution of:

\[-q_{TK}\]
\[
(4) \quad [p-b'(Re)]qRe = c'(K) + \rho.
\]

Note that if \( R = R_*(K) \), we have \( s(R_*(K)) = R_*(K)e^{-\rho} \). Now \( S' \leq S \leq R \) iff \( S_0 \leq R \leq R_*(K) \), so that the optimal escapement is \( S^* = S_0 \) in this case. Similarly if \( S_0 e^{-\rho} \leq R \leq R_*(K) \) then \( S^* = s(R) \). It is reasonable to assume, as we shall, that the benefits of escapement increase with escapement, but at a decreasing rate, so that \( b'(S) > 0 \) but \( b''(S) < 0 \). Hence \( b'(S) \) is decreasing, so that \([p-b'(S)]qS \) is increasing in \( S \). For \( S \leq R \), we then have that \( \text{Max}[[p-b'(S)]qS] \) occurs at \( S = R \). Furthermore for given \( R \), \( c'(1/qT)\log[R/S] \) is decreasing in \( S \). Thus one of the equations \( d\pi/dS = 0 \) must have a solution \( S^* \), unless either:
(i) \([p-b'(S)]qR < c'(E_0) + \rho \) so that \( R < S_0 \), in which case the optimal \( S^* = R \).

\[-q_{TK}\]
\[
(ii) \quad [p-b'(Re)]qRe > c'(K) + \rho \quad \text{so that} \quad R > R_*(K) \quad \text{and the optimal escapement is} \quad S^* = Re^{\rho}.
\]

For \( K \leq E_0 \), we have:
\[
\pi(R,K) = \text{Max}[p(R-S) - q^{-1}\log[R/S] \cdot c(K) / K + b(S) - (\rho/q) \log[R/S] - (K - \rho T)K]
\]
over \( S \) in the interval \( Re^{-\rho} \leq S \leq R \). Setting the derivative of
this expression (with respect to $S$) equal to zero defines a desired escapement $S = e(K)$ satisfying the equation $[p-b'(S)]qS = c(K)/K + \rho$. Incorporating the constraints on $S$ and defining $R^{**}(K) = e(K)e^{qTK}$, we conclude that if $K < E_0$ the optimal escapement is:

$$S^* = \begin{cases} 
R \cdot e^{qTK} & ; R > R^{**}(K) \\
e(K) & ; \text{otherwise} \\
R & ; R < e(K)
\end{cases}$$

$-qTE_0$

Defining $R^* = S_0 e^{-qTK}$, we can now summarize the optimal escapement and the yearly rents function, for any values of $R$ and $K$, in the following table:

(a) $K \geq E_0$

<table>
<thead>
<tr>
<th>Recruitment</th>
<th>$S^*$</th>
<th>Yearly Rents $\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R &lt; S_0$</td>
<td>$R$</td>
<td>$\pi_1 = b(R)$</td>
</tr>
</tbody>
</table>
| $S_0 < R < R^*$ | $S_0$ | $\pi_2 = p(R-S_0)-q^{-1}\log(R/S_0)c(E_0)/E_0$  
|           |      | $- (\rho/q)\log(R/S_0)+b(S_0)$            |
| $R^* < R < R^{**}(K)$ | $s(R)$ | $\pi_3 = p(R-s(R))-Tc([1/qT]\log[R/s(R)])$ 
|           |      | $- (\rho/q)\log[R/s(R)] + b(s(R))$            |
| $R > R^{**}(K)$ | $R_0$ | $\pi_4 = pR(1-e^{-qTK})-Tc(K)-\rho TK + b(Re_{TK})$ |

(b) $K < E_0$

<table>
<thead>
<tr>
<th>Recruitment</th>
<th>$S^*$</th>
<th>Yearly Rents $\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R &lt; e(K)$</td>
<td>$R$</td>
<td>$\pi_5 = b(R)$</td>
</tr>
</tbody>
</table>
| $e(K) < R < R^{**}(K)$ | $e(K)$ | $\pi_6 = p(R-e(K))-q^{-1}\log[R/e(K)]c(K)/K$  
|           |      | $- (\rho/q)\log[R/e(K)]+b(e(K))$            |
| $R > R^{**}(K)$ | $R$  | $\pi_7 = pR(1-e^{-qTK})-Tc(K)-\rho TK + b(Re_{TK})$ |
Now our objective becomes: \( \max_{K \geq 0} [\pi(R,K) - (\kappa - \rho T)K] \). In other words, we wish to maximize the yearly fishery rents net of capital (depreciation and 'rental') costs.

Consider first the range of values \( K \geq E_0 \). Note that, since \([p-b'(S)]qS\) is increasing in \( S \) and \( c'(K) \) is increasing in \( K \), differentiating equation (4) shows that \((d/dK)R_*(K) - qTR_*,>0\), so that \((d/dK)R_*(K) > qTR_*,>0\). Hence \( R_* \) is monotone increasing and we can define an inverse function \( k(R) \) such that \( K = k(R) \) iff \( R = R_*(K) \), and \( R < R_*(K) \) iff \( K > k(R) \). Since \( s(R_*(K)) = R_*(K)e^{-qTK} \), we have that \( s(R) = Re^{-qTK} \) for any \( R \geq S_o e^{-qTK} \). Now the optimal escapement \( s(R) \) can be attained with \( K = k(R) \), so we conclude that if \( R > S_o e^{-qTK} \), it will never be optimal to have \( K > k(R) \). Optimizing over the range \( K \leq k(R) \) \([R \geq R_*(K)]\), with \( R > S_o e^{-qTK} \), we have:

\[ -qTK \quad [p-b'(Re^{qTK})]qRe^{qTK} - Tc'(K) - \rho T = \kappa - \rho T \]

Simplifying this expression produces:

\[ -qTK \quad -qTK \]

\[ -qTK \quad [p-b'(Re^{qTK})]qRe^{qTK} = c'(K) + \kappa/T \]

Now we have assumed that \( \kappa > \rho T \), so that, for given \( K \), the right hand side in this equation is greater than that of equation (4). Since \( K = k(R) \) solves (4) and the optimal capacity \( K*(R) \) solves (5), it follows that we must have the optimal capacity \( K* < k(R) \).

Recall that we assumed \( K \geq E_0 \). If in fact

\[ -qTE_0 \quad -qTE_0 \]

\[ [p-b'(Re^{qTE_0})]qRe^{qTE_0} < c'(E_0) + \kappa/T \] then (5) cannot be solved for \( K* \geq E_0 \), and so the optimal capacity must be \( K* < E_0 \).
If $S_0 \leq R \leq S_0 e^{q_{TE_0}}$, $\pi(R,K)$ is independent of $K$, so that the optimal capacity amongst values $K \geq E_0$ is $\text{Min}\{K : K \geq E_0\} = E_0$. If $R \leq S_0$, the same result holds; $K^* = E_0$. However in both cases this is merely the optimum for $K \geq E_0$; a value $K^* < E_0$ will be preferable if $\frac{d\pi}{dK}(R,E_0) < K - \mu T$.

We now consider the possibility $K^* < E_0$. Unfortunately analysis in this case is complicated by the ambiguity resulting from the lack of concavity in the rents function; for example the sign of the derivative of $e(K)e^{q_{TK}}$ is ambiguous. Hence we shall content ourselves with partial results. First, note that since $c(E_0)/E_0 = c'(E_0)$, we must have $e(E_0) = S_0$, since both sides of this expression satisfy the same equation. Suppose that $R > S_0 e^{q_{TE_0}}$ and the solution $K^*$ of (5) is such that $K^* < E_0$ and $R e^{q_{TK}}(K^*) = R^{**}(K)$ (this will certainly be the case if $K^*$ is only slightly less than $E_0$). Then $K^*$ is the optimal capacity.

If, however, $R > e(K^*)e^{q_{TK}}$, then $K^*$ lies outside the range for which equation (5) applies, and hence the optimum must instead lie in the interval $[0,K']$, where $K'$ satisfies $e(K')e^{q_{TK'}} = R$. In addition, note that for $K < E_0$, $c(K)/K$ decreases with $K$, so that $e(K)$ must also decrease with $K$. Hence if $R \leq S_0$, $R < e(K)$ for all $K < E_0$, and so $K^* = 0$ in this case.

To summarize, the optimal capacity $K^*$ is given by:

- $K^* = \text{solution of (5)}$; if either (i) $K^* \geq E_0$ and $R \geq S_0 e^{q_{TK^*}}$
- or (ii) $K^* < E_0$ and $R \geq e(K^*)e^{q_{TK^*}}$
$K^* = 0$ ; if $R \leq S_0$

$0 \leq K^* \leq E_0$ ; otherwise

If we assume linear costs, $c(E) = cE$, then we can let $E_0 = 0$ and the solution simplifies to the following:

$$-qTK -qTK$$

$K^* = \text{solution of } [p-b'(Re)]qRe = c + \kappa/T$ or

$K^* = 0$ if $[p-b'(R)]qR < c + \kappa/T$

If in addition we ignore benefits of escapement,, so that $b(S) = 0$ for all $S$, we have:

$K^* = [qT]^{-1} \log[R/X_0]$ where $X_0 = (c + \kappa/T)/pq$

This simply states that sufficient capacity should be in place to harvest the biomass $R$ down to bionomic equilibrium $x^*$, where both fishing costs and time-scaled capital costs are included.

In cases where recruitment is thought to be approximately constant from year to year, the results presented in this section can be used to provide an estimate of the optimal capacity level. Even for fisheries with more complicated stock-recruitment relationships, these results provide a simple upper bound on the optimal fleet size (assuming the population dynamics parameters are known and stochastic variability is sufficiently low). In fisheries with more extensive randomness, a stochastic analysis is necessary - in Chapter III.B we consider such a stochastic analogue.
C. Heuristic Analysis

We now return to the general model, and in particular the general stock-recruitment function \( F(S) \), introduced in Section A. Our objective in this section is to gain qualitative information about the full optimal investment and escapement problem from an heuristic study of the dynamic programming equation (1). Let us assume for now that the fish stock displays pure compensatory population dynamics (as in the Beverton-Holt model) and that \( V > 0, \ V > 0, \ V > 0, \ V > 0 \) over all non-zero \( R, K, RK, KK \) values of \( R \) and \( K \) for which \( V \) is twice differentiable. The latter assumption simply states that more fish and more capital increase the value of the fishery, that more fish are more desirable the larger the capital stock, and that the fishery has decreasing marginal returns to capital. Although it is reasonable to make such an assumption about the function \( V \), for this to be useful we must know something about the existence of the partial derivatives.

Recall that equation (1) defines \( V \) implicitly by two maximizations, over escapement and investment. We shall see below that the resulting optimal escapement and investment functions, \( S^*(R,K) \) and \( I^*(R,K) \), are smoothly differentiable except along 3 curves in the biomass-capacity plane. Hence the value function \( V(R,K) \) will be at least \( C^2 \) in both variables, almost everywhere. In fact, we know more; since \( V \) and \( V \) \( R \) \( K \) represent the marginal economic benefits of extra biomass and extra capacity, respectively, it is clear that both quantities
exist, and it is reasonable to expect continuity as well. In any case, the important point is that in general the partial derivatives are evaluated at points \((F(S), h(S))\) lying away from the 3 curves on which \(S^*\) and \(I^*\) are non-differentiable (since \(F(S)\neq S\) in general). Thus questions of non-differentiability pose no serious difficulties.

Now performing the inner maximization in (1), for fixed \(S\), we obtain the optimality equation for investment:

\[
V_{(F(S),(1-\gamma)K+I)} = \delta/\sigma \quad \text{or} \quad I = 0 \text{ if } V_{(F(S),(1-\gamma)K)} < \delta/\sigma
\]

This states that, unless the fleet is temporarily overcapitalized, next year's optimal capacity, \((1-\gamma)K + I\), should be set such that the marginal benefit of an extra unit of capital equals its marginal cost.

Define \(K = h(S)\) to be the solution of the implicit equation

\[
V_{(F(S),K)} = \delta/\sigma
\]

so that \(h(S)\) is next season's optimal capacity, and \(V_{(F(S),h(S))} = \delta/\sigma\). Performing the total derivative of this latter equation produces:

\[
F'(S)V_{(F(S),h(S))} + h'(S)V_{(F(S),h(S))} = 0
\]

Rearranging this expression, we find that:

\[
h'(S) = \frac{F'(S) \cdot [V_{(F(S),h(S))}]/[-V_{(F(S),h(S))}]}{\delta/\sigma} > 0
\]

by our above assumptions. Hence the optimal capacity \(h(S)\) is an increasing function of escapement. While the concavity of \(h(S)\) cannot be deduced from the expression for \(h'(S)\), it can be expected since compensatory recruitment produces decreasing marginal returns to escapement.
Thus if \((1-\gamma)K > h(S)\), the optimal investment is \(I^* = 0\) (capital is already sufficiently abundant), while otherwise \(I^*\) is chosen so that \((1-\gamma)K + I^* = h(S)\). This can be written:

\[(7) \quad I^*(S,K) = \text{Max}\{h(S) - (1-\gamma)K, 0\}.
\]

Now inserting \(I^*(S,K)\) into (1), performing the outer maximization by taking a total derivative with respect to \(S\), and noting that for any \(S\) and \(K\), either \(I^*(S,K) = 0\) or \(aV(F(S),(1-\alpha)K + I^*(S,K)) - \delta = 0\), we obtain the optimality expression equating the marginal benefit and marginal cost of an incremental increase in escapement:

\[(8) \quad aF'(S)V(F(S),(1-\alpha)K + I^*(S,K)) R \]

\[= \begin{cases} 
  p(1-x_0^1/S) & \text{if } K < E^* \\
  p(1-x_0^2/S) & \text{if } K > E^*; \quad \text{Re} \leq S \leq R \\
  P(1-x_0^3(R,S)/S) & \text{if } K > E^*; \quad \text{Re} < S \leq \text{Re} 
\end{cases}
\]

where we define: \(x_0^1 = [c(K)/K + \rho]/pq\), \(x_0^2 = [c(E^*)/E^* + \rho]/pq\), \(x_0^3(R,S) = [c'([1/qT]\log(R/S) + \rho)]/pq\), and the constraint \(\text{Re} \leq S \leq R\) has been neglected temporarily.

There is no conceptual difficulty in using this optimality equation to derive an optimal escapement policy, which in general will depend on both current escapement and current capacity; \(S = s(R,K)\). However, note that if costs were linear in effort, \(c(E) = c \cdot E\), then we would have \(x_0^1 = x_0^2 = x_0^3\) and in particular \(x_0^3\) would be independent of \(R\), since in the linear cost case
c(E)/E=c'(E)=c for all values of E. This assumption would allow us to consider an escapement target curve S=s(K) rather than a surface s(R,K). Hence for descriptive simplicity and to compare our results to those of Clark, Clarke and Munro (where a similar assumption was used), we shall set c(E)=cE, and denote by x_0=c/pq the common quantities x_0=x_2^*=x_3^*, for the remainder of this work (apart from III.B).

We shall assume in addition that S=s(K) is the unique solution of (8). Consider the equation:

\[ \alpha F'(S)V(F(S),h(S)) = p(1-x_0/S) \]

If (9) has a solution (assumed unique), denote it by S^-. Then s(K)=S^- is the optimal escapement for all K <(1/[1-\gamma])h(S^-), since we have (1-\gamma)K +I*(S,K) = h(S) for all S near S^-, but amongst these values of S, only the value S^- satisfies the optimality equation (9).

Denote by S^+ the solution of:

\[ \alpha F'(S)V(F(S),\inf) = p(1-x_0/S) \]

This S^+ is simply the optimal equilibrium escapement for the discrete-time abundant-capital problem (c.f. \( \chi \) in CCM). To see this, define the function \( v(R) = V(R,\inf) \) and note that for K very large and \( R \geq s(\inf) \), we have:

\[ v(R+\delta R) = v(R+\delta R,\inf,s(\inf),0) + \alpha V[F[s(\inf)]] \]

\[ = p[R-s(\inf)] - (c/q)\log(R/s(\inf)) + p\delta R \]

\[ - (c/q)\log((R+\delta R)/R) + \alpha V[F[s(\inf)]] \]

\[ = v(R) + p\delta R - (c/q)(\delta R/R) \]

Therefore \( v(\inf) = p - c/qR = p(1-x_0/R) \) if \( R \geq s(\inf) \).
In fact, we wish to derive an expression for \( V(F(S), \inf) \), for \( S \) near \( S^* \). Note first that if \( S^0 \) is the unexploited equilibrium biomass, given by \( F(S^0) = S^0 \), and assuming that \( S^0 > x_0 \), then an optimal value of \( S^* \) must be such that \( S^* \leq S^0 \). [Otherwise the harvestable, and profitable, biomass between \( S^* \) and \( S^0 \) would be wasted.] Hence we must have \( F(S^*) \geq S^* = s(\inf) \), so we can in fact write: \( v(F(S^*)) = p(1-x_0/F(S^*)) \). Then equation (9) can be written: \( \alpha F'(S)(1-x_0/F(S)) = p(1-x_0/S) \) for \( S \) near \( S^* \) and hence \( S^- \) is given as the solution of the Modified Golden Rule equation:
\[
F'(S) \cdot \left[ \frac{(1-x_0/F(S))}{(1-x_0/S)} \right] = \frac{1}{\alpha}.
\]

Note that our assumption \( V(\cdot, \cdot) > 0 \) implies
\[
V(F(S), \inf) > V(F(S), h(S)) \text{ for any } S. \text{ Also, } (p/\alpha F'(S))(1-x_0/S) \text{ is an increasing function of } S. \text{ Hence we can conclude that } S^- < S^*.
\]

We wish now to consider the behaviour of the target escapement curve \( s(K) \). For \((1-\gamma)K < h(S^-)\), we have \( s(K) = S^- \) so that \( ds/dK = 0 \). For \((1-\gamma)K > h(S^-)\), \( S = s(K) \) is defined by:
\[
(11) \quad V(F(S), (-\gamma)K) = (p/\alpha)F'(S)(1-x_0/S).
\]

Differentiating this equation with respect to \( K \) and rearranging terms we obtain:
\[
(12) \quad s'(K) = (1-\gamma)F'(S) \frac{V(F(S), (-\gamma)K)}{R} \cdot \left\{ (p/\alpha)[x_0/S^2 - (F''(S)/F'(S))(1-x_0/S)] \right\}^{-1} - F'(S)^2 V(F(S), (1-\gamma)K)
\]

\[ \text{RR} \]
With the assumptions to date, the sign of $s'(K)$ is ambiguous. However we will be assured that $s'(K) > 0$ if $V(R,K)$ is either concave or not too convex in $R$, so that $V(\cdot, \cdot)$ is not too large. This would seem to be a likely situation, although it proves to be faulty to assume that $V < 0$ in all cases, as we shall see in section E.

The value of $s(K)$ is the target escapement given capacity $-qTK$. Due to the constraint $R \leq S \leq R$, this target may not be attainable. Hence the optimal escapement $S^* = S^*(R,K)$ must be defined as follows:

$$S^*(R,K) = \begin{cases} R \cdot \exp\{-qTK\} ; & R > s(K) \exp\{qTK\} \\ s(K) ; & R \text{ intermediate} \\ R ; & R < s(K) \end{cases}$$

(13)

We have now heuristically derived a synthesis of the optimal harvesting/investment policy, in the form of the two policy functions $s(K)$ and $h(S)$, giving the optimal action $(S^*, I^*)$ as a function of the state $(R,K)$. The optimal capacity $h(S)$ is likely to be zero below some minimum escapement $S > 0$ and concave increasing for $S > S^m$. The optimal escapement is expected to be non-decreasing as a function of capacity, approaching an asymptote $S^*$ as $K$ tends to infinity.

At the beginning of a season, given recruitment $R$ and capacity $K$, the fish stock is first harvested down to $S^*(R,K)$. Then depreciation and investment occur, such that if the depreciated capacity is $(1-\gamma)K < h(S^*)$, investment brings the capacity to $h(S^*)$ by the start of the next season. The process
is then repeated from the new point 
\((F(S^*(R,K), \text{Max}[(1-\gamma)K, h(S^*(R,K))])\). The resulting trajectory
eventually converges on a long run equilibrium point, discussed
in section E (see figures 1 and 2).

If instead of experiencing delays in bringing new
investment on-line, we allow instantaneous investment, and
maintain the assumption of linear costs, equation (1) must be
changed to become:

\[
(14) \quad V(R,K) = \max_{I \geq 0} \{ \max_{S} \{ p(R-S)-[(c+\rho)/q] \log(R/S)+\rho TK-6I \}
+ aV(F(S),(1-\gamma)(K+I)) \}
- qT(K+I)
\]

where the inner maximization is over the range \( R \leq S \leq R \).
Performing this inner maximization we obtain:

\[
(15) \quad aF'(S)V(F(S),(1-\gamma)(K+I)) = p(1-(x_0)/S)
\]

unless the constraint on \( S \) is binding.

Now (14) can be rewritten:

\[
(14') \quad V(R,K) = \max_{I \geq 0} \{ \max_{J \geq 0} \{ p(R,S,J)+aV(F(S),(1-\gamma)(K+I+J))-6I \} 
= \max_{I \geq 0} \{ V(R,K+I)-6I \}
\]

where now the maximization with respect to \( S \) is restricted to

\( R \leq S \leq R \). The optimal investment is then obtained by
maximizing the right hand side to obtain:

\[
(16) \quad V(R,K+I) = 6 \text{ or } I = 0 \text{ if } V(R,K) < 6
\]

Note that (16) can be written \( V(F(S),K^*) = 6 \), giving this
year's optimal capacity \( K^* \) implicitly as a function of the
previous season's escapement. Clearly (16) and (6) are equivalent, apart from the reduced unit capital cost in the instantaneous case (where investments do not have to be paid for in advance). Likewise, (15) and (8) are equivalent, giving the optimal escapement as a function of this year's optimal capacity, \( K+1 \). We can conclude therefore that the delayed investment case with unit capital cost \( \delta \) and the instantaneous investment case with unit capital cost \( \delta_0 \) should produce comparable results, as expected for a deterministic model.

**D. Numerical Method**

While the heuristic analysis of the previous section provided considerable information about optimal investment and escapement policies, numerical methods are required to produce quantitative results. We therefore turn now to a numerical solution of the dynamic programming equation (1). The two primary techniques for obtaining solutions in dynamic programming are policy iteration and value iteration, both of which are iterative procedures which, under appropriate conditions, converge to the optimal policy and value functions. While value iteration is useful for solving finite time horizon optimization problems, the policy iteration approach is more helpful for the infinite time case considered here, and has the advantage of concentrating explicitly on the objects of interest, namely the management policies. Procedures have been developed for speeding the convergence of policy iteration (Puterman and Shin, 1978), but in our case, reasonably rapid
convergence is obtained with simple policy iteration based on equations (6) and (8) in the previous section. The procedure used is as follows:

First, a set of uniformly-spaced mesh points was chosen for each dimension of the positive quadrant of the $R-K$ plane. The upper and lower limits for biomass and capacity were chosen arbitrarily but so as to include the $R-K$ region of interest: the lower capacity limit was $K_i = 0$ in all cases. Variation of these limits did not substantially affect the optimal policy or value functions. The effect of the number of mesh points used is discussed later in this section. An initial guess is made for the policy functions $s(K)$ and $h(S)$ and the value function corresponding to these policies is determined for each mesh point by solving a set of simultaneous equations of the form:

$$(17) \quad V(R_{i,j}, K_{i,j}) = \pi(R_{i,j}, K_{i,j}) + \alpha V(F(S^*(R_{i,j}, K_{i,j})), K'(R_{i,j}, K_{i,j}))$$

$$= \pi(R_{i,j}, K_{i,j}) + \alpha \delta_1 V(R_{i,j}, K_{i,j}) + \delta_2 V(R_{i,j}, K_{i,j})$$

$$+ \delta_3 V(R_{i,j}, K_{i,j}) + \delta_4 V(R_{i,j}, K_{i,j})$$

$$= \pi(R_{i,j}, K_{i,j}) + \alpha \delta_1 V(R_{i,j}, K_{i,j}) + \delta_2 V(R_{i,j}, K_{i,j})$$

where $R = \text{Max}\{R \geq F(S^*(R, K)), k=1, \ldots, M\}$, $K'(R, K)$ is next year's capacity, $K = \text{Max}\{K \leq K'(\cdot, \cdot): k=1, \ldots, N\}$ and the $\delta$ are determined by linear interpolation ($\delta \geq 0$ and $\sum_{i=1}^{4} \delta_i = 1$). The system (17) contains $MN$ equations in the $MN$ unknowns $V(R_{i,j}, K_{i,j})$.

In vector-matrix form, the system can be rewritten:

$$(18) \quad V = \pi + \alpha QV \quad \text{or}$$
(19) \((I - \alpha Q)V = \gamma\), where \(I\) is the identity matrix and \(Q\) is the appropriate transition matrix.

For \(\alpha < 1\), the matrix \(A = I - \alpha Q\) is diagonally dominant and hence (19) has a unique solution. The system (19) may be solved in a straightforward manner by Gaussian elimination, but for MN relatively large, two alternative procedures seem to be superior.

Firstly, if the depreciation rate \(\gamma = 0\), the capital stock cannot decrease from one year to the next, so \(K'(R,K) \geq K\) and hence \(m \geq j\) in (17). If we can assume that the maximum capacity value, \(K^N\), is sufficiently large that no investment occurs given this capacity then the equations in (17) can be divided into subsystems (with \(K\) fixed within each subsystem) and solved \(j\) consecutively, beginning with the \(M\) equations having \(K = K^N\), \(j\) followed by the equations for which \(j = N - 1\) and so on. If, for particular values of \(i\) and \(j\), we have \(m \geq j\) the equation (17) can be solved directly for \(V(R,K)\) using previously determined \(i\) \(j\) values on the right hand side. Furthermore if \(m = j\) the values of \(V(\cdot,K)\) can be immediately inserted into (17), having been \(m+1\) determined in the previous step. Thus the problem is reduced from solving one MN by MN system to solving \(N\) systems of maximum size \(M\) by \(M\).

A second approach is to notice that, for any values of \(i\) and \(j\), at most 5 of the \(MN\) entries in row \((ij)\) are non-zero.
Hence iterative sparse matrix methods are highly suitable for this problem. The diagonally dominant nature of the matrix \( A = I - \alpha Q \) enabled the use of a sparse symmetric over-relaxation (SSSOR) package written in /370 Assembler and available on the University of British Columbia computer system (Conrad & Wallach, 1977). The scheme can reduce the number of calculations per iteration below that required by standard SOR methods. As with other over-relaxation techniques, one must choose a relaxation parameter \( \omega (1.0 \leq \omega < 2.0) \), with \( \omega = 1.0 \) corresponding to the basic Gauss-Seidel method. The rate of convergence to the value function did not appear sensitive to the choice of \( \omega \): any value \( 1.0 \leq \omega \leq 1.2 \) seemed suitable. In our numerical work, the SSSOR method proved quite efficient, being as inexpensive in the \( \gamma = 0 \) case as the above step-by-step procedure.

Once the value function at the mesh points has been determined, a \( C^2 \)-cubic spline approximation method is used to fit a smooth \( V(R,K) \) surface and hence to obtain the first partial derivatives \( V \) and \( V \) at the mesh points. Tension was not introduced into the spline approximation so that in theory inaccuracies in the solution for the \( V(R,K) \) value could lead to unwanted curvature in the surface \( V(R,K) \). However, a fairly fine mesh \( (M=30, N=30) \) was used and no unreasonable behaviour of the \( V \) and \( V \) values could be detected.

The values of \( V \) and \( V \) are then used to solve equations (6) and (8), again resorting to linear interpolation where
necessary. The solutions are two new (and improved) policy functions \( h(S) \) and \( s(K) \) respectively. These are then used to derive a new value function, and the process is repeated until the policy functions become stationary. Specifically, for a prescribed fraction \( f, 0 < f \leq 1 \), and mesh sizes \( \Delta R, \Delta K \), stationarity of the policies is considered to have been achieved when:

\[
\begin{align*}
\max_{i \text{ new}} |h(i) - h(i)| &< f \cdot \Delta R \\
\max_{j \text{ new}} |s(j) - s(j)| &< f \cdot \Delta K
\end{align*}
\]

(20)

In most cases, depending on the initial guesses for the policy functions, no more than 5 iterations were necessary to obtain convergence, indicating that the numerical scheme is well-behaved.

While an examination of the Modified Golden Rule equation for optimal escapement in the case of Beverton-Holt dynamics [see equation (22) in the following section] shows that the optimal escapement at high capacities, \( S^* \), is unique, the heuristic analysis of section C does not rule out multiple solutions for the value of \( S^- = s(0) \). To check for such a possibility in our base case prawn fishery, the value of \( S^- \) was varied, while keeping \( S \) constant at its optimal value, and \( s(K) \) was formed by an appropriate continuous connection between \( s(0) = S^- \) and \( s(\text{inf}) = S^* \). For each \( S^- \), the value function was calculated (using the optimal capacity function). This value function reached its maximum when \( S^- \) was set at its optimal value, \( S^-* \), although it changed very little with variations in \( S^- \) near \( S^-* \). It therefore seems reasonable to assume that our
numerical scheme is determining a unique optimal escapement curve \( s(K) \).

The above numerical method applies to the case of delayed investment, but the procedure with instantaneous investment is very similar. The only difference arises in solving for the new \( h(R) \) function. In the instantaneous case, for any \( K_1 < K_2 \leq h(R) \), we have \( V(R,K_2) = V(R,K_1) + \delta \cdot (K_2 - K_1) \). Hence (16) is satisfied for any \( K+1 < h(R) = \text{Max}\{K+1 : V(R,K+1) = \delta\} \). Determining \( h(R) \) requires finding the value of \( K \) at which \( V \) begins to decrease from the \( K \) level \( \delta \). However values of \( V \) are only obtained at mesh points, \( K \) and \( h(R) \) cannot be determined by interpolation in this case, so an arbitrary linear extrapolation technique is used, based on the first two mesh points having \( V < \delta \). Specifically, if

\[
V(R,K_j) = \delta \quad \text{and} \quad V(R,K_{j+1}) < \delta , \quad \text{then}
\]

\[
h(R) = K_j \cdot \left[ \frac{(\delta - V_{j+1})}{(V_{j+1} - V_{j+1})} \right] \cdot (K_j - K_{j+1}),
\]

where \( V_i = V(R,K_i) \), unless this lies outside the interval \([K_j, K_{j+1}]\) in which case \( h(R) \) is set equal to the appropriate endpoint.

For the two fisheries considered here, this method performed well in one (the whale fishery -see section E) but proved ill-behaved in the other (the base case prawn fishery). In the latter case, it was not possible to achieve an accurate solution in a reasonable number of iterations. It is not clear, however, whether the numerical scheme is non-convergent or
merely subject to very slow convergence. As we are only interested in the instantaneous investment case as a bridge between the CCM results and our discrete-time delayed investment results, we have not investigated in depth the cause of these numerical difficulties. Instead we note that each policy iteration step produced $h(R)$ and $s(K)$ curves which agreed qualitatively with the heuristic analysis of section C. Furthermore, the best results we were able to obtain for the prawn fishery, when averaged over a number of policy iterations, agree closely with the transformation of the corresponding delayed investment curves (see section C for a discussion of this transformation). Hence we have reported these averaged results as an indicator of the optimal instantaneous investment policies for the prawn fishery, and in particular, the true optimal capacity curve in figure 1 should be considered to lie somewhere in the range $h(R) \pm 2$.

In all the results obtained in this paper, the number of mesh points for biomass and capacity were set at $M=30$ and $N=30$ respectively. The linear interpolation method used in our numerical scheme can lead to inaccuracies which decrease as the mesh becomes finer. To check the adequacy of the meshes used in our analysis, results for our base case fishery were also obtained with a finer mesh, $M=N=60$, and a coarser one, $M=N=10$. With the finer mesh, no change in the value function could be seen, and the optimal capacity function changed by less than 1 percent. The maximum change in the escapement function $s(K)$ was 4 percent but in most cases it changed by 1-1.5 percent. The
coarse mesh performed surprisingly well, with reductions in the value function of approximately 1 percent compared with our normal mesh. The function \( h(S) \) differed by up to 4 percent while changes of up to 13 percent occurred in the values of \( s(K) \). While it appears that a coarse mesh is useful to obtain good approximate results, the accuracy of our \( M=N=30 \) mesh makes it of more use in studying policy implication and parameter sensitivity. The \( M=N=60 \) case proved too expensive for the small increase in accuracy it affords.

E. Numerical Results.

We are now in a position to provide a full analysis of the deterministic investment and escapement model embodied in equation (1). Using the procedure of section D, numerical results have been obtained for each of two fisheries, namely (i) the Australian Gulf of Carpenteria banana prawn fishery (Clark & Kirkwood, 1979) and (ii) the aggregated pelagic whaling fishery (Clark & Lamberson, 1980).

The purpose of our numerical study was not to derive detailed solutions for these fisheries in particular but rather to gain understanding of the optimal investment problem. To this end, the received data was substantially simplified. In the case of the prawn fishery, only one type of vessel was considered, no alternative fishery was allowed \( (\rho = 0) \), an average prawn weight was used in lieu of intraseasonal growth, and natural mortality was constrained to occur at the end of each season.

The stock-recruitment functions \( F(S) \) were given by
R = \frac{aS}{1+as/b} \quad \text{and} \quad R = aS\exp\{-\left(\frac{a}{eb}\right)S\}

for the Beverton-Holt and Ricker cases respectively, where S is the escapement after fishing has taken place. The maximum possible recruitment, b, for the prawn fishery was set equal to the sample mean of recruitment data from Kirkwood (1980). The rate of growth, or fecundity, of the prawn stock (a) was set arbitrarily. The whaling data used by Clark & Lamberson was converted from continuous to discrete-time and, as in the continuous-time case, delays in recruitment were neglected (cf. Clark, 1976b).

The data used for each fishery are presented in Table (I), where the values given for a and b correspond to the Beverton-Holt stock-recruitment function. If S is the escapement after fishing, \(S_{-mT}\) is taken to be the end-of-year escapement after both fishing and natural mortality. An examination of the stock-recruitment functions indicates that the factor \(e^{-mT}\) can be directly incorporated by changing the value of a given in \(e^{-mT}\) Table (I) to \(ae^{-mT}\); this is done in our analysis.

It will be of interest to compare our optimal policies with the open-access scenario resulting from uncontrolled fisheries development. If we assume that in the open-access case, investment continues until the average net revenue (per unit capacity) just covers the unit capital cost, then in equilibrium we have:

\(\frac{a/[1-a]}{\pi(R,K,S,I)/K} = 6\)

where the left hand side represents the total present value of
discounted rents, per unit of capital. Setting $I = \gamma K$ to hold the capital stock constant in equilibrium, and assuming full utilization of the fleet, this can be written:

$$\left(\sigma/[1-\sigma]\right) \cdot \left[p(R-S)/K - cT - \gamma\delta\right] = \delta$$

$qTK$

with $R = F(S)/S e^{-qTK}$. This simplifies to:

$qTK$

$$(e^{-qTK})(S/K) = \left(\left[\left(1-\sigma\right)/\sigma + \gamma\delta + cT\right]/p \right) = RHS$$

Solving this equation simultaneously with $F(S)/S = e^{qTK}$ produces the open-access equilibrium capital stock and biomass.

If Beverton-Holt stock recruitment is assumed, so that $F(S) = aS/[1+(aS/b)]$, this solution can be simplified. In equilibrium we have $F(S)/S = a/[1+(aS/b)] = e^{qTK}$, so that

$$S = \left(b/a\right)(ae^{-qTK} - 1)$$

Hence the open-access capacity can be restated as the solution of the equation:

$$\left(21\right) \quad \left(b/a\right)(ae^{-qTK} - 1)(e^{-1})/K = RHS$$

where RHS is as above. This equation can be solved iteratively, and is applied in part (b) below.

Results in the form of feedback control diagrams are given for the above prawn fishery data in figures 1 and 3, corresponding to the cases of instantaneous and delayed investment respectively. Similar results are shown in figures 2 and 4 for the whale fishery. [As discussed in section D, the instantaneous investment results for the prawn fishery are only approximate, due to numerical difficulties. However this does not affect the qualitative discussion below.]
(a) The Instantaneous Investment Case

Interpretation of these results is facilitated by comparing them with those obtained by CCM, who assumed that investment occurs instantaneously. Let us first concentrate on our figures 1 and 2, which are precisely the discrete-time analogues of the CCM results, in particular their figure 2. Our \( s(K) \) and \( h(S) \) curves correspond closely with their switching curves, \( \sigma_1 \) and \( \sigma_2 \), respectively. To the left of the \( s(K) \) curve, no harvesting should take place, while above and to the left of the \( h(R) \) curve, no investment should be undertaken. If \( K < h(R) \), immediate investment should occur until \( K = h(R) \). To the right of the \( s(K) \) curve, harvesting should take place until the stock is reduced to \( s(K) \), or as close as possible to that escapement, given the level of capacity available.

A sampling of possible trajectories is shown in each figure. Note that all trajectories eventually converge on a single long-run equilibrium point, \( (R,K) \), given in terms of recruitment and capacity (after depreciation and reinvestment). The equilibrium point corresponds to the point \( (x^*,K^*) \) in CCM and represents the optimal equilibrium in the case where capital is totally malleable but not "abundant", so that the rental cost of capital must be included in variable costs. This is discussed in more detail in Appendix A. Unlike the CCM case, this equilibrium point is not apparent from the synthesis diagram. In the continuous-time case, when a trajectory crosses the line \( x = x^* \) below \( (x^*,K^*) \), the optimal policy is an instantaneous investment to the capacity level \( K = K^* \), thereafter remaining at
\((x^*,K^*)\). In discrete time, however, almost all trajectories approach and/or cross \(R= R\) , rather than touch that line, so the use of a single final impulse control at \(R\) is not sufficient to reach the equilibrium point.

Furthermore, whereas equilibrium is reached in finite time with the continuous-time CCM model, in our discrete-time situation the approach to equilibrium is asymptotic. This can be seen by setting \(r = R - R\), \(k = K - K\), linearizing about the equilibrium and assuming that \(k << (\gamma/[1-\gamma])K\). We obtain the system 

\[
\begin{align*}
\dot{r} &= a_1 r - a_1 b_1 r = (a_1 - a_2 b_1) r \\
\end{align*}
\]

where

\[
\begin{align*}
a_1 &= F'(R \cdot \exp\{-qTK\}) \cdot \exp\{-qTK\} , \quad a_2 = qTR \cdot \exp\{-qTK\} \cdot F'(R \cdot \exp\{-qTK\}) \\
b_1 &= h'(R).\end{align*}
\]

and hence \((r,k)\) tends asymptotically to \((0,0)\) as long as \(a_1 - a_2 b_1 = \exp\{-qTK\} F'(R \cdot \exp\{-qTK\} \cdot [1-qTRh'(R)]) < 0\), that is \(h'(R) > 1/qTR\). This asymptotic convergence can be seen in the sample trajectories: indeed, equilibrium points were verified by determining the common point of convergence of a number of sample trajectories.

Apart from the differences mentioned above, the behaviour of our instantaneous investment model and that of CCM are quite similar, reflecting the pure compensatory nature of both the Beverton-Holt function and the continuous-time compensatory function used by CCM.
(b) **Delays in Investment**

As discussed in section C, the introduction of a delay between the time an investment decision is made and the time the corresponding new capacity becomes available produces little change in the desired escapement and capacity in any given year, except in as much as payment for the new capacity must be made earlier than would be the case for instantaneous investment, and hence effective capital costs are higher. However the appearance of the optimal policies can be substantially different, with optimal capacity given as a function of escapement rather than recruitment.

Figures 3 and 4 depict the optimal policies for the prawn and whale fisheries respectively, with delayed investment, but otherwise unchanged parameters. The policy curves for the prawn fishery were derived using both our usual mesh of M=N=30 and a finer mesh M=N=60. As discussed in section D, the differences are minimal; changes which occur when the finer mesh is used are indicated in figure 3.

The introduction of delays in investment does not change the s(K) curves, reflecting the common optimality equations for escapement, derived in section C. Little qualitative change in the h(S) curve is noticeable between the instantaneous and delayed investment cases for a fishery based on a slow-growing stock (whales). However, in the case of fast-growing prawns stocks, a low escapement this year can still produce a large recruitment next year. With delayed investment, it may be optimal to plan and pay for investment this year, even though
stocks seem low, since the new capacity will not be available until next year, when a higher capacity can be used to harvest the much larger stock. This can lead to the situation shown in figure 3, where positive investment can be optimal even in some situations where no harvesting takes place.

In situations where the optimal escapement and optimal capacity policy functions intersect, the optimal escapement \( s(K) \) must be constant, i.e. a vertical line, below the intersection point, as indicated in figure 3. Mathematically this can be seen from the derivation of the value \( S^- \) in section C. Intuitively, the rationale for this is as follows. If, for a given escapement \( S \), the current capacity is relatively low, investment will take place up to the capacity level \( h(S) \), a level dependent solely on the escapement. Next year's recruitment, \( F(S) \), is also dependent on the escapement. Hence current capacity is irrelevant to the determination of optimal escapement, which is therefore independent of \( K \); that is, \( s(K) = \text{constant} \).

Apart from the differences in appearance and interpretation of the optimal policies described above, the cases of instantaneous and delayed investment are quite similar. In particular, trajectories approach a long run equilibrium, derived in a manner similar to that of the instantaneous investment case—see Appendix A. These long run equilibrium points are shown in figures 3 and 4; comparing with the corresponding open-access results obtained using equation (21), we find that for the prawn fishery, the open-access capacity, 16.8 standardized vessels, is roughly double the optimal level.
In the whale fishery, however, equilibrium biomass is very sensitive to the capital stock. Hence the open-access and optimal capacities cannot differ by much in this case; in fact they are almost identical, at 2505 and 2250 catcher days/year respectively. The indication of these results is that the extent by which open-access conditions leads to over-investment can vary considerably. Of course, the actual open-access investment behaviour may be quite complicated, so that our model only approximates the true situation.

For the remainder of this paper we shall concentrate on the delayed investment model, turning now to results relating to the comparative dynamics for several economic and ecological parameters.

(c) Fecundity and Carrying Capacity of the Stock

Using the Beverton-Holt stock-recruitment function

\[ R = F(S) = ae^{-mT} S/(1 + ae^{-mT} \frac{S}{b}) \]

a measure of the maximum fecundity \( F'(0) = ae^{-mT} \). As a increases, holding the maximum recruitment \( b \) constant, recruitment becomes less and less dependent on escapement. One would expect that the higher is the fecundity, the better off is the fishery and hence the higher is the optimal capacity. This is confirmed for the prawn fishery in figure 5, where optimal policy functions are shown for each of \( a = 3.5, 14, 42 \) and 560, with \( b = 7.0 \times 10^6 \) fixed. With \( a = 3.5, \) we have \( ae^{-mT} = 0.95 < 1 \), so the stock size will decline towards zero even without fishing. Not surprisingly, a zero investment level is optimal in this case and harvesting should take place down to
s(K) = x₀ if possible. The optimal policy functions for the case a = 14 resemble those of the relatively low-fecundity whale fishery, while a = 42 corresponds to our base case for the prawn fishery. As fecundity increases, the upward-shifting of the h(S) optimal capacity curve continues. The limiting case of recruitment being independent of escapement is approximated by setting a = 560; the optimal capacity curve is fairly flat, with h(S) ≈ 12, for all but the lowest escapements.

Note that as fecundity decreases, the optimal escapement at low fleet capacity approaches x₀, i.e. s(0) tends to x₀. This is a confirmation of the reasoning that with a slow-growing stock and a low level of capacity one has little incentive to conserve the current stock: the future is not rosy in such a situation. On the other hand, at high capacity levels the optimal escapement is determined from the Modified Golden Rule equation (see section C), which for the Beverton-Holt F(S) function described above can be written:

\[-mT \quad -mT\]
\[(22) \quad (b + aeS)^2(S-x₀) - ae b(b-x₀)S + ax₀b^2 = 0\]

where b = 7.0x10⁶, x₀ = 0.993x10⁶ and a, M, T are as before. For each value of a, there is a unique positive root of (22). This optimal escapement s(K), for large K, depends on a in a rather complicated way, as indicated in figure 5. As in the low capacity case, s(K) ≈ x₀ if a is very small. As fecundity increases, the reproductive potential of the stock is improved, and higher escapements s(K) are desirable. When a becomes very large, recruitment becomes less dependent on escapement, so that s(K) can be lowered with little effect on future stocks.
The maximum recruitment level, \( b \), serves as a suitable indicator of the carrying capacity of a fish stock with Beverton-Holt dynamics. As stated earlier, our base case used the value \( b = 7.0 \times 10^6 \), derived from the sample mean of Kirkwood's (1980) recruitment data. This data had been substantially revised and extended from that used in the analysis of Clark and Kirkwood: their older data produces the value \( b = 11.3 \times 10^6 \). Figure 6 indicates the effect of the revision in the value of the carrying capacity. The fecundity is \( a = 42.0 \) for both cases. Clearly the change produces substantial movement in both optimal policy functions. The optimal equilibrium capacity declines from 14.5 to 8.2 if the new data is used in place of the old. Further results (not shown here) indicate that this relative decrease in optimal capacity holds also for higher values of fecundity, e.g. \( a = 560 \).

In summary, the fecundity and carrying capacity of the fish stock can have substantial effects on the optimal policies, in particular the optimal capacity function. This is especially of interest in such cases as the banana prawn fishery, where little is known about the stock-recruitment relationship. Dealing with parameter uncertainty in these fisheries becomes an important problem for further research.

(d) The Depreciation Rate

The value of the parameter \( \tau \) used by Clark and Lamberson, namely \( \tau = 0.15 \), was utilized in our base case runs for the whaling fishery. The same value was chosen as reasonable for the prawn fishery as well. In a capital investment model, it is
certainly of interest to examine the effect on optimal policies of variations in the depreciation rate. Figure 7 indicates the optimal \( s(K) \) and \( h(S) \) curves for the \( (a=42) \) prawn fishery with \( \gamma = 0, 0.05, 0.15 \) and \( 0.20 \).

The results for the fishery are intuitively appealing. A decrease in the depreciation rate leads to an upward shift in the investment curve \( h(S) \), reflecting the increased life and hence the increased value of a new unit of capacity. On the other hand, an increase in \( \gamma \) increases the desire to use capacity before it depreciates, leading to a shift in the \( s(K) \) curve to lower escapements. This latter shift is less pronounced at high capacity values, where capital is relatively abundant in the "near future" even for \( \gamma = 0.20 \).

With no depreciation \( (\gamma = 0) \), the capacity \( K \) can never decrease. Hence any point \((S, K)\) which satisfies
\[
\text{\( S = \max\{s(K), F(S)e^{-\gamma T}\} \)} \quad \text{(see Appendix A)}
\]
and which lies above the curve \( K = h(S) \) is an equilibrium point. These points form an equilibrium curve (shown in figure 7) upon which all trajectories will converge. For the particular case shown in figure 7, the optimal capacity curve \( h(S) \) is very flat for \( S > 2.2 \times 10^6 \), with \( h(S) \approx 15.2 \). If initially \( K < 15.2 \), the prawn stock will eventually reach a point \((S, K)\), with \( S > 2.2 \times 10^6 \), and investment will occur until \( K \approx 15.2 \). Thereafter an equilibrium point, close to that shown for \( K = 15.2 \), will be approached.

The corresponding results for the whale fishery are shown in figure 8, where the optimal policy functions for \( \gamma = 0 \) and \( \gamma = 0.15 \) are given. The variation of the \( s(K) \) curves with \( \gamma \) is
qualitatively similar to that of the prawn fishery. In the \( r = 0 \) case, the long run equilibrium will again lie on that part of the curve defined by \( S = \text{Max} \{s(K), \text{solution of } S = F(S)e^{-qTK}\} \) which lies above the curve \( K = h(S) \). However the \( h(S) \) curve is now sufficiently steep that if capacity is initially low, a wide range of equilibrium points may be reached, depending on the initial recruitment value. The equilibrium curve for \( r = 0 \) and equilibrium point for \( r = 0.15 \) are indicated in figure 8.

The unusual aspect of the whaling fishery results is the intersection of the two \( h(S) \) investment curves and in particular the fact that for \( S > 12.2 \times 10^8 \), the investment curve for \( r = 0.15 \) lies above the \( r = 0 \) curve. As described above, one would expect that if a unit of investment is profitable at any fixed point in time, given a relatively high depreciation rate, then that same unit of capacity is even more desirable if there is no depreciation. Indeed, this is the case with the prawn fishery results above.

To examine this phenomenon further, consider figure 9, depicting the effect of depreciation on a lower-fecundity \((a=14)\) prawn fishery (with otherwise unchanged parameters). The change in fecundity has not altered the relationship between the \( r = 0 \) and \( r = 0.15 \) investment curves determined in figures 7. However, note that to this point, no change has been made in the other key investment parameter, unit capital cost. Consider the relative magnitudes of unit capital costs, \( c \), and maximum yearly unit variable costs, \( c \cdot T \). For the prawn fishery the ratio \( \delta/c \cdot T = 11.3 \) while \( \delta/c \cdot T = 2.0 \) for the whale fishery. If we reduce the
capital cost in the prawn fishery so that $\delta/c \cdot T=2.0$, implying $\delta=\$0.0832 \times 10^6$, we obtain the optimal policies shown in figure 10, which are qualitatively similar to those of the whale fishery. An analysis of trajectories for the policy functions of figure 10 indicates that the relative heights of the $\gamma=0$ and $\gamma=0.15$ optimal capacity curves are determined not simply by the ratio $\delta/c \cdot T$ but rather by a more complicated comparison of the present values of investment costs vs. rents for various investment strategies. The no-depreciation $h(S)$ curve represents a balance between investment costs and the natural preference of having higher capacity to enable more rapid accumulation of rents from harvesting the stock down to equilibrium. Depreciation introduces two new factors: (i) the need for future investment to overcome depreciation and (ii) the desire to "beat" depreciation by harvesting the stock before the fleet "wears out". It is this latter effect which appears responsible for the $\gamma=0.15$ $h(S)$ curve lying above the corresponding zero depreciation curve at high escapement levels. However as depreciation increases beyone 15 percent, the optimal capacity curve drops as the yearly costs of overcoming depreciation predominate. When $\gamma=1.00$, so that vessels last for only one season, the $h(S)$ curve lies completely below its no-depreciation counterpart. This rather complicated response to the depreciation rate seems to depend critically on actual parameter values, necessitating careful treatment of the data in specific applications.
(e) **Capital Cost, Fish Price and Discount Rate**

One would expect intuitively that the attractiveness of developing a fishery is enhanced by low unit capital costs, high selling price, and a low discount rate. These expectations are confirmed here, and the sensitivity of optimal investment levels to these parameters is deduced.

The result of a decrease in unit capital cost is shown in figure 11 for our base case prawn fishery, and by comparing figures 9 and 10 for an a=14 (low fecundity) prawn fishery. In the latter case, a reduction to almost one-sixth the usual capital cost, from $0.47x10^6$ to $0.0832x10^6$, produces a 3.5-fold increase in the equilibrium capacity (with $r=0.15$). In the former case, a halving of the capital cost resulted in a 1.7-fold increase in equilibrium capacity. In both cases, optimal escapement at low capacities increased as capital cost decreased, reflecting the increased benefit in saving more of the fish stock for the future, at which time capacity will be higher.

The variation of the optimal policy functions with fish price is shown in figure 12 for our usual (a=42) prawn fishery. A doubling of the price, from its actual level of $0.9/kg to $1.8/kg, produced more than a doubling in equilibrium capacity, while a halving of the price made investment entirely uneconomic, so that depreciation will slowly reduce the fleet size to zero. However, harvesting still takes place in this low-price case, whenever $R>s(K)$, although the $s(K)$ curve itself has shifted to the right relative to the base case. A harvesting
domain will exist as long as the price is high enough that \( x_0 = c/(pq) < a/b \), the maximum possible recruitment with our Beverton-Holt model.

The optimal policy functions obtained for the (a=42) prawn fishery with discount factors (and corresponding discount rates) of \( \sigma = 0.99 \) (1 percent), 0.90 (11 percent) and 0.8 (25 percent) are shown in figure 13. Naturally, the lower the rate of discounting, the higher is the benefit from investing in capacity for the future (to become available next year) and the higher is the desired escapement, \( s(K) \), to be left at the end of the current season. It is interesting to note that while optimal escapements (for fixed \( K \)) increase with \( \sigma \), the equilibrium escapement decreases with \( \sigma \), reflecting the optimality of using the increased capacity which becomes available with low discounting.

(f) **Ricker Stock-Recruitment**

While the concave, monotone properties of the Beverton-Holt stock-recruitment function make it pleasant to deal with, it is but one of several commonly-used functions. The Ricker form, \( R = F(S) = aS \cdot \exp\{-(a/eb)S\} \), is used extensively in studying salmon stocks (Ricker, 1975), provides reasonable fits to data for gadoid stocks, including haddock and Arcto-Norwegian cod (Cushing and Harris, 1973) and has recently been applied to northeast Pacific herring populations (Walters, 1981). It has the property that recruitment attains a maximum value of \( R=b \) at \( S=eb/a \), and thereafter declines roughly exponentially. Since \( K=h(S) \) is determined from the equation \( V(F(S),K) = \delta/\sigma \) it would
be expected that the optimal capacity will follow the behaviour of $F(S)$, initially increasing to a maximum and thereafter decreasing.

Numerical results, shown in figure 14, confirm this expectation. The parameters of the Ricker function used in this example, namely $a = 11.639$ and $b = 7.0 \times 10^6$, were chosen so that the maximum recruitment is identical to that of the Beverton-Holt form for our base case prawn fishery, and occurs at $S = 6.0 \times 10^6$ (to produce a reasonable agreement with the Beverton-Holt curve for low and medium escapements). Other model parameters are unchanged from those given in Table (I) for the prawn fishery, and delayed investment is assumed.

Note that there is a threshold level of escapement, and hence of $F(S)$, below which no investment occurs. At high escapements, $F(S)$ decreases and eventually drops below its threshold value: thereafter we have $h(S) = 0$. (The horizontal scale in figure 14 has been changed from that used in previous results in order to include this upper cutoff.)

Comparing figure 14 with figure 3, it can be seen that optimal capacity levels are approximately equal in the neighbourhood of $S = 6.0 \times 10^6$, where the Ricker curve peaks. In figure 14, the $h(S)$ curve increases rapidly for $S > 6.0 \times 10^6$ and declines rather slowly to the right of the maximum value, mimicking the behaviour of the Ricker curve itself. The optimal escapement curve $s(K)$ behaves similarly to those of Beverton-Holt cases, except the optimal high-capacity escapement has substantially increased, reflecting reduced fecundity at low
escapements for our particular Ricker curve.

Further results (not shown here) indicate that as the Ricker curve shifts to lower escapements (with its maximum value unchanged), the optimal policy functions shift similarly. It becomes desirable to leave escapements which will produce high recruitments next season, and to invest only if the current escapement will result in such high recruitments. In addition, optimal capacity levels increase, reflecting the importance of harvesting sufficiently to reduce high recruitments down to relatively low escapements.

(g) The Value Function

To this point we have dwelt on deriving and examining the optimal policy functions, s(K) and h(S), under various assumptions and parameter combinations. However the dynamic programming approach produces not only the optimal policies but also the optimal value function. Indeed for any policies s(K) and h(S), the corresponding value function is the solution of the equation:

\[ V(R,K) = \pi(R,K,S^*(R,K), \text{Max}\{h(S^*(R,K)) - (1-\gamma)K, 0\} \]

\[ +\alpha V(F(S^*(R,K)), \text{Max}\{h(S^*(R,K)), (1-\gamma)K\}) \]

where \( S^*(R,K) \) depends on \( s(K) \) through equation (13).

A sample value function, corresponding to the optimal policy functions for our base case prawn fishery, is represented in Table (II). At low levels of the capital stock, the value function is quite insensitive to the level of recruitment, R. This reflects the fact that, with low fleet capacity, increased recruitment has little effect on rents for the current season,
and since the prawn stock is fast-growing, differences in this year's stock size tend to substantially disappear by next year.

An examination of Table (II) shows that for Beverton-Holt population dynamics, we have $V > 0$, $V > 0$, $V > 0$ and $V > 0$, as expected, throughout the $R-K$ range considered. The curvature of the value function in the $R$-direction, measured by $V_{RR}$, varied considerably, from negative to positive values. However $V$ was not "too convex" in $R$ since the function $s(K)$ was in every case non-decreasing (to within the accuracy of the numerical scheme). As expected, the value function behaved approximately linearly in $R$ when both $R$ and $K$ were large.

Of particular interest is the sensitivity of the value function to the optimal policy functions: to what extent can the policy curves be altered (perhaps to take into account other objectives) without seriously reducing rents from the fishery? This important question will be considered in some detail in Chapter III.

F. Summary and Discussion

Using a dynamic programming approach, heuristic analysis and numerical methods, we have studied a deterministic model of optimal fisheries investment. The model, while similar to that of Clark, Clarke and Munro, differs in a number of respects: in particular, our model is based on a seasonal, discrete-time fishery and allows for delays in bringing new investment into service. In addition, the flexibility of the dynamic programming
method permitted consideration of the important Ricker stock-recruitment function as well as the Beverton-Holt form. However their key assumption regarding the irreversibility of capital investment has been retained.

Results in the form of optimal investment and escapement policy functions have been obtained numerically using policy iteration. These results were compared to those of Clark, Clarke and Munro, and the effects of a discrete-time analysis, delayed investment and variations in a number of model parameters were examined, using data from the Gulf of Carpenteria prawn fishery and the aggregated whaling fishery.

In the instantaneous investment case, our results were found to correspond closely to those of CCM, apart from natural differences between discrete- and continuous-time analysis. In particular their important conclusion regarding optimality of "a complex pattern of expansion, overcapacity, and gradual contraction via depreciation" towards an "optimal sustained yield" equilibrium holds for our discrete-time model as well.

The introduction of delays in investment added further realism to our model and changed the appearance of the optimal capacity function. However for the deterministic model of this chapter, it was shown that a simple transformation exists between the optimal policy functions in the cases of delayed and instantaneous investment. Delays in bringing new investment on-line necessitate advance planning, so that it can be optimal to invest even in years of low stock size, knowing that stocks will have recovered when the new capital becomes available.
The effects of changes in the values of fecundity, maximum recruitment, fish price, capital cost, discount rate and depreciation rate were investigated. In most cases the qualitative effects of these parameters were as expected. The optimal capacity function proved to be particularly sensitive to the fish price and stock-recruitment parameters, indicating the potential importance of including parameter uncertainty in the analysis.

The effect of variations in the depreciation rate was found to be rather complicated, depending both on the magnitude of unit capital costs (relative to operating costs and other economic factors) and on the actual values of the depreciation rate being considered. It was suggested that if capital is relatively inexpensive and if the depreciation rate is not too high, the optimal investment at high stock sizes can be greater than in the zero depreciation case. An explanation was put forward for this effect, which runs counter to the usual idea that depreciation, as a type of fishing cost, should lead to lower investment.

The use of a Ricker stock-recruitment form produced little change in the structure of the optimal escapement curve. However, the optimal capacity function adopted an appearance mimicking that of the stock-recruitment curve itself: increasing rapidly at low escapements and declining relatively slowly at higher escapements. Clearly this effect is important for investment strategies in those fisheries where over compensation exists.
Chapter III:
Optimal Fisheries Investment Under Uncertainty

It has long been observed that fisheries, perhaps as much as any natural resource, exhibit remarkably high levels of uncertainty, arising from economic and biological factors, and affecting not only the participants in the fishery but the resource managers as well. Hence it has seemed obvious that attempts should be made to incorporate uncertainty into fishery management models.

Walters & Hilborn (1978) distinguish three general classes of uncertainty in fisheries management:
1. Random effects whose future frequency of occurrence can be determined from past experience;
2. Parameter uncertainty that can be reduced by research and acquisition of information through future experience;
3. Ignorance about the appropriate variables to consider and the appropriate form of the model.

Most studies of the optimal management of fisheries under uncertainty have concentrated on the first type of uncertainty. Typically the corresponding deterministic dynamics of the fishery are transformed to a stochastic analogue, and the resulting stochastic optimization problem is analyzed using dynamic programming.

In a realistic multi-species, multi-cohort multi-parameter fishery, stochastic effects can enter in a number of ways. Typically, the source of uncertainty is taken to be
environmental fluctuations affecting the population dynamics of the fish stocks, although Lewis (1975) and Andersen (1982) allow for price variability as well. Dudley & Waugh (1980) study a simulation-optimization model in which yearly recruitment, mortality rate and catchability fluctuate simultaneously. While fluctuations are usually taken to be independent from one point in time to another, they may in fact form a Markov process (Spulber, 1981). Each species or each age-cohort may respond to randomness differently (Mendelssohn, 1978, 1980a; Spulber, 1978). Nonconvexities and risk aversion can affect substantially the role of uncertainty (Lewis, 1981).

Andersen (1981) reviews research on the behaviour of competitive firms under uncertainty, and applies these results to optimal management of fishing firms faced with stock and price uncertainties. Pindyck (1982) examines the interaction of ecological uncertainty, demand elasticity and the biomass growth function in the context of renewable resource markets, extending the analysis of the exhaustible resource case in Pindyck (1980).

Methods of solving such stochastic optimization problems have also varied considerably. Jaquette (1972, 1974) and Reed (1974, 1978, 1979) used analytic approaches to study the effects of uncertainty on discrete-time dynamic programming models of fisheries. Walters (1975) and Walters and Hilborn (1976), using a dynamic programming approach with Kalman filter techniques, examined both stochastic effects and problems of parameter uncertainty. Beddington and May (1977) studied the effect of uncertainty on Maximum Sustainable Yield policies using
characteristic return times and the coefficient of variation in fisheries yields. Ludwig (1979a) formulated a continuous-time stochastic control model to which he applied perturbation techniques, while Ludwig & Varah (1979) used numerical methods to study the same model. Smith (1978) looked at continuous time models where the optimal policy was not a bang-bang control. Mendelssohn & Sobel (1980) placed the fisheries problem in the context of capital accumulation and used discrete-time dynamic programming techniques to obtain theoretical results for a fairly general fisheries model. Both May et al. (1978) and Spulber (1978) have emphasized the role of steady-state probability distributions for optimally harvested fish stocks. Aron (1979) formulated and analysed a compromise harvesting policy which performed well according to a number of indicators and which in addition was robust to lack of knowledge of yearly biomass levels.

Two useful surveys of the literature on fisheries under uncertainty are Andersen & Sutinen (1981) and Spulber (1982). There is a growing body of work in the areas of behavioural modelling with uncertainty (see, for example, Bochstael & Opaluch, 1981) and predictive modelling (Eswaran and Wilen, 1977). In this chapter we examine the role of stochastic biomass fluctuations in determining optimal fisheries investment strategies. Section A describes the stochastic model, which extends the deterministic model of Chapter II by allowing the resource stock to fluctuate randomly from year to year. As in Chapter II, sections B, C, D and E contain, respectively; analytic
results, heuristic analysis, numerical method and numerical results. The chapter ends with a summary and discussion in section F. Again, for the reader wishing to move quickly to the main results of the chapter, sections B and D can be bypassed without loss of continuity.

A. The Model

The stochastic model presented here is similar to that described in the previous chapter, with the exception that the biomass is assumed to follow a stochastic stock-recruitment relationship. Specifically the recruitment in year \( n \), \( r_n \) is governed by a lognormal probability distribution with mean \( F(S_{n-1}) \), where \( S_{n-1} \) is the previous year's escapement and \( F(\cdot) \) is the corresponding deterministic stock-recruitment function.

If the biomass has been aggregated over a number of fish stocks, one is faced with the possibility that environmental fluctuations will affect the various stocks to different extents (particularly if stocks are geographically separated). We shall assume here that the randomness is perfectly correlated across stocks, or alternatively that some average noise level has been determined. If in fact the stocks behaved independently, we would expect that the overall effect of environmental fluctuations on the aggregated biomass would tend to be relatively small.

The intraseasonal dynamics of the fish stock and the yearly dynamics of the capital stock are as in the deterministic case.
In particular, the biomass \( x \) in year \( n \) is governed by the differential equation \( \frac{dx}{dt} = -h(t) = -qE(t)x(t) \) with \( x(0) = R \) and the instantaneous fishing effort \( E(t) \) subject to \( 0 \leq E(t) \leq K \).

Hence the optimal escapement \( S^* \) must be chosen subject to

\[
R \exp\{-qTK\} \leq S^* \leq R,
\]

where \( T \) is the maximum season length. Given \( S^* \), the optimal addition to fleet capacity \( K \) desired to become available at the beginning of the next season (namely \( I_{n+1} \)) is determined before the end of the current season. Again the resale value of the vessels is assumed to be zero, so the capital stock can be reduced only through depreciation, which occurs at the end of each season. Note that the inherent delay in bringing new investment on-line, while of little consequence in the deterministic case, contributes to the fishery manager's uncertainty in a stochastic environment.

A major assumption in this model, as in practically all other fishery optimization models, is the observability of the yearly recruitment. In the real world, there is of course no way of knowing how many fish are available at the beginning of a fishing season, unless some test fishing has taken place. Typically a rough guess is made of the available stock, openings are made accordingly, and the stock is re-estimated periodically on the basis of new information. While this form of adaptive in-season management is clearly desirable, and has recently been studied in some detail [Ludwig and Walters (1981), Walters and
Ludwig (1981), Mangel & Clark (1982), it is difficult to include in a model of inter-seasonal optimal management. We shall henceforth make the simplifying assumption that, by some means such as a test fishery, the recruitment at the start of each season is known precisely.

In Chapter II, we adopted as the social objective maximization of the present value of the fishery. However, in formulating the objective of the fishery manager one should consider the attitude of society towards risk. Mendelssohn (1980b) has studied the Pareto optimal tradeoff between economic returns and variance in these returns, while Lewis (1977) has pointed out the merits of incorporating risk preference or risk aversion in a utility function of yearly rents accruing to the fishery, risk aversion being reflected in a concave utility function. [In fact such a utility function might be expected to depend also on past rents (Ryder & Heal, 1973), as well as past and present catches.] We shall adopt the utility function approach here, with \( U = U(\pi) \), but assume that society is risk neutral with respect to investment costs. Hence the utility function applies to yearly rents net of investment expenditures, a quantity which serves as a proxy for the current health of the fishery. This realistically reflects society's aversion to low stock levels and to low incomes for fishermen, rather than to low yearly rents per se. Naturally, utility must be measured in the same units as investment costs, since yearly benefits are now given by:

\[
B(R,K,S,I) = U(\pi(R,S)) - 6I
\]
using the same notation as in Chapter II, except with

\[ \pi(R,S) = p(R-S)-(c/q)\log(R/S), \]

where we have assumed linear costs, \( c(E) = cE \), and no alternative fishery, in order to make the
description more transparent. Note that the intraseasonal
fishery is assumed to be deterministic, and hence is unaffected
by year-to-year biomass fluctuations (given the initial
recruitment). The stochastic control problem can now be stated
as follows:

\[
\text{Max } \quad \left[ \sum_{i=1}^{n-1} \sigma \cdot \mathbb{E}\{B(R,K,S,I)\} \right] \quad \text{subject to:} \\
\{S_1; I_2; S_3; \ldots\} \quad n \geq 1 \quad n \quad n \quad n \quad n+1 \\
R \sim \phi(\cdot), \quad K = (1-\gamma)K + I, \quad R \cdot \exp(-qTK) \leq S \leq R, \quad \text{and} \\
R \cdot \exp(-qTK) \leq S \leq R, \quad R(n+1) \quad \sigma \quad n \quad n \quad n+1 \quad n \quad n \\
I \geq 0, \quad \text{where } \phi(R) = (\nu \sigma R)^{-1} \exp\{-\left(\log R - \log \bar{R} + \sigma^2/2\right)^2/2\sigma^2\} \\
\bar{R} ; \sigma
\]

is a lognormal density with mean \( \bar{R} \) and uncertainty parameter \( \sigma \)
[so that the variance is \( \bar{R}^2(e^{-1}) \), \( \sigma \) is the annual discount
factor, and it is understood that \( S \) and \( I \) are to be chosen
in year \( n \), given the state \((R,K)\) in that year. (Here and
throughout the remainder of this work, we define \( \nu \) to be the
square root of \( 2\pi \).)]

The dynamic programming equation corresponding to this
problem is the following [equation (1)]:

\[
V(R,K) = \text{Max } \quad \text{Max } \quad \mathbb{E}\{B(R,K,S,I) + \sigma \mathbb{E}\{V(R',(1-\gamma)K+I)\}\} \\
R \cdot \exp(-qTK) \leq S \leq R \quad I \geq 0
\]

where \((R,K)\) is the 'state' this year, \((S,I)\) are the controls
(decision variables), and \( R' \) is next year's recruitment (lognormally distributed as above). Clearly this equation has a form very similar to equation II(1). As in Chapter II, we now proceed to consider analytic, heuristic and numerical treatments of equation (1), in the following three sections.

B. **Analytic Results.**

Before examining the full model we shall follow the procedure of Chapter II and first look at some simplifications of the model which permit a more analytic treatment. In particular, recruitment will be considered here to follow a stationary time-independent probability distribution; this is equivalent to asserting that recruitment is independent of previous escapement levels, an assumption which was used for the deterministic case in II.B. As before, we shall compensate for this extreme assumption by adding a "benefits of escapement" function \( b(S) \) to the yearly rents; this provides an incentive to avoid overly low biomass levels. As a further simplification, we assume risk neutrality, \( U(\pi) = \pi \), but in this section we include nonlinear costs and an alternative fishery. Hence the two-stage optimization problem can be stated as follows [equation (2)]:

\[
\max \mathbb{E}\{ \max_{K \geq 0} \left[ \pi(R-S)-r(c(E)+b(S))-(\rho/q)\log(R/S)-(\pi-\rho T)K \right] \exp\{-qTK\} \leq S \leq R \}
\]

where the expectation is with respect to an arbitrary density function \( \phi(R) \) (not necessarily lognormal). For given \( R \) we have, as before, \( x(0)=R \) and \( h=-qEx \), with other notation as in II.B.

The inner optimization (which is strictly deterministic due to our assumption of perfect knowledge of the stock size at the
beginning of the season) was solved in Chapter II, where we obtained the optimal escapement and the yearly rents function in terms of \( R \) and \( K \):

(a) \( K \geq E_0 \)

<table>
<thead>
<tr>
<th>Recruitment</th>
<th>( S^* )</th>
<th>Yearly Rents ( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R &lt; S_0 )</td>
<td>( R )</td>
<td>( \pi_1 = b(R) )</td>
</tr>
<tr>
<td>( S_0 &lt; R &lt; R^* )</td>
<td>( S_0 )</td>
<td>( \pi_2 = p(R-S_0)-q^{-1}\log(R/S_0)c(E_o)/E_0 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( -(\rho/q)\log(R/S_0)+b(S_0) )</td>
</tr>
<tr>
<td>( R^* &lt; R &lt; R^<em>_</em>(K) )</td>
<td>( s(R) )</td>
<td>( \pi_3 = p(R-s(R))-\text{Tc}([1/qT]\log[R/s(R)]) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( -(\rho/q)\log[R/s(R)]+b(s(R)) )</td>
</tr>
<tr>
<td>( R &gt; R^<em>_</em>(K) )</td>
<td>( Re )</td>
<td>( \pi_4 = pR(1-e^{-qTK})-\text{Tc}(K)-\rho TK+b(Re) )</td>
</tr>
</tbody>
</table>

(b) \( K < E_0 \)

<table>
<thead>
<tr>
<th>Recruitment</th>
<th>( S^* )</th>
<th>Yearly Rents ( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R &lt; e(K) )</td>
<td>( R )</td>
<td>( \pi_5 = b(R) )</td>
</tr>
<tr>
<td>( e(K) &lt; R &lt; R^<em>^</em>(K) )</td>
<td>( e(K) )</td>
<td>( \pi_6 = p(R-e(K))-q^{-1}\log[R/e(K)]c(K)/K )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( -(\rho/q)\log[R/e(K)]+b(e(K)) )</td>
</tr>
<tr>
<td>( R &gt; R^<em>^</em>(K) )</td>
<td>( Re )</td>
<td>( \pi_7 = pR(1-e^{-qTK})-\text{Tc}(K)-\rho TK+b(Re) )</td>
</tr>
</tbody>
</table>

where as before:

(i) \( R^* \) is the solution of \([p-b'(R,e^{-qTK})]qR.e^{-qTK} = c'(K)+\rho\)

(ii) \( S_0 \) is the solution of \([p-b'(S_o)]qS_0 = c'(E_o)+\rho\)

(iii) \( R^* = S_0 e \)

(iv) \( s(R) \) is the solution of \([p-b'(S)]qS = c'(\text{T}[1/qT]\log(R/S))+\rho\)

(v) \( e(K) \) solves \([p-b'(S)]qS = c(K)/K + \rho \) for \( K < E_0 \),

(vi) \( R^*^*(K) = e(K)\exp\{qTK\} \) and
(vii) $E_0$ is the solution of $c'(E_0) = \tau E_0$.

In addition, note that $S_0$ and $R^*$ are independent of $K$.

Then the optimization problem reduces to:

(3) \[ \max \{ E[\pi(R,K)] - (\tau - \psi T)K \} \]
\[ K \geq 0 \]

where $E[\pi(R,K)]$ is given in the case $K \geq E_0$ by:

(4) \[ K \geq E_0; \quad E[\pi(R,K)] = \begin{cases} \pi_1 \phi(R) dR & \text{if } S_0 \leq R^* \\ \pi_2 \phi(R) dR & \text{if } S_0 \leq R^* \end{cases} \]

and in the case $K < E_0$ by the following:

(5) \[ K < E_0; \quad E[\pi(R,K)] = \begin{cases} e(K) \pi_5 \phi(R) dR & \text{if } e(K) \leq R^* \\ \pi_6 \phi(R) dR + \pi_7 \phi(R) dR & \text{if } e(K) \leq R^* \end{cases} \]

Now if $K > E_0$ we have:

\[ \frac{dE[\pi]}{dK} = \pi_3(R_+,E) \phi(R_+) (dR_+/dK) + \int_{S_0}^{R_*} (d\pi_3/dK) \phi(R) dR \\
- \pi_4(R_+,E) \phi(R_+) (dR_+/dK) + \int_{R_*}^{R^*} (d\pi_4/dK) \phi(R) dR \]

But by definition of $R_*$, $\pi_3 = \pi_4$ at $R = R_*, E = K$. Furthermore, by inspection, $\pi_3$ is independent of $K$, so that $d\pi_3/dK = 0$.

Hence the original problem is solved either by an interior maximum in the interval $[E_0, \text{infinity})$ or by some value $0 \leq K^* < E_0$; in other words, $K^*$ is either $< E_0$ or is given by the solution of:

(6) \[ \begin{cases} (d\pi_4/dK) \phi(R) dR = \tau - \psi T & \text{if } R > R_* \\ 0 & \text{if } R = R_* \end{cases} \]

The quantity defined by $dp = \{ d\pi_4 \text{ if } R \geq R_*, 0 \text{ if } R < R_* \}$ represents the infinitesimal gross yearly benefits to an
increase in fleet capacity of dK. The quantity \( \kappa dK \) is the effective yearly payment for the additional capital dK, while \( \rho T \) is that part of the capital cost \( \kappa dK \) which could be offset by rents from the alternative fishery. Thus equation (1) can be written:

\[
E\{ \text{marginal benefit} \} = \text{marginal (capital) cost}
\]

This result is as expected, being the usual economic optimality condition. Equation (6) can be expressed as follows:

\[
(7) \quad [\left[ p - b'(R) \right] q R - c'(K) - \rho \} \cdot \phi(R) dR = \kappa / T - \rho
\]

Note that \( d\pi_e / dK = \left[\left[ p - b'(R) \right] q R - c'(K) \right] > 0 \) for \( R > R_e \). Furthermore \( dR_e / dK > 0 \) and for fixed \( R_e \), \( d(d\pi_e / dK) / dK < 0 \). Hence the lefthand side of (7) is a decreasing function of \( K \), so that (7) has at most one solution \( K^* \). If we set \( K = E_0 \) in the lefthand side of equation (7) and find that:

\[
(8) \quad \left( \frac{1}{T} \right) dE_0 / dK = \left( \frac{1}{qTK} \right) [c(K)/K - c'(K)] \log[R/e(K)] \phi(R) dR
\]

\[
\begin{align*}
&\left[\left[ p - b'(R) \right] q R - c'(E_0) \right] \cdot \phi(R) dR < \kappa / T - \rho \\
&\quad R > R_0
\end{align*}
\]

then the optimal capacity \( K^* \) must be less than \( E_0 \). In that case we wish to solve \( dE_0 / dK = \kappa - \rho T \) for \( K < E_0 \). Note first that \( \pi_5 = \pi_6 \) at \( R = e(K) \), \( \pi_6 = \pi_7 \) at \( R = R^{**} (K) \) and \( e(K) \) is such that the partial derivative of \( \pi_6 \) with respect to \( e(K) \) is 0 by construction of \( e(K) \). Hence we find that for \( K < E_0 \):

\[
(9) \quad \left( \frac{1}{T} \right) dE_0 / dK = \left( \frac{1}{qTK} \right) [c(K)/K - c'(K)] \log[R/e(K)] \phi(R) dR
\]

\[
\begin{align*}
&\left[\left[ p - b'(R) \right] q R - c'(K) - \rho \} \cdot \phi(R) dR = \kappa / T - \rho \\
&\quad R > R^{**}
\end{align*}
\]

If we know that \( K^* < E_0 \), but (8) has no non-negative solution,
then we conclude that the optimal capacity is \( K^* = 0 \); the fishery is not worth developing.

A simpler form of the solution can be obtained by neglecting benefits of escapement, so \( b(\cdot) = 0 \), and assuming linear variable costs, \( c(E) = cE \). In this case \( E_0 = 0 \) so that we do not have to be concerned about the regions \( K < E_0 \) and \( S_0 \leq R \leq S_0 \), and the solution becomes:

\[
qTE_0
\]

\[
\int_{R > R_*} [pQR - c'(K)] \phi(R) dR = \frac{\kappa}{T}
\]

The term \( R_* \) is as before but is now given by the simpler form

\[
qTK
\]

\[
R_* = x_0 e^{\frac{qT}{K}}
\]

where \( x_0 = c/(pq) \). Equation (9) can be rewritten:

\[
(R_*)^{-1} \int_{R > R_*} (R - R_*) \phi(R) dR = \frac{\kappa}{cT}
\]

or equivalently:

\[
(R_*)^{-1} \int_{R > R_*} R \phi(R) dR + \phi(R_*) = 1 + \frac{\kappa}{cT}
\]

Here \( \phi(\cdot) \) is the cumulative distribution function for \( \phi(R) \). Note that the right hand side is \( 1 + [\gamma + (1 - \alpha)/\alpha](\delta/cT) = 1 + [\gamma + r](\delta/cT) \)

where \( r \) is the discount rate. In this simple case we can note that:

(i) The depreciation rate and the discount rate play identical roles in determining the optimal capacity,

(ii) The ratio of unit capital cost to maximum yearly variable costs \( (cT) \) is the critical cost parameter, and

(iii) Differentiating (11) with respect to \( \theta = [\gamma + r](\delta/cT) \), using

\[
R_* = x_0 E^{\frac{qT}{K}}
\]

and simplifying, we obtain:
\[ \frac{dK^*}{d\theta} = \frac{1}{[QT(\theta+1-\theta(R_\ast))] < 0} \]

so that, as expected, the optimal capacity decreases with the depreciation rate, the discount rate and the relative cost of capital.

The more interesting question is the effect of uncertainty on the optimal capacity. Differentiating (10) with respect to \( \sigma \), we obtain:

\[ -(R_\ast)^{-2} \left[ R-R_\ast+R_* \right] \phi(R)dR \left( dR_\ast/d\sigma \right) + (R_\ast)^{-1} \left[ (R-R_\ast) \phi(R)dR \right] = 0 \]

where \( \phi \) is the partial derivative of \( \phi \) with respect to \( \sigma \).

Simplifying and using (10) as well as the relationship \( dR_\ast/dK^* = qT R_\ast \), we have:

\[ \frac{dK^*}{d\sigma} = \left\{ qT R_\ast \left[ 1 + \kappa/cT - \phi(R_\ast) \right] \right\} \left[ (R-R_\ast) \phi(R)dR \right] \]

Since \( 1 + \kappa/cT \geq 1 \), we can conclude that the optimal capacity increases with uncertainty whenever:

\[ (R-R_\ast) \phi(R)dR > 0. \]

Since \( R \geq R_\ast \), we have:

\[ \phi(R_\ast) = \left\{ \begin{array}{ll} R_\ast/R & \text{if } R_\ast \leq R \; \text{ and } R_\ast > R \end{array} \right\} \]

and

\[ (R_\ast)^{-1} \left[ R \phi(R)dR = \{ (R^2-R_\ast^2)/(2R,R) \} \right] \]

If \( \theta > 0 \), substitution of these quantities into equation (11) shows that we must have \( R \leq R_\ast \), and \( R_\ast \) must satisfy:

Examples

(1) As a first example of the use of equation (11), consider a fish stock following a uniform probability distribution on \([0,R]\), so that its mean value is \( R/2 \) and its variance is \( R^2/12 \). (This is not claimed to be a realistic situation but will allow us to obtain an algebraic solution.) Then we have the following:

\[ \phi(R_\ast) = \left\{ \begin{array}{ll} R_\ast/R & \text{if } R_\ast \leq R \; \text{ and } R_\ast > R \end{array} \right\} \]

If \( \theta > 0 \), substitution of these quantities into equation (11) shows that we must have \( R \leq R_\ast \), and \( R_\ast \) must satisfy:
\[(R^2-R_*^2)/(2R_*^2) + R_*/R = 1 + \frac{k}{cT}\]

This is a quadratic equation for \(R_*\) which can be easily solved to produce:

\[R_* = \frac{1}{2}[(1+\theta) - \sqrt{(1+\theta)^2 - 1}]R\]

Then, from the definition of \(R_*\), the optimal capacity is given by \(K^* = (qT)^{-1}\log(R_*/x_0)\). In Chapter II we showed that the optimal capacity in a deterministic environment with constant yearly recruitment \(R/2\) is given by \(K^*(\text{det}) = (qT)^{-1}\log\left(\frac{R/2}{x_0(1+\theta)}\right)\). Now define \(X = 2[1+\theta]/([1+\theta] + \sqrt{([1+\theta]^2 - 1)})\). Note that since \([1+\theta]^2 - 1 < [1+\theta]^2\), we must have \(X > 1\). Then comparison of the expressions for \(K^*\) and \(K^*(\text{det})\) gives:

\[K^* = K^*(\text{det}) + (qT)^{-2}\log(X)\]

Therefore in this example we have \(K^* > K^*(\text{det})\) in all cases (assuming \(\theta > 0\)). In particular, it can be shown that for \(\theta\) very small, we have the result:

\[(K^* - K^*(\text{det}))/K^*(\text{det}) \sim \log \left( \frac{2}{\log \left( \frac{R}{2x_0} \right)} \right)\]

so that the relative increase in the optimal capacity from the deterministic to the stochastic case can be substantial if the mean recruitment is not too high relative to \(x_0\).

(2) Next, let us consider the more realistic case of a lognormal probability density with mean \(E(R) = R\) and standard uncertainty parameter \(\sigma\). In this case we have:

\[
\int_{R > R_*} R \phi(R) dR = \int_{x > \log(R_*)} \left(\frac{\mu}{\sigma}\right)^{-1} e^{-\frac{(x-b)^2}{2\sigma^2}} e^x dx
\]

\[= R \int_{z > z_*} e^{-\frac{z^2}{2}} dz \quad \text{where} \quad z_* = \left(\frac{\log(R_* - b - \sigma^2)}{\sigma}\right)\]

\[= R \left[ 1 - \Phi \left( \frac{\log(R_*/R)}{\sigma - \sigma/2} \right) \right]_N \]

\[= R \left[ 1 - \Phi \left( -x/\sigma - \sigma/2 \right) \right]_N\]
where \( b = \log(R e^{-\sigma^2/2}) \), \( x = \log(R/R_*) = \log(R/x_0) - qTK^* \),

\( \Phi(z) = \Pr\{N(0, 1) \leq z \} \) (i.e. the cumulative distribution function for a standard normal random variable) and \( \theta \) is as before.

Similarly, \( \Phi(R_*) = \Phi(-x/\sigma^2 + \sigma/2) \) and \( (R_*)^{-1} = e^{x/R} \). Hence (11) simplifies to:

\[
\text{(12)} \quad e^{\left[ 1 - \Phi(-x/\sigma^2 + \sigma/2) \right]} + \Phi(-x/\sigma^2 + \sigma/2) = 1 + \theta
\]

Now \( (R-R_*)\Phi(R)\,dR = \int_{R > R_*} R\Phi(R)\,dR - R_* \int_{R > R_*} \Phi(R)\,dR \)

\[= R[1 - \Phi(-x/\sigma^2 + \sigma/2)] - R_*[1 - \Phi(-x/\sigma^2 + \sigma/2)]\]

The partial derivative of this quantity with respect to \( \sigma \) is then as follows:

\[
\text{(13)} \quad -R(x/\sigma^2 - 1/2)\Phi(-x/\sigma^2 - \sigma/2) + R_* (x/\sigma^2 + 1/2) \Phi(-x/\sigma^2 + \sigma/2)
\]

\[= (R_*/\nu) \left\{ -e^{-(x^2/\sigma^2 - 1/2 - x^2/\sigma^2 + \sigma^2/4)/2} + (x/\sigma^2 + 1/2) e^{-(x^2/\sigma^2 - x^2/\sigma^2 + \sigma^2/4)/2} \right\} \]

\[= (R_*/\nu) e^{-(x^2/\sigma^2 - x^2/\sigma^2 + \sigma^2/4)/2} \times [x/\sigma^2 - x/\sigma^2 + (1/2) + (1/2)] \]

\[= R_* \Phi(-x/\sigma^2 + \sigma/2) > 0 \]

Hence we can conclude that in the simplified model of this section, the optimal capacity must increase with uncertainty if recruitment follows a stationary lognormal distribution. As we shall see, no such simple statement can be made in the general stock-recruitment situation. The following section returns to this general model, deducing qualitative behaviour of optimal
investment policies under uncertainty.

C. Heuristic Analysis

Our analysis in this section proceeds in a manner analogous to that of section II.C. We begin by performing the inner maximization in equation (1), for fixed $S$, to obtain the optimality equation for investment:

$$E\{V(R,(1-\gamma)K+I)\} = \delta/\alpha \text{ or } I = 0 \text{ if } E\{V(R,(1-\gamma)K)\} < \delta/\alpha$$

where the expectation over $R$ is with respect to the lognormal density $F(S), \sigma$. This states that, unless the fleet is temporarily overcapitalized, next year's optimal capacity, $(1-\gamma)K+I^*$, should be set such that the expected marginal benefit of an extra unit of capital equals its marginal cost.

Now define $K = h(S)$ to be the solution of the equation $E\{V(R',K)\} = (1/\alpha)\delta$, so that $h(S)$ represents next season's optimal capacity. We can observe that this equation can be written

$$E\{V(F(S)e^{-\delta z-\sigma^2/2},K)\} = \delta/\alpha$$

and note the similarity to equation (6) in Chapter II. Differentiating with respect to $S$, we obtain:

$$h'(S) = [F'(S)/F(S)] \cdot EV(R',h(S))/EV(R',h(S)) > 0$$

given the reasonable assumptions (for the Beverton-Holt model) that:

(i) there are decreasing marginal benefits to increasing the capital stock $[V < 0]$
(ii) the marginal benefit of increasing the capital stock increases with the current resource stock size \( [V_{RK}>0] \), and

(iii) \( F(\cdot) \) is a positive increasing function.

Hence the optimal capacity for next season is an increasing function of the current escapement, as in the deterministic case. Thus once again we have that if \((1-\gamma)K > h(S)\), the optimal investment is \( I^* = 0 \) (capital is already sufficiently abundant) while otherwise \( I^* \) is chosen so that \((1-\gamma)K + I^* = h(S)\). This is precisely equation (7) of Chapter II.C, which we repeat here:

\[
(16) \quad I^*(S,K) = \max\{h(S)-(1-\gamma)K,0\}
\]

Now performing the outer maximization we obtain the optimality expression equating the expected marginal benefit with the marginal cost of an incremental increase in escapement:

\[
(17) \ldots \]

\[
\alpha[F'(S)/F(S)] \mathbb{E}[R' \cdot V(R', (1-\gamma)K + I^*(S,K))] = p(1-x_0/S)U'(\pi(R,S))
\]

where \( x_0 = c/(pq) \) represents bionomic equilibrium and the \((-qTK)\) constraint \( R_{\pi} \leq S \leq R \) has been neglected temporarily. Note that equation (17) involves \( R, S \) and \( K \) (unless \( U' \) is identically a constant). Hence the introduction of a nontrivial utility function implies that it is no longer possible to deduce a target escapement curve \( S = s(K) \). Instead we will have \( S = s(R,K) \).

While this more complicated case could still be analyzed numerically, we shall opt in this study for the descriptive simplicity arising from the existence of the target curve \( s(K) \). Therefore, for the remainder of this chapter, let \( U(\pi) = \pi \) for all \( \pi \). Then (17) takes a form similar to that of equation II(8); we
shall assume as before that (17) has a unique solution $S = s(K)$.

In Chapter II, the following results were obtained for the deterministic model:

(i) If $K = h(S)$ and $S = s(K)$ intersect at $S = S^-$, then the optimal escapement is independent of fleet capacity if $K$ is relatively small, that is; $s(K) = S^-$ for all $K < \left(\frac{1}{1-\gamma}\right) \cdot h(S^-)$. Thus $S^-$ represents the optimal escapement at low fleet capacities.

(ii) As fleet capacity $K$ becomes very large, the optimal escapement $s(K)$ approaches or reaches the level $S^+$ representing the abundant-capital equilibrium and corresponding to $X$ in Clark, Clarke and Munro (1979).

(iii) $S^- < S^+$

(iv) $s(K)$ is "likely" to be an increasing function of $K$ throughout.

Results (i), (iii) and (iv) carry over to the stochastic case by entirely analogous reasoning, replacing $v, v, v$ by $E\{v\}, E\{v\}, E\{v\}$ respectively, wherever they arise. To derive the stochastic equivalent of result (ii), we proceed in a manner analogous to that of II.C, letting $v(R) = V(R, \inf)$ as before and noting that if $K = \inf$ then $S = s(\inf)$ for any $R$-value, so we have:

$$v(R + dR) = \pi(R + dR, \inf, s(\inf), 0) + \sigma v(F[s(\inf)])$$

and

$$v(R) = \pi(R, \inf, s(\inf), 0) + \sigma v(F[s(\inf)])$$

Hence $v(R + dR) - v(R) = p dR - (c/q) \log([R + dR]/R) = p(1 - x_0/R) dR$

so that $v(R) = p(1 - x_0/R)$ if $R > S' = s(\inf)$

Setting $R = F(S)$ and substituting into (17) this expression...
for \( v \) produces:

\[
\sigma[F'(S)/F(S)]E[R' \cdot p(1-x_0/R')] = p(1-x_0/S)
\]

Now \( E[R'] = F(S) \), so this equation reduces to:

\[
p\sigma[F'(S)/F(S)](F(S)-x_0) = p(1-x_0/S) \quad \text{or:}
\]

\[
F'(S) \cdot [1-x_0/F(S)]/[1-x_0/S] = 1/\sigma
\]

which is again the Modified Golden Rule equation. However, this is incorrect since the expression for \( v \) derived above applies only for \( R>S' \). While this inequality was valid for a deterministic fishery, it cannot be assumed in the stochastic case. Instead, we can write \( E[R' \cdot v (R')] \) as follows:

\[
E[R' \cdot v (R')] = E[R' \cdot p(1-x_0/R')]
\]

\[
S^+ + \int_0^R [R' \cdot v (R') - p(R' - x_0)] \phi(R')dR'
\]

Now substituting into (17) and rearranging, we have the optimality expression:

\[
F'(S)[1-x_0/F(S)]/[1-x_0/S] - (1/\sigma) =
\]

\[
- [F'(S)/pF(S)(1-x_0/S)] \int_0^S [R' \cdot v (R', \inf) - (R' - x_0)] \phi(R')dR'
\]

For the Beverton-Holt \( F(S) \) function, the left hand side of this equation is a decreasing function of \( S \), so we can conclude that in the stochastic case, the abundant-capital optimal escapement is \( S^+ (<S^+) \) according as the integral on the right hand side of this equation, evaluated at \( S=S^+ \), is \( 0 (<0) \).

Now incorporating the constraint \( \Re e^{\frac{-qTK}{R}} \leq S \leq R \), we obtain
equation (13) of Chapter II, namely:

\[
S^*(R,K) = \begin{cases} 
R \cdot \exp\{-qTK\} & \text{if } R > s(K) \exp\{qTK\} \\
S(K) & \text{if } R \text{ intermediate} \\
R & \text{if } R < s(K) 
\end{cases}
\]

Thus the two policy functions \( h(S) \) and \( s(K) \) are sufficient to determine the synthesis of our stochastic control problem. Since the heuristic analysis has followed so closely the deterministic case, resulting in similar policy functions \( h(S) \) and \( s(K) \), we expect the numerical results to mimic the deterministic case, at least qualitatively. The important question then, is the extent to which randomness affects the quantitative aspects of the optimal policies; this will be addressed in section E. First, we describe the numerical scheme used to produce our main results.

D. Numerical Method

The dynamic programming equation (1) cannot be solved analytically, so one must resort to numerical methods. As in the previous chapter, policy iteration is used to derive the optimal functions \( h(\cdot) \) and \( s(\cdot) \). However in this chapter we develop a more accurate numerical method, to obtain a consistent solution using more of the information contained in the dynamic programming equation.

In Chapter II, a 2-dimensional mesh was set up in the positive quadrant of the \( R-K \) plane. For a given pair of policy functions, the objective was to determine the matrix of values \( V \). A smooth surface was then formulated to fit these points, \( V_{ij} \) and the matrices of partial derivatives, \( V_{(R)ij} \) and \( V_{(K)ij} \).
calculated. These were then used in the policy improvement stage to determine the optimal policies for the current value function. These new (improved) policies were then used to calculate a new value function.

This method, while accurate for a sufficiently dense mesh, suffers from the disadvantage that not all available information is being used. Specifically, equations involving the partial derivatives of the value function can be obtained by differentiating equation (1) with respect to R and K (in turn). Thus at each mesh point, one desires to determine 3 unknowns, namely \( V_{ij} \), \( V_{(R)ij} \) and \( V_{(K)ij} \). This is a consistent method for calculating the partial derivatives directly, rather than deducing them indirectly as in Chapter II.

A second improvement can be obtained by writing \( V(\cdot,\cdot,\cdot) \) as a product of cubics in \( R \) and in \( K \), with coefficients depending on the mesh point being considered. This ensures that a \( C^1 \) surface will be generated in \( R-K \) space. Finally, as pointed out by Ludwig (1979a), when faced with lognormal stochastic effects, it is convenient to change variables and deal with the natural logarithm of the biomass in place of the biomass per se as a state variable. This latter change simplifies the numerical work but naturally does not affect the results—indeed the results presented in the following section have been transformed back to the original variables for ease of understanding.

We now diverge from the discussion of the numerical method to examine heuristically the expected optimal policies obtained using \( x = \log(R) \) in place of \( R \), \( \xi = \log(S) \) in place of \( S \), and
defining the transformed log(stock)-log(recruitment) function
\( f(\xi) = \log(F(e^\xi)) \). The value function can then be written \( V(x,K) \)
and is given by the equation:
\[
(19) \quad V(x,K) = \max_{x-qTK \leq \xi \leq x} \max_{I \geq 0} \left[ \pi(x,K,\xi,I) + \alpha \mathbb{E}[V(x',(1-\gamma)K+I)] \right]
\]
where \( \pi(x,K,\xi,I) = p(e^{-e}) - (c/q)(x-\xi) - \delta I \), and the
expectation is with respect to \( x' \sim \mathcal{N}(f(\xi)-\sigma^2/2, \sigma^2) \).
Performing the maximizations on the right hand side, we obtain
the optimality relationships:
\[
(20) \quad \mathbb{E}[V(x',f(\xi)-\sigma^2/2+z,(1-\gamma)K+I)] = \frac{\delta}{\sigma} \text{ or } I = 0 \text{ if } \mathbb{E}[V] < \frac{\delta}{\sigma} ;
\]
\[
\mathbb{E}[V(x',f(\xi)-\sigma^2/2+z,(1-\gamma)K+I*)] = p(e^{-x_0})/f'(\xi)
\]
By the reasoning of Section C, we deduce the existence of
optimal policy functions \( u(\xi) \) and \( w(K) \) such that:
\[
\xi^*(x,K) = \begin{cases}
    x \geq qTK ; x > w(K) + qTK \\
    w(K) ; \text{otherwise}
\end{cases}
\]
\[
\xi^* = \mathbb{I}^*(\xi,K) = \max\{u(\xi)-(1-\gamma)K,0\}
\]
represent the optimal log(escapement) as a function of the state
\((x,K)\) and the optimal investment as a function of
log(escapement) and current capacity, respectively. The
behaviour of the policy function \( u(\cdot) \) can be described in part
by setting \((1-\gamma)K+I=0 \text{ or } u(\xi) \) in (20) and differentiating with
respect to \( \xi \), to produce:
\[
(21) \quad u'(\xi) = f'(\xi) \cdot \mathbb{E}[V(x',u(\xi))]/\mathbb{E}[V(x',u(\xi))] > 0
\]
where again we have assumed \( V > 0 \) and \( V < 0 \). The concavity of
\( u(\cdot) \) cannot be deduced directly from (21), but would seem to be
likely if $f''<0$ (c.f. Chapter II). However, since $f(\xi) = \log(F(e^\xi))$, it can be shown that:

$$f''(\xi) = (S^2/F(S))[F'(S)[(F(S)-S)/SF(S)] + F''(S)]$$

where $S=e^\xi$. This indicates that the concavity of $f$ is ambiguous, since the first term in the parentheses is positive while the second is negative, unless $S$ is so large that $F(S)<S$, in which case we clearly have $f''(\xi)<0$.

For the Beverton-Holt stock-recruitment function, this expression for $f''(\xi)$ can be considerably simplified. Using $F(S)=aS/(1+dS)$, where $d=a/b$, it is easily shown that:

$$f''(\xi) = -[(d^2S^2+(2+a)dS-(a-1))/(a(1+dS)^2)]$$

Since $a>1$, we can conclude that $f$ will be concave whenever the quadratic within the parentheses is positive. This will be the case for all $S>S_*$, the positive solution of $d^2S^2+(2+a)dS-(a-1)=0$. For example, in the base case prawn fishery, $f$ will be concave for all $S>0.877\times 10^6$, and with $a=14$ this result holds for all $S>0.48$ (i.e. essentially all $S$-values).

Hence in general we can expect to find the optimal capacity function increasing throughout, but concavity may be restricted to some range $S>S_*$, which may in fact include the relevant range of escapements.

We return now to a detailed discussion of the numerical method used (i) to obtain the value function corresponding to a given pair of policy functions, and (ii) to determine the optimal policies for a given value function. These two steps
performed consecutively will converge to the optimal policies and value function.

We shall first describe the procedure in step (i), to solve for the value function given any pair of policy functions. The initial step is to form a mesh of $M$ values in the $x$-direction and a mesh of $N$ values in the $K$-direction; these mesh points can be chosen more or less arbitrarily, but numerical difficulties might be expected if the meshes are very non-uniform. As in Chapter II, the lower limit of the $K$-values was set at $K=0$ in every case. The upper $K$-value was set at a convenient value above the optimal capacity level determined in Section B, since this represents the maximum possible level to which a manager would invest, occurring only in the extreme case of stock/recruitment independence. The upper and lower limits of the $x$-mesh must be determined by examining the maximum and minimum "reasonable" recruitments which can arise given stock-recruitment parameters and the maximum level of noise to be considered. For example, in the case of the stochastic ($\sigma=0.58$) prawn fishery discussed in section E, suitable limits for the $x$-mesh were $x_-=12.206$ and $x_+=16.811$ (corresponding to $R_-=0.2\times10^6$ and $R_+=20.0\times10^6$). While one wishes to include "most" of the $x$-line within the mesh, in theory arbitrarily large and/or small recruitments are possible (with lognormal noise), so it becomes necessary to approximate the value function for $x$-values above $x_+$ and below $x_-$. These extrapolation schemes are described below. Varying the $x$-mesh limits $x_-$ and $x_+$ naturally has some effect on the precise location of the optimal policies.
h(S) and s(K). However, experiments with several x-meshes indicated that quantitative changes in the results were minor, and the qualitative conclusions were never affected.

Given any policy functions w(K) and u(\xi), the value function satisfies the following integral equation:

\[
V(x,K) = \pi(x,K,\xi(x,K),\Max\{u(\xi(x,K))-(1-\gamma)K,0\})
+ \alpha E\{V(f[\xi(x,K)]-\sigma^2/2+z,\Max\{u(\xi(x,K)),(1-\gamma)K\})\}
\]

where \( z \sim N(0,\sigma^2) \) and \( \xi(x,K) = \begin{cases} x-qTK & x>\Min\{\lambda,K\}+qTK \\ \Min\{\lambda,K\} & \text{otherwise} \end{cases} \)

This equation must hold for all values of x and K, but may not be everywhere differentiable. However at suitable points, one may differentiate both sides of (22) with respect to x and with respect to K, to obtain two new equations (where we define \( K'=\Max\{u(\xi(x,K)),(1-\gamma)K\} \) and \( I(\xi,K)=\Max\{u(\xi)-(1-\gamma)K,0\} \)):

\[
V(x,K) = (\pi + \pi \xi + \pi I \xi x)
+ \alpha[f'(\xi(x,K))\xi ] \cdot E\{V(f(\xi)-\sigma^2/2+z,K')\}
\]

\[
+ \alpha[I \xi ] \cdot E\{V(f(\xi)-\sigma^2/2+z,K')\}
\]

\[
(24) \quad V(x,K) = (\pi \xi + \pi I \xi + \pi I K)
+ \alpha[f'(\xi(x,K))\xi ] \cdot E\{V(f(\xi)-\sigma^2/2+z,K')\}
\]

\[
+ \alpha[(1-\gamma)+I \xi + I K] \cdot E\{V(f(\xi)-\sigma^2/2+z,K')\}
\]
From the definition of $\xi$, we observe that:

$$
\xi(x,K) = \begin{cases} 
0 & \text{if } x \leq w(K) + qTK < w(K) \\
1 & \text{otherwise}
\end{cases}
$$

$$
I(\xi,K) = \begin{cases} 
0 & \text{if } K = u(\xi)/(1-\gamma) \\
\frac{dh}{d\xi} & \text{if } K \neq u(\xi)/(1-\gamma)
\end{cases}
$$

Now, assuming that $w(K)$ and $u(\xi)$ are both $C^1$, as appears reasonable from section C, we can conclude that $V(x,K)$ is differentiable, so that (23) and (24) are valid, everywhere except possibly along the curves $x = w(K)$, $x = w(K) + qTK$ and $K = u(\xi(x,K))/(1-\gamma)$, where the policy functions change their form (c.f. CCM, p.45). Since equations (22)-(24) are valid for almost all points in the positive $x-K$ quadrant, one would hope that they would hold in particular at every mesh point $(x_i, K_j)$. If we are so unlucky as to have one of the 3 "switching curves" passing through a mesh point, we have adopted the rather arbitrary convention of using (22)-(24) with the appropriate left-hand or right-hand partial derivatives in place of $\xi$, $\xi_x$, $\xi_K$, $I_\xi$, and $I_K$. (A partial inspection of the numerical results did not uncover occasions where this measure had to be taken.)

We thus have $3MN$ equations, from which we desire to obtain the $3MN$ values representing $V(x_i, K_j)$, $V(x_i, K_j)$ and $V(x_i, K_j)$ at each mesh point. This apparently simple task is complicated by the existence of a further set of $MN$ unknowns, to be discussed below.

Note at this point that these equations can be easily
modified to the deterministic case by setting \( \sigma = 0, \ z = 0 \) and removing the expectation operator. In order to compare results for the stochastic and deterministic cases, we shall develop the numerical method for the two possibilities concurrently; few changes will be required.

Consider the rectangle in \( x-K \) space defined by the 4 corners \( (x, K_i), (x, K_{i+1}), (x, K_{j+1}), (x, K_j) \). Our first objective is to define the value function on this rectangle in terms of data at the corners and such that \( V \) is continuous and has continuous first partial derivatives across each edge; this is desired so that the policy improvement process makes sense. Following Ludwig (1979b), we define functions \( g_i(x) \) as follows:

\[
\begin{align*}
g_{1i}(x) &= (3/\Delta^2)(x-x_i)^2 + (2/\Delta^3)(x-x_{i+1})^3 \\
g_{2i}(x) &= (1/\Delta^2)(x-x_i)^3 + (1/\Delta)(x-x_{i+1})^2 \\
g_{3i}(x) &= (3/\Delta^2)(x-x_i)^2 - (2/\Delta^3)(x-x_{i+1})^3 \\
g_{4i}(x) &= (1/\Delta^2)(x-x_i)^3 - (1/\Delta)(x-x_{i+1})^2
\end{align*}
\]

where \( \Delta = x_{i+1} - x_i \). Then \( g_{1i}(x) = 1, \ g'(x_i) = 0, \ g(x_{i+1}) = 0 \) and \( g'(x_i) = 0 \), and similarly for the other functions. These \( g_i(x) \) functions can be rewritten as follows:

\[
\begin{align*}
g_{\text{ni}}(x) &= \sum_{\gamma=1}^{4} \sum_{\gamma=1}^{4-\gamma} g(i, n, \gamma) x \\
&= \sum_{\gamma=1}^{4} \sum_{\gamma=1}^{4-\gamma} g(i, n, \gamma) x
\end{align*}
\]

for \( i = 1, \ldots, M \) and \( n = 1, 2, 3, 4 \). A straightforward exercise in
algebra produces the values of the $g(x,\cdot,\cdot,\cdot)$ terms, displayed in Table (III). Now we define functions $h(K)$ in an exactly analogous way, substituting $K$ for $x$, $j$ for $i$, and letting $\Delta=K(j+1)-K(j)$. Finally for notational convenience we adopt a convention for depicting the unknowns $V, V', V$ and $V^{'}$ at the mesh points on the rectangle:

$$v_{11}=V(ij), \\ v_{12}=V(ij), \\ v_{13}=V(i,j+1), \\ v_{14}=V(i,j+1)$$

$$v_{21}=V(ij), \\ v_{22}=V(ij), \\ v_{23}=V(i,j+1), \\ v_{24}=V(i,j+1)$$

and the equivalent definitions for $v_{3\cdot}$ and $v_{4\cdot}$, with $i$ replaced by $i+1$.

Let us note at this stage that if our surface is to be represented by a product of cubics in $x$ and in $K$, each cubic containing 4 unknown constants, we must of necessity deal with $4\times4=16$ unknowns on the rectangle, representing 4 unknowns per corner. This corresponds to $4MN$ unknowns in all, and yet we have only $3MN$ equations. In theory this problem can be overcome in a very straightforward manner by differentiating equation (24) with respect to $x$ to obtain a fourth set of equations for the cross partial derivative. Unfortunately this produces a very complicated equation involving the second partial derivative with respect to $x$, so that in effect one must obtain equations for all the second partial derivatives. To avoid this problem, we shall keep the number of equations at $3MN$ and instead reduce the number of unknowns by approximating $V^{'}$ terms by the slope
of cubics in the x-direction fitted to values of $V_K$. This is described further below.

With the above definitions we are now in a position to define the value function on the $(i,j)$th rectangle:

$$V_{x}(x,K) = \sum_{i,j} \sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} g(i,\alpha,\gamma) g(j,\beta,\delta) x^{4-\gamma} x^{4-\delta} (\alpha \beta)$$

By simply differentiating this value function with respect to either $x$ or $K$, we can deduce the expressions for $V_x$ and $V_K$ on the rectangle:

$$V_{x}(x,K) = \sum_{i,j} \sum_{\alpha=1}^{4} \sum_{\beta=1}^{3} g(i,\alpha,\gamma) g(j,\beta,\delta) x^{3-\gamma} x^{4-\delta} (\alpha \beta)$$

$$V_{K}(x,K) = \sum_{i,j} \sum_{\alpha=1}^{4} \sum_{\beta=1}^{3} g(i,\alpha,\gamma) g(j,\beta,\delta) x^{4-\gamma} x^{3-\delta} (\alpha \beta)$$

The derivation of the above 3 equations was independent of the stochastic nature of the problem and hence the equations apply in the deterministic case as well. In fact, if $i$ and $j$ are chosen such that $x \leq f(\xi) \leq x$ and $K \leq K' = (1-\gamma)K + I(\xi, K) \leq K$, then the future value of the fishery, given current escapement $\xi$ and current capacity $K$, is given by $V_{ij}(f(\xi), K')$ in the $\sigma=0$ case.

Before proceeding further we take up the question of extrapolations to high ($x>x_\ast$) and low ($x<x_\ast$) recruitment values in the stochastic case. First consider the range $x>x_\ast$.

Assume that $\xi(x,K)=x-qTK$ for $x>x_\ast$, so that harvesting is at full
capacity if the recruitment is sufficiently high. Assume also that the capacity desired for next season, \( u(x-qTK) \), is approximately equal to \( u(x,-qTK) \) for \( K<u(x,-qTK) \), i.e. \( u(\cdot) \) is flat for \( x>x_*-qTu(x,-qTK) \). Together these assumptions imply that the optimal investment and capacity for next season are given by the corresponding values at \( x_* \). Assume in addition that \( f(x-qTK)=f(x,-qTK) \), a reasonable assumption for a Beverton-Holt model, since the function \( f \) approaches an asymptote at high \( x \)-values. [Note, however, that the assumption that \( f \) and \( u \) are flat for \( x>x_*-qTK \), while likely to be accurate for low \( K \)-values, will be less accurate as \( K \) is increased. Hence the values \( x_* \), \( K \), should be chosen so that \( x_*-qTK \) is sufficiently large.]

Then we have the following equations for \( V(x,K) \) and \( V(x_*,K) \):

\[
V(x,K) = \pi(x,K,I(x,K),u(x,K)) + \alpha\mathbb{E}\{\int_{x,K} \xi(x_*,K) \}
\]

\[
V(x_*,K) = \pi(x_*,K,I(x_*,K),u(x_*,K)) + \alpha\mathbb{E}\{\int_{x,K} \xi(x_*,K) \}
\]

where \( \pi \) is given by \( \pi(x,K,I)=p(e^{-e^{-x}})-cTK-6I \), and \( I = \text{Max}\{u(x,-qTK)-(1-K)K,0\} \). Hence \( V(x,K) \) can be approximated by:

\[
V(x,K) \approx V(x_*,K) + p(1-e^{-x_*)}(e^{-e^{-x}})
\]

Clearly it also follows that \( V(x,K) \approx p(1-e^{-x_*})e^{-qTKx} \) and

\[
V(x,K) \approx V(x_*,K) + pqTe^{-qTKx}(e^{-e^{-x}}). \text{ The integral from } x_* \text{ to } K \text{ infinity in equation (22) can now be written as follows:}
\]
(27) \[ \phi(x')[V(x_*,K') + p(1-\exp{-qTK'})](\exp{x'} - \exp{x_+})]dx' \]
\[\int_{x_+}^{x} \]
\[= [1-\Phi((x_+ - \bar{x})/\sigma)]V(x_*,K') \]
\[+ p(1-\exp{-qTK'})\exp{\bar{x}+\sigma^2/2}[1-\Phi((x_+ - \bar{x} - \sigma^2)/\sigma)] \]
\[-e^{-\bar{x} [1-\Phi((x_+ - \bar{x})/\sigma)]} \text{ where } \bar{x} = f(\xi) - \sigma^2/2. \]

Similarly, the integrals involving \( V \) and \( V \) become:
\[\int_{x}^{K} \]
(28) \[ \phi(x')V(x',K')dx' \]
\[\int_{x_+}^{x} \]
\[= p(1-\exp{-qTK'})\exp{\bar{x}+\sigma^2/2}[1-\Phi((x_+ - \bar{x} - \sigma^2)/\sigma)] \]

(29) \[ \phi(x')V(x',K')dx' \]
\[\int_{x_+}^{x} \]
\[= [1-\Phi((x_+ - \bar{x})/\sigma)]V(x_*,K') + pqT\exp{-qTK'} \cdot \]
\[\int_{x_+}^{x} \]
\[\exp{\bar{x}+\sigma^2/2}[1-\Phi((x_+ - \bar{x} - \sigma^2)/\sigma)] - e^{-\bar{x} [1-\Phi((x_+ - \bar{x})/\sigma)]} \]

Now consider the approximation of the value function for values of \( x < x_+ \). Assume that \( x_+ \) is chosen sufficiently low so that for any \( x < x_+ \), the escapement function satisfies \( \xi(x,K) = x \) and the required investment \( I(\xi,K) = 0 \) for all values of \( K \). [The latter condition implicitly assumes that, for given \( x_+ \), the parameter \( a \) in the stock-recruitment function is not too large, so that \( u(\xi) = 0 \) for \( \xi < x_+ \). To deal with large \( a \)-values, the value of \( x_+ \) can be decreased accordingly. For the extreme case of \( a = \infty \), the approximation will be somewhat inaccurate, but
this will have negligible effect if \( x \) is sufficiently small.]

Then the dynamic programming equation for \( V \) in this case is:

\[
(30) \quad V(x, r, y) = \min_{f(x)} \left\{ \int_{x'} f(x') V(x', (1-r)K) \right\} dx'
\]

For \( x \) small enough, \( x' \) will also be small for 'most' \( z \)-values. Thus, following Ludwig and Walters (1982) we assume linear population dynamics: \( F(S) = aS/(1+aS/b) \) "=" \( aS \). This produces the approximation \( f(x) = \log(a) + x \), or \( f(x) = x + (f(x) - x) \) since \( f(x) = \lambda x \). Assume that \( V \) can be represented by \( V(x, r, y) = \lambda x \) for small \( x \). Then we have the following (see Ludwig and Walters, 1982):

\[
(31) \quad V(K) = \int_{x} \min_{f(x)} \left\{ \int_{x'} \lambda x \right\} dx'
\]

where the integrals are over the real line. Simplifying, we obtain:

\[
\exp\left( -\frac{\lambda^2 \sigma^2}{2} - \frac{\lambda \sigma^2}{2} + \lambda \left[ f(x) - x \right] \right) = \frac{v(K)}{v([1-r]K)}
\]

Unlike the situation in Ludwig and Walters, our model involves capital dynamics, which leads to the problem of dealing with the term \( v(K)/v([1-r]K) \). For the purposes of this extrapolation we use the simplification (only required at low \( x \)-values) that \( v(K) \) is proportional to \( K \); hence we obtain \( v(K)/v([1-r]K) = 1/(1-r) \).

Thus the coefficient \( \lambda \) satisfies:

\[
(\sigma^2/2)\lambda^2 + [f(x) - x - \sigma^2/2] \lambda + \log(\sigma(1-r)) = 0
\]

which can be seen to have exactly one positive root \( \lambda^* \) as desired. The integral between \(-\infty\) and \( x \) can now be approximated as follows:
(32) \[ \int_{x<x_0} \phi(x')v(x',K')dx' \]
\[ = v(K') \int_{x<x_0} \phi(x') \exp\{\lambda x'\} dx' \]
\[ = v(K') \exp\{\sigma^2/2\lambda^2 + \overline{x}\lambda\} \phi((x_0 - \overline{x} - \lambda^2)/(2\sigma)) \]
\[ = V(x_0, K') \exp\{\sigma^2/2\lambda^2 + (\overline{x} - x_0)\lambda\} \phi((x_0 - \overline{x} - \lambda^2)/(2\sigma)) \]

where we have used the property:

\[ v(K) = e^{-\lambda x}. \]

\[ v(K) = v(x_0, K) \]

The corresponding integrals involving $V$ and $V$ are also given by (32), but with $V$ replaced by $V$ and $V$ respectively. This can be seen by use of the following consequences of our approximation method:

(i) \[ V(x, K) = \int_{x_0}^{x} [\lambda V(K)] e^{-\lambda x} dx \]

(ii) \[ V(x, K) = v(K) e^{-\lambda x} \]

(iii) \[ [\lambda V(K')] = e^{-\lambda x} V(x, K) \]

(iv) \[ v'(K) = e^{-\lambda x} V(x, K) \]

This completes the extrapolations to high and low recruitment values. We remark again that the use of a logarithmic variable $x = \log(R)$ has the advantage that very low and very high recruitments can be included in a fairly uniform mesh by judicious choice of $x_0$ and $x_1$, and hence the $x$-regions where the extrapolations apply can be ones of very low probability.
Note that the extrapolations for the integrals from -infinity to \( x \) and from \( x \) to infinity include only the unknown values \( V(x, \cdot) \), \( V(x, \cdot) \), \( V(x, \cdot) \), \( V(x, \cdot) \) and \( V(x, \cdot) \), together with some known constants. Hence these \( K \) extrapolations can be included within the system of equations for \( V \), \( V \) and \( V \) at the mesh points by suitably altering the \( x \) coefficients found above. The numerical constants can be included in the expressions for \( \pi \), \( \pi \) and \( \pi \). Thus these regions \( x \) \( K \) of high and low \( x \) need not concern us further. (Of course, in the \( \sigma = 0 \) case, these extrapolation quantities are set equal to zero as desired.)

The equations for \( V \), \( V \) and \( V \) can now be summarized as follows:

\[
\begin{align*}
(33) \quad \sigma^{-1}V(x,K) &= \sigma^{-1} \pi + \sum_{k=1}^{M-1} \int_{x(k)}^{x(k+1)} \phi(x')V(x',K')dx' \\
(34) \quad \sigma^{-1}V(x,K) &= \sigma^{-1} \pi + \sum_{k=1}^{M-1} \int_{x(k)}^{x(k+1)} \phi(x')V(x',K')dx' \\
&+ T_2 \left[ \int_{x(k)}^{x(k+1)} \phi(x')V(x',K')dx' \right] \\
(35) \quad \sigma^{-1}V(x,K) &= \sigma^{-1} \pi + \sum_{k=1}^{M-1} \int_{x(k)}^{x(k+1)} \phi(x')V(x',K')dx' \\
&+ T_4 \left[ \int_{x(k)}^{x(k+1)} \phi(x')V(x',K')dx' \right]
\end{align*}
\]

where \( T_1, T_2, T_3, T_4 \) are given by:
\[ T_1 = f'(\xi(x,K))\xi_x \]

\[ T_2 = I\xi \xi_x \]

\[ T_3 = f'(\xi(x,K))\xi_K \]

\[ T_4 = \{(1-\gamma)+I\xi+I\}^{\xi_K}_K \]

[In the deterministic case \( \phi \) is given by the delta function centred at \( \bar{x} = f(\xi(x,K)) \)]. Defining \( l \) by \( K \leq K' < K \) and replacing \( j \) with \( l \) in equation (25) and the corresponding expressions for \( V_{ij}^{(x)} \) and \( V_{ij}^{(K)} \), the integrals of \( \phi V \) and \( \phi V \) in the interval \((x_k, x_{k+1})\) can be written in terms of integrals of \( \phi x \) for \( n = 1, 2, 3, 4 \). For example:

\[
(36) \quad |\phi(x')V(x',K)dx'| = \sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} \sum_{\gamma=1}^{4} \sum_{\delta=1}^{4} g(i, \alpha, \gamma)g(j, \beta, \delta) \int_{x_k}^{x} \int_{x_{k+1}}^{x_{K'}} \int_{\bar{x}}^{4-\delta(\alpha\beta)} \int_{(K')}^{4-\gamma} \phi(x')(x') dx' 
\]

Define \( \text{XINT}(k,i) = |\phi(x')(x')dx' \) where the lower and upper limits of integration are \( x_k \) and \( x_{k+1} \), respectively. These quantities can be evaluated explicitly, in terms of cumulative distribution function values. To accomplish this, make the following definitions:
\[ y(k) = (x(k) - \bar{x})/\sigma \] for each \( k \).

\[ A(k) = \Phi(y(k+1)) - \Phi(y(k)) \]

\[ B(k) = \exp\{ -y(k)^2/2 \} - \exp\{ -(y(k+1)^2)/2 \} \]

\[ C(k) = y(k) \cdot \exp\{ -y(k)^2/2 \} - y(k+1) \cdot \exp\{ -(y(k+1)^2)/2 \} \]

\[ D(k) = y(k)^2 \cdot \exp\{ -y(k)^2/2 \} - y(k+1)^2 \cdot \exp\{ -(y(k+1)^2)/2 \} \]

Then we have:

\[ \text{XINT}(k,0) = A(k) \]

\[ \text{XINT}(k,1) = (\sigma/\nu)B(k) + \bar{x}A(k) \]

\[ \text{XINT}(k,2) = (\sigma^2/\nu)C(k) + (2\sigma \bar{x}/\nu)B(k) + (\sigma^2 + \bar{x}^2)A(k) \]

\[ \text{XINT}(k,3) = (\sigma^3/\nu)D(k) + (3\sigma^2 \bar{x}/\nu)C(k) + ((2\sigma^3 + 3\sigma \bar{x}^2)/\nu)B(k) + (3\sigma^2 \bar{x}^2 + \bar{x}^3)A(k) \]

(Note that in the deterministic case we have \( \text{XINT}(k,i) = x \) if \( x \leq \bar{x} < x \) and \( = 0 \) otherwise. From this point on, the deterministic and stochastic numerical methods for part (i) are identical.)

Now define \( G, \) \( GPR, \) \( H, \) \( HPR, \) \( GH, \) \( GPRH \) and \( GHPR \) as follows:

\[ G(k, \sigma) = \sum_{\gamma=1}^{4} g(k, \sigma, \gamma) \text{XINT}(k, 4-\gamma) \]

\[ \text{GPR}(k, \sigma) = \sum_{\gamma=1}^{3} g(k, \sigma, \gamma) \text{XINT}(k, 3-\gamma) \cdot (4-\gamma) \]

\[ H(1, \rho) = \sum_{\delta=1}^{4} g(1, \rho, \delta)(K')^{4-\delta} \]

\[ \text{HPR}(1, \rho) = \sum_{\delta=1}^{3} g(1, \rho, \delta)(K')^{3-\delta} \cdot (4-\delta) \]
GH(k, l, α, β) = G(k, α)H(l, β)
GPRH(k, l, α, β) = GPR(k, α)H(l, β)
GHPR(k, l, α, β) = G(k, α)HPR(l, β)

Then for any values of x and K we have the equations:

\[(37) \quad \alpha^{-1}V(x, K) = \alpha^{-1}x + \sum_{k=1}^{M-1} T_1 \Sigma \Sigma \frac{G(k, \alpha, \beta) v_{kl}}{\beta = 1} \]

\[(38) \quad \alpha^{-1}V(x, K) = \alpha^{-1}x + \sum_{k=1}^{M-1} T_2 \Sigma \Sigma \frac{GPR(k, \alpha, \beta) v_{kl}}{\beta = 1} \]

\[(39) \quad \alpha^{-1}V(x, K) = \alpha^{-1}x + \sum_{k=1}^{M-1} T_3 \Sigma \Sigma \frac{GHPR(k, \alpha, \beta) v_{kl}}{\beta = 1} \]

where the T values are as above. For completeness, we must set

GH(M, α, β) = 0, GHPR(M, α, β) = 0 and GPRH(M, α, β) = 0 for all α and β.

The next step is to combine the GH, GPRH and GHPR terms suitably

\[(a\beta)\]

so that each v term occurs only once on the right hand sides

kl

of each of equations (37)-(39). This is the purpose of the following, where \(\beta = 1, 2, 3, 4\) and \(\alpha = 1, 2\):

co(1, k, α, β) = GH(k, α, β) + GH(k-1, α+2, β)

co(2, k, α, β) = GPRH(k, α, β) + GPRH(k-1, α+2, β)

co(3, k, α, β) = GHPR(k, α, β) + GHPR(k-1, α+2, β)

for \(k = 2, \ldots, M\) while for \(k = 1\) we have:
\[ c_1(1, \alpha, \beta) = \text{GH}(1, \alpha, \beta) \]
\[ c_2(2, \alpha, \beta) = \text{GPRH}(1, \alpha, \beta) \]
\[ c_3(3, \alpha, \beta) = \text{GHPR}(1, \alpha, \beta) \]

To this point we have maintained 4MN unknowns, including the cross partial derivatives of \( V \) at the mesh points. We now approximate \( V \) at \((i,j)\) by the slope of the cubic at \((x_{K})\) fitted through the points \((x_{i-1}, V_{i-1,j}), (x_{i}, V_{i,j})\) and \((x_{i+1}, V_{i+1,j})\). The basic equations used are as follows:

\[ V_{xK} = c_1(1) V_{xK} + c_2(1) V_{xK} + c_3(1) V_{xK} \]
\[ (x_{K})_{11} \quad (K)_{11} \quad (K)_{21} \quad (K)_{31} \]

\[ V_{xM} = c_1(M) V_{xM} + c_2(M) V_{xM} + c_3(M) V_{xM} \]
\[ (x_{K})_{M1} \quad (K)_{M-2,1} \quad (K)_{M-1,1} \quad (K)_{M,1} \]

where, letting \( \Delta = x_{i+1} - x_{i} \), the \( c_i(\cdot) \) values are given by:

\[ c_1(1) = -(2\Delta_1 + \Delta_2)/(\Delta_1[\Delta_1+\Delta_2]) \]
\[ c_2(1) = (1/\Delta_1) + (1/\Delta_2) \]
\[ c_3(1) = -\Delta_1(\Delta_2[\Delta_1+\Delta_2]) \]
\[ c_1(M) = \Delta(M-1)/(\Delta(M-2)[\Delta(M-2)+\Delta(M-1)]) \]
\[ c_2(M) = -(1/\Delta(M-2))-(1/\Delta(M-1)) \]
\[ c_3(M) = [2\Delta(M-1)+\Delta(M-2)]/(\Delta(M-1)[\Delta(M-2)+\Delta(M-1)]) \]

At the end points \( x_1 \) and \( x_m \), the cubic obtained using the endpoint and its two nearest neighbours is used. In each case the coefficients \( c_0(\cdot,\cdot,\cdot,\cdot) \) must be modified accordingly. Now a simple reordering of the coefficients is applied for notational convenience:
\[ co'(j,k,1,1) = co(j,k,1,1) \]
\[ co'(j,k,1,2) = co(j,k,2,1) \]
\[ co'(j,k,1,3) = co(j,k,1,2) \]
\[ co'(j,k,2,1) = co(j,k,1,3) \]
\[ co'(j,k,2,2) = co(j,k,2,3) \]
\[ co'(j,k,2,3) = co(j,k,1,4) \]

where \( j = 1, 2, 3 \) and \( k = 1, \ldots, M \). Finally we define the coefficients \( coef(\cdot, \cdot, \cdot, \cdot) \) as follows:

\[ coef(1, k, m, n) = co'(1, k, m, n) \]
\[ coef(2, k, m, n) = T_1 co'(2, k, m, n) + T_2 co'(3, k, m, n) \]
\[ coef(3, k, m, n) = T_3 co'(2, k, m, n) + T_4 co'(3, k, m, n) \]

for \( k = 1, \ldots, M \) and \( m = 1, 2 \) and \( n = 1, 2, 3 \). This produces the final form of the equation relating \( V \) at any point \((x, K)\) to the \( 6M \) unknowns \( v_{kl}, v_{(x)kl}, v_{(K)kl}, v_{k,l+1}, v_{(x)k,l+1}, v_{(K)k,l+1} \) \((k=1, \ldots, M)\), where \( l \) is such that \( K < K'(x, K) < K \):

\[ V(x, K) = \sum_{k=1}^{M} \left( coef(1, k, 1, 1)v_{kl} + coef(1, k, 1, 2)v_{(x)kl} + coef(1, k, 1, 3)v_{(K)kl} + coef(1, k, 2, 1)v_{k,l+1} + coef(1, k, 2, 2)v_{(x)k,l+1} + coef(1, k, 2, 3)v_{(K)k,l+1} \right) \]

Precisely analogous equations for \( \sigma^{-1} V(x, K) \) and \( \sigma^{-1} V(x, K) \) are obtained simply by replacing \( \pi \) by \( \pi \) or \( \pi \) respectively, and replacing \( coef(1, \cdot, \cdot, \cdot) \) with \( coef(2, \cdot, \cdot, \cdot) \) or \( coef(3, \cdot, \cdot, \cdot) \) respectively throughout equation (40).

Choosing \( x = x_i \) and \( K = K_j \) for \( i = 1, \ldots, M \) and \( j = 1, \ldots, N \) now gives us
3MN equations in the 3MN unknowns \( V \), \( V \), and \( V \); the 
\( \text{xij} \) \( \text{(x)ij} \) \( \text{(K)ij} \) 
solution of part (i) of our numerical problem is now drawing near.

Consider now the choice of \( M \) and \( N \), the number of mesh 
points in the x- and K- directions respectively. In chapter II, 
we used \( M=N=30 \), which provided a suitable balance between cost 
and accuracy with the numerical method used there. On the other 
hand, Ludwig (1979b) found that \( M=5 \) was sufficient for his 
one-dimensional cubic spline method. In this chapter we adopt 
the values \( M=N=8 \), which results in computing costs and accuracy 
comparable (or superior) to the case \( M=N=30 \) of chapter II. The 
sensitivity of the solution to the values of \( M \) and \( N \) was tested 
by producing results with \( M=N=12 \). These results were very 
expensive to obtain, requiring up to 80 seconds of CPU time and 
1000 page-minutes of CPU storage VMI, to obtain one pair of 
optimal policy functions (as opposed to 17 seconds CPU time and 
50 page-minutes of CPU storage VMI in the \( M=N=8 \) case). For the 
base case prawn fishery, the maximum difference between the 
\( M=N=8 \) and \( M=N=12 \) results for the \( u(S) \) curves was approximately 
1.4 percent for both \( \sigma=0 \) and \( \sigma=0.58 \), while the \( w(K) \) curves 
differed negligibly. For the whale fishery, differences were 
more substantial: up to 6 percent for the \( u(\xi) \) curve and 11 
percent for the \( w(K) \) function. It appeared that an \( M=N=8 \) mesh 
was sufficient for all but the whale fishery results, for which 
\( M=N=12 \) was used.

The 3MN by 3MN system of linear equations which must now be 
solved has the feature that in any row at most \( 2 \cdot 3 \cdot M+1 \) of the
entries can be nonzero, since for given \((i,j)\) only entries with \(K=K\) or \(K=K^1\) are of interest, corresponding to the \(V\), \(V^l\) and \(x^l\) terms at next season's level of capital stock. If \(3MN\) is \(K\) large compared to \(6M+1\), sparse matrix methods can be used to advantage to solve the system. However \(6M/(3MN) = (2/N)\) so to have as few as 10 percent of the entries nonzero would require \(N>20\). Since the \(x\)-mesh is at least as critical as the \(K\)-mesh (since we are approximating a doubly infinite \(x\)-interval), we would want \(M>N>20\). The resulting system of equations would be at least \(1200x1200\), which while manageable is rather cumbersome. In any case, as discussed above, the expected accuracy of our cubic spline method (as suggested by the results of Ludwig and Walters, 1982) indicates that a much coarser mesh provides sufficient accuracy, so that sparse methods are not particularly suitable. Note in addition that while the simpler system described in Chapter II, involving only the value function at each mesh point, \(V(i,j)\), is diagonally dominant, this is not true in the present case since the sign of the terms involving \(V\) and \(V\) need not be positive and clearly the sum of the terms on the right hand side need not equal one. As well, the system is not and cannot be made block diagonal since both \(x\) and \(K\) change from year to year and the extent of that change depends on the current value of \((x,K)\) in a non-trivial manner. Indeed apart from limited sparseness there appear to be no aspects of the structure of the system that can be utilized to advantage in the solution; we therefore resort to an efficient package
routine for solving general systems $Ax=b$. The package used, SLIMP/DSLIMP, is available on the U.B.C computer system, and is written in both Assembler and Fortran. It uses Gaussian elimination (LU decomposition) with partial pivoting, together with forward and backward substitution. There is a built-in iterative procedure which calculates the residual vector $r=b-Ax_i$ at each step $i$, solves $Ae=r$ for the error vector $e$ and obtains an improved solution $x_{i+1}=x_i+e$. Iterations continue until the error vector satisfies a user-specified convergence criterion. It was found in most cases that fewer than 5 iterations were required to obtain a very accurate solution. An important remark at this point is that our program functions entirely in double precision; initial attempts to use single precision produced misleading results which were correct in some instances but completely wrong in other cases. The problem appeared to arise in the use of SLIMP, the single precision version of the routine, which seems to only produce correct single precision results for relatively small systems.

Once the values of $V$, $V_i$ and $V$ have been obtained at each mesh point, the policy improvement stage can begin.

Improvement of the $u(\xi)$ curve utilizes equation (20); for each value $\xi=x_i$, the objective is to determine the $K'$ which solves the optimality equation:

$$E[V(x,K')] = \delta/\sigma \text{ where } x \text{ has mean value } f(\xi).$$

Then $u(\xi)=K'$. The procedure used was based on the expectation
that $E\{V\}$ decreases with $K$: mesh values $K_1, K_2, K_3, \ldots$ were checked consecutively until $E\{V\} \leq 6/\alpha$. If this occurs at $K_j$, we set $K' = 0$. If it occurs at $K_j$, then we know that $K_{j-1} \leq K_j$.

The solution is approached by alternation of linear interpolation and interval-halving methods; this procedure was used to speed up the rather slow convergence of the linear interpolation method alone. At each step, we have an upper and a lower bound on $K'$, say $K^{(-)}$ and $K^{(+)}$. A test value of $K$, $K(?)$, is obtained either by taking $K(?) = [K^{(-)} + K^{(+)}/2$ or by finding the point where the linear interpolant between $E\{V \cdot, K^{(+)}}$ and $E\{V \cdot, K^{(-)}\}$ is equal to $6/\alpha$. Then $E\{V \}$ is determined at this test value and compared to $6/\alpha$. Depending on the result, $K(?)$ then becomes the new upper or lower bound on $K'$. At the next step, the method to obtain $K(?)$ is alternated; this process is continued until an accuracy criterion $|K'_{i+1} - K_i| < \varepsilon$ is reached.

At each stage, the quantity $E\{V\}$ is found as follows:

$$
E\{V\} = \sum_{k=1}^{M} \{ c(3,k,1,1)v_{k1} + c(3,k,1,2)v_{k1} \}^{(x)k1} + c(3,k,1,3)v_{k1}^{(K)k1} + c(3,k,2,1)v_{k1}^{k,l+1} + c(3,k,2,2)v_{k1}^{(x)k,l+1} + c(3,k,2,3)v_{k1}^{(K)k,l+1} \}
$$

Determination of an improved $w(K)$ curve is accomplished in a similar manner. For each mesh value $K_j$, $\varepsilon$ is increased through
the values \( x_1, x_2, x_3, \ldots \) until we have:

\[
\mathbb{E}\left\{ V(x', K') \right\} < (p/s)(e^{-x_0}/f'(\xi))
\]

where \( K' = \text{Max}\{ u(\xi), (1-\gamma)K \} \), using the new \( u(\xi) \) curve. To narrow in on the solution, alternation of linear interpolation and interval-halving is again used, until a similar accuracy criterion is attained. The quantity \( \mathbb{E}\{V \} \) is calculated using equation (41), with \( c_0'(3, \ldots, \cdot) \) replaced by \( c_0'(2, \ldots, \cdot) \) throughout.

Given the new \( u(\cdot) \) and \( w(\cdot) \) curves, we determine the corresponding \( V(\cdot, \cdot) \) function, and then check the convergence criterion:

\[
\text{Max } |u(i) - u(i)| < \epsilon_1 \quad \text{and} \quad \text{Max } |w(j) - w(j)| < \epsilon_2
\]

where \( i = 1, \ldots, M \) and \( j = 1, \ldots, N \); "new" and "old" refer to the current and previous values respectively. If this criterion is met, so that little change occurs in the policy functions between iterations, we have found the optimal policies. If not, another policy improvement iteration is performed, until convergence is obtained.

This numerical scheme was tested using the simplified stock-recruitment function \( F(S) = b = \text{constant} \), and comparing results with those obtained using the analytic methods of II.B and III.B. We performed 6 test runs, based each time on the parameters for the prawn fishery in Table (I), but varying the unit capital cost, the depreciation rate and the noise level.

With \( \epsilon = 0 \), it was found in II.B that the optimal capacity \( K^* \)
is given analytically by \( K^* = \left[ qT \right]^{-1} \log \left[ b / X_0 \right] \) where \( X_0 = (1+\theta)x_0, \) with \( \theta = \frac{(1-\alpha)cT}{\sigma^2} \). For the \( \sigma > 0 \) case, the analytic optimal capacity \( K^* \) was found iteratively, using equation (12) and a hand calculator. The analytic and numerical results, together with the percentage difference between the two, are as follows:

<table>
<thead>
<tr>
<th>Parameter Values</th>
<th>Analytic ( K^* )</th>
<th>Numerical ( K^* )</th>
<th>( \Delta )Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta = 0.470 \times 10^6, \ \sigma = 0 )</td>
<td>12.48</td>
<td>12.44</td>
<td>0.3</td>
</tr>
<tr>
<td>( \delta = 0.470 \times 10^6, \ \sigma = 0.58 )</td>
<td>12.53</td>
<td>12.48</td>
<td>0.4</td>
</tr>
<tr>
<td>( \delta = 0.0832 \times 10^6, \ \sigma = 0 )</td>
<td>32.92</td>
<td>32.93</td>
<td>0.04</td>
</tr>
<tr>
<td>( \delta = 0.0832 \times 10^6, \ \sigma = 0.58 )</td>
<td>34.53</td>
<td>34.63</td>
<td>0.3</td>
</tr>
<tr>
<td>( \delta = 0.0832 \times 10^6, \ \sigma = 0, \ \tau = 0 )</td>
<td>36.27</td>
<td>37.65</td>
<td>3.8</td>
</tr>
<tr>
<td>( \delta = 0.0832 \times 10^6, \ \sigma = 0.58, \ \tau = 0 )</td>
<td>40.24</td>
<td>42.17</td>
<td>4.8</td>
</tr>
</tbody>
</table>

Clearly the numerical method performed very well for the first 4 runs. Note that for the first pair of runs, although the numerical inaccuracy is of the same magnitude as the difference between the deterministic and stochastic results, nevertheless the effect is to shift both of the latter results by roughly the same amount. Hence the desired conclusion that \( K^* \) (stochastic) > \( K^* \) (deterministic) can still be made. Obviously, however, one can feel more confident with the numerical results when the difference is larger between the \( K^* \) values in the deterministic and stochastic cases, as in runs 3 and 4.

When depreciation is omitted, so that \( \tau = 0 \), we find that the results are still satisfactory, but substantially less accurate.
There seems to be a fundamental problem with the numerical scheme when $\gamma = 0$; in most cases we have considered, convergence of the policy iteration method for $\gamma = 0$ runs required considerably more iterations. In addition, a non-convergent cycling phenomenon was found in some circumstances, with the policy improvement scheme jumping between 2 pairs of control functions at consecutive iterations. The latter effect occurred for the one $\gamma = 0$ case we have reported here (in section E(c)), but the jumping occurred only for one of the 16 policy values being calculated, and the difference between the two values at this point was sufficiently small that it did not cause any substantial ambiguity in the results, which agreed with those obtained in Chapter II. Although disconcerting from a numerical point of view, it is at least comforting to remind oneself that the $\gamma = 0$ case is an extreme, economically unlikely situation.

E. Numerical Results.

The numerical methods described in the previous section can be used to study the effect of uncertainty on both the prawn and whale fisheries outlined in Chapter II. Following the approach of that chapter, we have used the general form of the prawn model as the primary source of data but have varied the parameters to study comparative dynamics. This is done for convenience but there is nothing in the structure of the prawn model to detract from its general applicability. The base parameters are as in Table (1), with the value $\sigma = 0.58$ used for the uncertainty parameter (representing the standard deviation
of the logarithm of recruitment). This value of \( \sigma \) was the maximum likelihood estimate obtained by fitting a lognormal distribution to recruitment data of Kirkwood (1980), with the mean value of the distribution set equal to the sample mean of the data (implicitly assuming no dependence of recruitment on escapement). Our approach in most cases was to compare the optimal policy functions derived for a fishery subject to this fairly high degree of uncertainty with the corresponding optimal policies in the extreme case of no randomness (\( \sigma=0 \)). The deterministic results correspond to those found in Chapter II, but are obtained using the more accurate numerical method described in III.D. There are no qualitative differences, and only minor quantitative differences, between these deterministic results and those of Chapter II. In some cases we have also considered other levels of uncertainty (besides \( \sigma=0.58 \)), and have thus been able to gain information regarding the rate at which the optimal policies change as the degree of uncertainty varies. The deterministic and stochastic policies have been compared under a number of parameter combinations. In particular we have considered the role of:

(i) the intrinsic biomass growth rate, (ii) the capital cost (relative to variable costs), (iii) the discount rate, and (iv) the depreciation rate.

We begin by describing the behaviour of an optimal fishery operating in a stochastic environment. Figure 15 depicts accurate piecewise linear approximations to the optimal policy functions \( h(S) \) and \( s(K) \) for the base-case prawn fishery with
\( s = 0.58 \). As in the deterministic case, and as deduced in III.C, the curve \( h(S) \) represents the optimal capacity for next season given escapement \( S \) this year. In other words, since it is necessary to order new capacity one season in advance of delivery, one should purchase \( \text{Max} \{ h(S) - (1 - \gamma)K, 0 \} \) in new fleet capacity, even though in the stochastic case recruitment next season can only be predicted roughly (i.e. in the mean) at the time of ordering the investment. The \( s(K) \) optimal escapement curve is entirely analogous to its deterministic counterpart; given capital stock \( K \), the objective is to harvest down to the escapement \( s(K) \), or as close to that target as possible. If \( R < s(K) \), no harvesting takes place.

The point \( (4.3 \times 10^6, 7.75) \) marked in Figure 15 represents the equilibrium point for the fishery if there were no random fluctuations. As pointed out by May et al (1978) and Spulber (1978), the deterministic equilibrium point translates into the steady state probability distribution over stock levels in the stochastic case. In our 2-dimensional model, any steady-state would also be 2-dimensional. The existence of such an equilibrium distribution for the optimally-managed fishery has not been examined for our model. Instead we have simulated a steady-state distribution by plotting the end points of a large number (160) of 40-year sample paths emanating from the quasi-equilibrium point; these end points are depicted in figure 15. (Note that the biomass and capacity scales have been changed substantially from previous figures in order to include a larger range of biomass values and to show in more detail the
fluctuations in capacity.)

The cloud of points in figure 15 can be interpreted as follows: the denser the points in a given region of the S-K plane, the more likely is the fishery to lie in that region (i.e. to have that escapement and that capacity) over the long term. One can observe a considerable spread both in biomass and capacity values about the quasi-equilibrium point. The spread in biomass values is due simply to the stochastic nature of the resource. Variation in the capital stock, on the other hand, is an induced phenomenon; fluctuations in recruitment lead directly to variations in escapement, which in turn cause dispersion in fleet capacity, through the investment function $K^* = h(S)$. This effect will be even more pronounced with slower-growing stocks, where particularly good or bad escapement levels will tend to influence the fishery for longer periods of time, and will therefore have a greater effect on desired fleet capacity.

Since the resource is fairly fast-growing ($a=42$), few points are found at low [$S<s(K)$] escapement levels. In fact the distribution of points resembles a lognormal distribution in the S-direction, truncated below at $S=s(K)$. This is unlikely to be the case precisely, however, since $(S,K)$ rather than $(R,K)$ points have been plotted, resulting in a tighter distribution (lognormally distributed R-values which are sufficiently high are reduced by fishing pressure to relatively lower S-values). In addition, the spread of points in the S-direction can be seen to be smaller at high capacities, since in this case fishing effort is sufficient to reduce even high recruitments down to
escapements fairly near the s(K) curve.

Since the end-points of 40-year sample paths have been plotted, one would expect that the simulated distribution so produced should be independent of the starting point. This is verified in figures 16 and 17, where similar simulated distributions have been plotted, but originating at (2.0x10^6,6.0) and (6.0x10^6,9.0) respectively. No significant differences can be seen; hence one can conclude that figure 15 provides a reasonable approximation to the steady-state distribution.

Figure 18 shows the effect of reducing the noise level from σ=0.58 to σ=0.2 in the prawn fishery. As expected, the steady state distribution collapses to within a much smaller neighbourhood of the quasi-equilibrium point. One would expect stochastic effects to be relatively unimportant at such low σ-values; however, as shall be seen, the values of σ which can be considered "low" depends on the other fishery parameters. In the whale fishery, σ=0.2 can be a substantial level of noise.

To see more vividly the actual process of managing a fishery in a stochastic environment, we have plotted in figure 19(a) a set of 8 20-year sample paths for an optimally managed base case prawn fishery, with the recruitment chosen each year from a lognormal (σ=0.58) density centred on F(S), where S is the previous year's escapement. Lines are drawn joining successive (S,K) points, beginning at the quasi-equilibrium point. As above, it can be seen that optimal risk-neutral management results in considerable variation in the
fleet capacity, as well as the biomass, over time.

Figure 19(b) shows this effect in more detail, for a single 15-year realization of the fishery's development. The 4 processes of (stochastic) recruitment, harvesting, investment and depreciation combine to determine the movement from one \((S,K)\) point to the next, governed by the policy curves. Since the resource is fast-growing and highly variable in this example, there is no apparent trend to return to the quasi-equilibrium point.

We now examine the extent to which each of the fishery parameters outlined above serves to determine how uncertainty affects the fishery's optimal management. In particular we wish to study whether investment increases or decreases with increasing uncertainty, and how the parameters of the fishery affect this behaviour.

(a) The cost of capital.

Consider Figure 20 where the optimal capacity curves \(h(S)\) are shown for 10 different cases, with \(a=42\) throughout but with varying levels of uncertainty and unit capital cost:
\[
\begin{align*}
\delta &= 0.1175 \times 10^6 ; \sigma = 0, 0.58 \\
\delta &= 0.2350 \times 10^6 ; \sigma = 0.29, 0.58 \\
\delta &= 0.4700 \times 10^6 ; \sigma = 0.58, 0.80 \\
\delta &= 0.7050 \times 10^6 ; \sigma = 0, 0.58
\end{align*}
\]

To concentrate on the investment problem, we have omitted the optimal escapement curves \(s(K)\) from figure 20. In fact, the difference between optimal escapements in the deterministic and
stochastic cases was negligible, as indicated by the following numerical results, giving the optimal values of $s(K_j)$ and $h(\xi_i)$ for the base case prawn fishery with $\sigma=0$ and $\sigma=0.58$:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x(i)$</th>
<th>$h(\cdot) \ (\sigma=0)$</th>
<th>$h(\cdot) \ (\sigma=0.58)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12.206</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>13.122</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>13.816</td>
<td>0.861</td>
<td>0.928</td>
</tr>
<tr>
<td>4</td>
<td>14.509</td>
<td>5.375</td>
<td>5.310</td>
</tr>
<tr>
<td>5</td>
<td>14.914</td>
<td>7.215</td>
<td>6.850</td>
</tr>
<tr>
<td>6</td>
<td>15.320</td>
<td>8.363</td>
<td>7.809</td>
</tr>
<tr>
<td>7</td>
<td>15.761</td>
<td>9.150</td>
<td>8.670</td>
</tr>
<tr>
<td>8</td>
<td>16.811</td>
<td>10.076</td>
<td>9.450</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$j$</th>
<th>$K(j)$</th>
<th>$s(\cdot) \ (\sigma=0)$</th>
<th>$s(\cdot) \ (\sigma=0.58)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>14.006</td>
<td>14.005</td>
</tr>
<tr>
<td>2</td>
<td>3.0</td>
<td>14.019</td>
<td>14.015</td>
</tr>
<tr>
<td>3</td>
<td>6.0</td>
<td>14.117</td>
<td>14.113</td>
</tr>
<tr>
<td>4</td>
<td>9.0</td>
<td>14.193</td>
<td>14.191</td>
</tr>
<tr>
<td>5</td>
<td>12.0</td>
<td>14.257</td>
<td>14.257</td>
</tr>
<tr>
<td>6</td>
<td>15.0</td>
<td>14.316</td>
<td>14.310</td>
</tr>
<tr>
<td>7</td>
<td>18.0</td>
<td>14.365</td>
<td>14.352</td>
</tr>
<tr>
<td>8</td>
<td>21.0</td>
<td>14.407</td>
<td>14.384</td>
</tr>
</tbody>
</table>

This insensitivity of the optimal escapement level to the degree of uncertainty was found for most of our parameter combinations and is in accordance with the results of other researchers. There are, however, cases where uncertainty does affect the $s(K)$ curves: these are discussed below. For the remainder of this chapter, $s(K)$ curves are shown only in such cases.

It was seen in Chapter II that the important cost parameter in the investment problem is neither capital cost nor operating cost alone, but rather the ratio of the two. Specifically, a useful quantity to study appears to be $\delta/cT$, the ratio of unit capital cost to maximum yearly operating cost (per unit of
capital). In a sense, this measures the capital intensity of the fleet; \(1 + (\delta/cT)\) is the fraction of total fleet costs that can be attributed to costs of capital. The important aspect of Figure 20 is the relative position of the \(h(S)\) curve between the deterministic and stochastic cases, as this ratio of costs, \(\delta/cT\), varies. However, since \(cT\) is fixed here, it suffices to speak in terms of varying the capital cost. It can be seen that at low capital costs, the optimal capacity is substantially higher in a fluctuating environment. As the unit capital cost increases, the difference between the deterministic and stochastic cases decreases, so that at \(\delta=\$0.235\) million (\(\delta/cT=5.65\)), the optimal capacity with \(\sigma=0.58\) still lies above its deterministic counterpart, but this relationship is reversed at lower levels of uncertainty (\(\sigma=0.29\)). As the unit capital cost is increased further, to \(\delta=\$0.47\) million (the base case, \(\delta/cT=11.3\)) or \(\delta=\$0.705\) million (\(\delta/cT=16.95\)), the optimal capacity decreases with increasing uncertainty. The small crossover in the (\(\delta=\$0.705\) million) \(h(S)\) curves, a symptom of the complex interaction between the escapement level, the degree of uncertainty and other model parameters, is discussed below. Figure (20) also indicates that the optimal escapement curves \(s(K)\) move to lower escapements as \(\delta\) increases, as was the case in Chapter II, but the difference between optimal escapements in the deterministic vs stochastic cases was negligible.

Figure 21 shows similar results, again varying the level of uncertainty and the unit capital cost, but this time for a slower growing (\(a=14\)) fishery. The key results are unchanged in
this case; at low relative capital costs (δ/cT = 2.0), investment should be higher under uncertainty whereas this is reversed if the fishery is subject to high unit capital costs.

At the intermediate level $δ = $0.235 million, the optimal h(S) curves in the deterministic and stochastic cases exhibit a crossover, as in the $δ = $0.705 million case of figure 20, so that the introduction of randomness increases optimal investment at low biomass levels while decreasing investment at higher stock sizes. This result can be explained by considering two opposing effects: the "downside" problem of suffering idle excess capacity in bad years, and the "upside" problem of lacking sufficient capacity to take full advantage of good years. With these particular parameter combinations, it appears that the role of uncertainty depends on the escapement level; at high biomass levels, the variance in recruitment is also high, so that the downside problem predominates. Hence investment is lower under uncertainty; the balancing act tilts towards caution in investment. On the other hand, if escapement is already relatively low, and the stock tends (in the mean) to grow reasonably rapidly, then the potential benefits to extra ("upside") investment outweigh the downside risk. Increased investment under uncertainty becomes optimal. Where the crossover will occur, if it does at all, seems rather difficult to predict. Indeed in most cases where a crossover appears, the difference between the h(S) curves in the deterministic and stochastic cases tends to be small.

With regard to the s(K) policy functions, the
slower-growing $a=14$ fishery shows a slightly greater effect of randomness on the optimal escapement levels than in the $a=42$ case: this is indicated for the $a=14$, $\sigma=0.0832 \times 10^6$ fishery in figure 23 and is discussed further below.

To summarize the interplay between the level of uncertainty and the unit capital cost it can be said that, ceteris paribus, the optimal fleet capacity increases with uncertainty given a relatively low ratio of capital to operating costs. The balance in this case is tilted towards overcoming the upside problem by investing in extra capacity at relatively low cost in order to benefit from exceptionally high recruitments. However, if unit capital costs are sufficiently high relative to operating costs, investment will decrease with the level of uncertainty. In this case the downside risk of more frequent bad years (when there is little or no return on the expensive investment) now outweighs the advantages of having extra capital available to profit from good years.

(b) The intrinsic biomass growth rate.

Figure 22 shows the optimal capacity functions for each of $\sigma=0$ and $\sigma=0.58$ and for each of the growth rate parameters $a=14$, $42$, $140$, $560$, with other parameters as in the base case fishery. (The value $a=560$, corresponding to a very high growth rate, was chosen to approximate a situation of independence between recruitment and escapement.)

Examining Figure 22, one can observe a uniform progression from high to low growth rates. When $a=\infty$, we have shown in
III.B that the optimal capacity must increase with the level of uncertainty. The case $a=560$ follows this result, at least for stock levels $S>1.3 \times 10^6$, where the optimal capacity at $\sigma=0.58$ is slightly higher than in the deterministic case. However with $a=140$, the optimal capacity is lower under uncertainty, and this effect increases as the growth rate is decreased to $a=42$ and then to $a=14$. (As before, in each case, there is negligible difference between the stochastic and deterministic optimal escapements.)

If the unit capital cost is reduced to $0.0832 \times 10^6$, results remain qualitatively unchanged. Figure 23 shows the deterministic ($\sigma=0$) and stochastic ($\sigma=0.58$) optimal policy functions for the two cases $a=14$ and $a=3.82$, with this lower capital cost. When natural mortality is taken into account the latter $a$-value corresponds to a maximum net growth rate of 4 percent annually. This value was chosen to equal that of the whale fishery, for comparative purposes discussed below. In this case, even at such a low $a$-value, the unit capital cost is sufficiently low that the optimal capacity is positive for both the deterministic and stochastic cases, and is higher under uncertainty, apart from a very small crossover region. As the biomass growth rate is increased to $a=14$, the optimal capacity under uncertainty rises even further above its deterministic counterpart.

While one can determine optimal policies for the $a=3.82$ fishery, in fact this fishery would not be economically sustainable in the deterministic case; the equilibrium occurs at
$R=S=0.28 \times 10^6$, which is well below biologic equilibrium. Hence no fishery will exist unless for some reason the biomass is increased to above 3.15 million. In the stochastic case there is always a possibility of the biomass reaching a sufficiently high level to warrant investment in the fishery. In the example shown here, if the stock is at its deterministic equilibrium (which would only occur in the mean), stocks would only increase to levels warranting investment approximately once in every $1/\Pr\{ R>3.2 \times 10^6 \} = 1/\Pr\{ z>4.49 \} = 270000$ years—not a particularly promising fishery. However, this obviously represents an extreme case of the general proposition that fisheries can exist which should only be developed periodically, when exceptionally high recruitments appear.

With regard to optimal escapement levels, when $a=3.82$ we have $s(K)=x_0$ (for all $K$) in the deterministic case, and $s(K)>x_0$ for $\sigma=0.58$. The $a=14$ optimal escapement curves exhibit rather complicated behaviour, with the stochastic $s(K)$ curve lying above the $\sigma=0$ curve, except in an intermediate range between $K=10$ and $K=24$. This intermediate phase appears to be due to the considerable variation in the fleet investment policies with uncertainty, since $I^*(S,K)$ enters into the optimality equation which determines $s(K)$. However in any case, the maximum difference between the $s(K)$ curves with $a=14$ is only $\Delta s=0.15 \times 10^6$.

The interplay between the degree of randomness and the intrinsic growth rate can be explained by appealing to the "downside vs. upside" argument discussed above. The
slower-growing is the resource stock, the greater is the connection between recruitment and the previous season's escapement. In effect, the memory of the system is longer. Hence if a low recruitment occurs in any given year, this tends to persist for a longer period of time (although of course stochastic fluctuations will cause deviations from this trend). Thus the downside risk is greater the higher the level of uncertainty, since this raises the probability of suffering bad years. On the other hand, a slower-growing stock does not need to be harvested quickly in good years since a high recruitment will tend to persist over several seasons, whereas in the fast-growing case recruitment is more independent of escapement, so that a good year must be utilized immediately or forever lost. (Note that the key parameter that decides the effective intrinsic growth rate is $a \cdot \exp\{-mT\}$ (see Chapter II), so that the natural mortality rate is also of importance. However $m$ is considered fixed here.) Hence the upside benefits are lower in the slow-growing case. On balance therefore, the stochastic optimal capacity will exceed its deterministic counterpart if $a=\infty$, but this difference will decrease and likely become negative as the intrinsic growth rate decreases.

(c) The depreciation rate.

Figure 24 shows that in the $a=42$ case, the depreciation rate plays a role similar to that of the unit capital cost described above. In the absence of depreciation ($r=0$), the optimal capacity under uncertainty exceeds that of the
deterministic case. Since capital is infinitely long-lived, the effective yearly rental cost of capital is relatively low. Hence the downside risk of an increased capital stock (above that of the deterministic case) is relatively small. As the rate of depreciation increases, the effective time horizon for a given unit of capacity is shortened, so that the upside benefits of an extra unit of capacity are reduced, since there are likely to be fewer years in which the fishery could take advantage of a higher level of capacity. Thus it is not surprising to see that the optimal capacity decreases with uncertainty when $\gamma = 0.15$ or $\gamma = 0.20$. In each of the three cases, the optimal escapement was insensitive to the level of uncertainty.

With a slower growing ($a = 14$), lower capital cost ($\delta = 0.0832 \times 10^6$, $\delta/cT = 2.0$) fishery, we found in II.E(d) that for the deterministic case, optimal investment levels can actually increase with the depreciation rate if escapement is sufficiently high. We argued that this result was due to an incentive to harvest the resource relatively quickly, before the fleet depreciates. This incentive outweighs the effective increase in unit capital cost due to depreciation, if capital is cheap initially. Figure 25 shows this result again, this time for $\gamma = 0.05$ and $\gamma = 0.15$ cases. However, the same behaviour does not carry over to the stochastic fishery; from figure 25, one can see that for $\sigma = 0.58$, optimal capacity decreases as the depreciation rate is increased, for all escapement values. This may be due to one of two reasons: either (i) the stochastic investment levels are already sufficiently high that the
resource can be harvested as rapidly as necessary, or (ii) stochastic fluctuations add sufficient unpredictability to the fishery that extra investment in order to harvest quickly in the face of depreciation is not warranted.

Results for this fishery also support the discussion concerning the base case fishery; the lower the depreciation rate, the more likely is investment to be higher under uncertainty. Indeed, we have the result that:

(stochastic optimal capacity)-(deterministic optimal capacity)
increases as the depreciation rate decreases.

(d) The discount rate.

Figure 26 depicts the optimal policy curves with and without randomness for 4 values of the discount factor (α) and corresponding discount rate (r=[1-α]/α): α=0.80 (r=0.25), α=0.90 (r=0.11), α=0.97 (r=0.03) and α=0.99 (r=0.01). The results are perhaps most interesting for the observation that the qualitative effects of varying the level of uncertainty seem quite independent of the rate of discounting. While it is true that: (i) the relative difference between optimal deterministic and stochastic capacities diminishes as the discount rate decreases towards zero, and (ii) as α increases, the region of escapements for which investment should be greater under uncertainty broadens, nevertheless the optimal stochastic capacity is never substantially above its deterministic counterpart. It appears that, at least for these parameter combinations, there is little interplay between the discount
rate and the level of uncertainty. However, the role of the discount rate in resource management is a complicated one; if it affects other economic parameters (such as the cost of capital), its interaction with the level of uncertainty may be considerably changed. Hence our results here should not be extrapolated too far.

(e) **Ricker Stock-recruitment**

If instead of using Beverton-Holt population dynamics, we adopt a Ricker reproduction function, we saw in II.E(f) that the optimal investment curve $h(S)$ mimics the Ricker form, rising to a peak and then declining to zero. As indicated in Figure 27, the introduction of uncertainty ($\epsilon=0.58$) does not change this qualitative behaviour. In terms of the future of the fish stock, very large escapements are as bad as very small ones: if $K=0$ but escapement $S>14.0 \times 10^6$, the expected value of a unit of investment is less than its cost, and hence $I^*=0$.

Comparing the deterministic and stochastic optimal capacity functions, we see that if the future of the fish stock is relatively bright ($3.25 \times 10^6 < S < 9.75 \times 10^6$) then investment is lower under uncertainty, while the reverse is true if expected future stock sizes are relatively small. This is equivalent to a single crossover in the $h(S)$ curves for the Beverton-Holt case (for example, compare the $a=14$, $6=0.235 \times 10^6$ results shown in Figure 21), and hence can be explained by the same reasoning as above.
(f) The whale fishery.

To this point in our stochastic results we have concentrated on modifications of the prawn fishery data. For completeness we now look at the effect of uncertainty on the base-case whaling fishery. Figure 28 shows the optimal capacity and escapement curves (derived using an M=N=12 mesh) for the cases of $\sigma=0$, $\sigma=0.1$ and $\sigma=0.2$, which covers a reasonable range of uncertainty for the aggregated whale fishery.

It can be seen that increasing the level of uncertainty decreases the optimal capacity for low stock sizes [$S<36\times10^4$] but increases optimal capacity for larger biomass levels. In other words investment under uncertainty should respond to the state of the fishery: if the stock is relatively healthy it is worthwhile taking advantage of good years by investing in capacity above the deterministic level, while for lower stock sizes a more conservative investment policy is to be preferred. Note, however, that the deterministic equilibrium point for the whale fishery lies at a low biomass level ($S=11.4\times10^4$, $K=2250$), so that even with $\sigma=0.2$ one will rarely see escapements $S>36\times10^4$ in the long term. On the other hand, the unexploited deterministic equilibrium lies at $S=46.2\times10^4$, so that for this example the optimal stochastic policy faced with a virgin stock calls for an initial investment above the deterministic optimum, but a more conservative investment strategy in the long term.

This result is consistent with the discussion presented above, where it was suggested that fisheries with slow-growing stocks will tend to have optimal capacities which decrease with
uncertainty. Note that both the maximum biomass growth rate and the ratio of unit capital cost to maximum yearly variable cost (δ/cT=2.0) for the base-case whale fishery is the same as that of a "prawn fishery" with a=3.82 and δ=83200 (by construction - see Chapter II.E). Comparing figure 28 with results for the a=3.82, δ=0.0832x10^6 prawn fishery shown in figure 23, one can see that the optimal policy curves are similarly behaved in most respects, although the location of the crossover between the deterministic and stochastic h(S) curves is somewhat different in the two fisheries. This may be due to the difference in noise levels being considered or may concern the carrying capacity of the fishery [i.e. the solution of F(S)=S], since the whale fishery has a carrying capacity equal to 46.2x10^4 >> x_0=5.5x10^4 while the "prawn fishery" described above has carrying capacity 0.28x10^6<x_0=0.99x10^6. In any case the general result appears to be that investment is either zero or decreases with uncertainty for most reasonable escapement values in a slow-growing fishery; only in the case of exceptionally high escapements is it optimal for the social manager to invest in capacity above the deterministic optimum (assuming a low unit capital cost).

The optimal escapement for the whale fishery increases with increasing uncertainty, as in the a=3.82, δ=0.0832x10^6 prawn fishery discussed above. It appears that management of a very slowly growing fishery should be more conservationist the higher the level of uncertainty; this is in accordance with previous research results. However, the difference between the c=0 and c=0.2 s(K) curves is never very great, particularly for K-values
near the (quasi-) equilibrium point.

Perhaps the most interesting result in comparing the deterministic and stochastic whale fisheries is the location of these (quasi-) equilibrium points: when $\sigma=0.2$, the whale stock steady-state is centred on $S=14.8\times10^4$, 30 percent higher than the deterministic equilibrium. Hence, although there is little change in the $s(K)$ curves, use of the optimal stochastic policy can effectively lead to a substantially larger stock of whales (on average), while decreasing the (mean) optimal capacity by only 11 percent, from $K=2250$ to $K=2000$.

(g) Performance of deterministic vs stochastic strategies.

At this point we address two fundamental questions: How sensitive is the value of the fishery to changes in the policy curves $s(K)$ and $h(S)$ away from their optimum positions? How well do the policy functions obtained as optimal for a deterministic environment compare to "true" optimal policies for the stochastic fishery?

The very nature of optimal controls suggests that small variations in the controls should have even smaller effects on the value function (Ludwig, 1980). This indeed appears to be the case in our model. Using the deterministic version of the model (II.D), the optimal policy function $h(S)$ for the base case prawn fishery was perturbed first upwards and then downwards by 10 percent. The reduction in the value function was less than 1.0 percent in both cases, a result in agreement with Ludwig's point that the variation in the value function should be proportional
to the square of the deviation in the policies.

We have seen that policies which take into account fluctuations in the fishery's environment can differ from their deterministic counterparts by as much as 30-40 percent, for reasonable parameter combinations. Optimal fisheries investment, then, can be significantly higher, or significantly lower, under uncertainty. However, as above, the optimal value function is rather insensitive to changes in the policy functions away from their optimal positions. Table (IV) gives the value function (at mesh points in the x-K plane) for an a=14, $=0.0832\times10^6$ fishery, and for two sets of policy functions: (i) the optimal policies for the stochastic ($=0.58$) fishery, and (ii) the optimal policies for the corresponding deterministic fishery, but evaluated in a stochastic ($=0.58$) environment. By comparing these value functions point by point, one can see that the loss from using the deterministic policy is never more than $0.20 million. For example, with $S=2.1\times10^6$ (the quasi-equilibrium escapement) and $K=0$, the optimal investment for the stochastic fishery is $I^*=12.5$, while using the deterministic policy we will have $I=8.8$, a 30 percent under-investment. However the reduction in value of the fishery caused by using the deterministic policy is roughly 3 percent, or only $0.16$ million, a rather negligible amount when one considers the overall lack of precision attainable in real-world fisheries.
F. Summary and Discussion

A stochastic version of the optimal fisheries investment model has been constructed by allowing yearly recruitment to follow a lognormal probability distribution, with mean value given by the deterministic stock-recruitment function. Heuristic analysis of the stochastic model indicated that optimal policies under uncertainty should not differ qualitatively from their deterministic counterparts; numerical results confirmed this expectation. However, by simulating stochastic sample paths and steady state distributions, it was shown that in practice a stochastic fishery appears quite different from a deterministic one, with considerable fluctuations in fleet capacity being optimal in the stochastic case.

In addition, this chapter has examined the role of the unit cost of capital, the intrinsic biomass growth rate, the depreciation rate and the discount rate in determining the effect of uncertainty on optimal investment and escapement strategies. It appears that of these, the key parameters are $\delta$, the unit cost of capital (relative to operating costs, $cT$) and the intrinsic biomass growth rate ($a$). Figure 29 is a schematic showing $(a, \delta)$ combinations (with $cT$ fixed) for which optimal capacity is generally higher (+) or lower (-) under uncertainty ($\sigma=0.58$), together with a rough curve dividing the two regions. In general investment will be higher under uncertainty if the resource is fast-growing and capital is relatively cheap. The reverse will be true for a slow-growing stock with expensive capital. With regard to other model parameters, our results
indicate in particular that the lower the depreciation rate, the more the tendency for optimal investment to be higher under uncertainty.

Finally in this chapter we have discussed the optimal investment and escapement policies for the case of Ricker stock-recruitment and for the aggregated whaling fishery.

It was found that for moderately high levels of variability, the relative difference between stochastic and deterministic optimal capacities could reach 30-40 percent. This results in substantial over- or under-investment in fleet capacity when the deterministic model is used in place of a full stochastic model. Target escapements, on the other hand, tended to be remarkably insensitive to the level of uncertainty in the fishery; this is in agreement with previous results, although the possibility that optimal escapements can be lower under uncertainty is contrary to most previous results (where investment was not considered).

The relative performance of deterministic vs. stochastic policies was also discussed, and the difference found to be very small in comparison to differences in the policies themselves; the implications of this result are discussed in Chapter IV.
Chapter IV. Conclusions

(a) Review

In renewable resource industries, two simultaneous investment problems must be addressed: investment in the resource stock (the biomass) and investment in the capital stock (harvesting capacity). The latter investment problem has been suppressed in most analyses to date, both deterministic and stochastic, by assuming either a fixed capital stock or the perfect malleability of capital. However, in many fisheries, and other natural resource industries, capital is in fact nonmalleable, with few if any alternative uses. In this context, investment is irreversible, and optimal management of the industry must involve control of both state variables, the resource stock and the capital stock, through appropriate investment strategies. Clark, Clarke and Munro (1979) have studied this irreversible investment problem, obtaining a full analytic solution in the continuous-time deterministic case.

This thesis has explored various aspects of the joint investment problem, building upon the work of Clark, Clarke and Munro, while maintaining the key assumption of irreversible investment. We have used a seasonal (discrete-time) stochastic fisheries model in which the resource stock and the capital stock vary over time, controlled by two decision variables, end-of-season escapement and yearly investment. The model includes delays in investment, reflecting the reality that
investment decisions must usually be made well in advance of the time at which the corresponding new capital is required.

In Chapter II, the deterministic version of the model was studied, using analytic, heuristic and numerical methods to produce comparative dynamics results. In Chapter III, we recognized that the resource stock fluctuates randomly from year to year. The resulting stochastic model required a more refined numerical method, but analytic and heuristic techniques carried over from the deterministic case in a fairly straightforward manner.

While numerical results have been obtained for two specific fisheries, the methodology and the qualitative results can be expected to apply in many fisheries, as well as in forestry and agricultural investment problems. It is clear from both qualitative and quantitative results presented here that a full analysis of renewable resource management must include questions of optimal investment strategies. Indeed for many of the cases studied, the investment aspect is substantially more complex than the more widely studied optimal harvesting problem.

As in the Clark, Clarke and Munro work, we have found that fisheries will tend to move between three primary regimes:
(i) a high-biomass, low-capacity regime, with both harvesting and investment being desirable,
(ii) a high-biomass, high-capacity situation, where investment is unwarranted but harvesting takes place, and
(iii) a low-biomass case in which the fishery is essentially shut down, with neither harvesting nor investment being desired.
In addition the existence of delays in bringing investment on-line leads to the possibility of a fourth regime, in which the resource stock is too low to permit harvesting, but is expected to recover during the "investment delay" period. Hence planning and payment for investment becomes desirable if current capacity is sufficiently low.

The reader is referred to the final section of each chapter for a detailed summary of the results. Here we shall simply list the more important of these results and then proceed to a discussion of our methods of analysis, implementation of optimal investment programs, and the role of uncertainty in optimal fisheries investment.

1) With linear variable costs and risk neutrality, optimal management is characterized by capacity and escapement target curves, \( h(S) \) and \( s(K) \), representing the optimal capacity for given escapement \( S \), and the optimal escapement for a given fleet capacity \( K \), respectively. These curves have the same qualitative form in both the deterministic and stochastic cases.

2) The primary difference between the use of seasonal and continuous-time models in studying the fisheries investment problem is the asymptotic approach to equilibrium in the former case, as opposed to the final "impulse" investment which arises in the continuous-time solution. However, a stochastic version of the continuous-time model would be unlikely to maintain this impulse investment, since a specified "equilibrium" biomass level cannot be readily attained, as we saw with our discrete-time model.
(3) The optimal capacity function mimics the qualitative behaviour of the underlying stock-recruitment function. This effect is due to the delay in bringing new investment on-line; desired capacity for next season depends on the current escapement, acting through the reproduction function F(S).

(4) Comparative dynamics results obtained in Chapters II and III provide insights into both qualitative and quantitative behaviour of the economic and ecological parameters arising in fisheries investment problems. Optimal investment appears to be particularly sensitive to fish price and stock-recruitment parameters. In addition we have shown that depreciation must be treated as more than simply an extra cost of capital, and must be analysed carefully in determining optimal investment policies.

(5) In an optimally-managed fishery where the biomass fluctuates randomly from year to year, considerable variation can also be expected in the optimal capital stock, which responds to end-of-season biomass through the function h(S).

(6) The intrinsic biomass growth rate and the cost of capital (relative to variable costs) seem to be the key fishery parameters in determining whether investment increases or decreases with the level of uncertainty. In general, we found that investment will be higher under uncertainty if the resource is fast-growing and capital is relatively cheap. The reverse will be true for a slow-growing stock with expensive capital.

(7) While target escapements tended to be insensitive to the level of uncertainty, the relative difference between stochastic
and deterministic (σ=0) optimal capacities could reach as much as 30-40 percent. On the other hand, the "optimal" policies obtained using the deterministic assumption performed nearly as well (in terms of the expected value of the fishery) as the true optimal (stochastic) policies.

(b) Methods of Analysis

The general approach to solving our optimal investment and escapement problem has been to adopt a dynamic programming formulation, obtain analytic results for the simple case of density independent stock-recruitment, analyse the model heuristically, and then proceed to develop a numerical scheme to solve the problem completely. It was found that the heuristic analysis, although relatively simple, was particularly useful in suggesting the qualitative behaviour of the optimal policies. This proved helpful on its own and as a guideline for the numerical scheme.

Two approaches to the numerical solution of our optimization problem have been developed here. The first, used for the deterministic case in Chapter II, was relatively straightforward, involved simple linear or cubic interpolation, but required a 30x30 R-K mesh for suitable accuracy, and did not use the full information contained in the dynamic programming equation. The second approach, discussed in Chapter III, is a rather more complex and consistent procedure based on a 2-dimensional C' cubic spline surface for the value function, and involving a simultaneous solution for the value function and
its first partial derivatives. The latter method required only an 8x8 mesh, but since 3 values are determined at each mesh point and sparse matrix methods are not useful in this case, the resulting cost per run of the deterministic model was found to be roughly equivalent for both approaches (and the corresponding results were in very close agreement). In the stochastic case, however, the simpler scheme of Chapter II failed to produce accurate results; hence we conclude that the more refined procedure is to be preferred in general.

Finally, we note that an important advantage of the dynamic programming approach is its versatility; the numerical methodology developed herein to solve our specific dynamic programming problem can be quite easily adapted to handle other related models (for example, the possibility of alternative investment assumptions).

(c) Implementation

Although the model discussed in this thesis represents an abstraction of real-world fisheries, it is sufficiently general to have some applicability in a variety of circumstances. The possibility of implementing the optimal program suggested both by our results and the continuous-time results of Clark, Clarke and Munro raises several questions for fisheries management in such cases.

When the capital stock is considered explicitly, it becomes apparent that the target escapement should depend on the current level of capitalization in the fishery [i.e. \( S^* = s(K) \)], and that,
perhaps contrary to intuition, management should be somewhat more conservationist when the industry has a high level of capacity. However, in the cases studied here, this effect tended to be rather minor, so that a fixed escapement target, equal to the abundant-capital Modified Golden Rule escapement, will be close to optimal.

On the other hand, for all but the most rapidly-growing resource stocks, the optimal capacity of the fishery depends strongly on the current status of the fish stock. In our model this is measured by the escapement; in any case what is required is sufficient information to predict the average future stock size, for this is the quantity which determines the desirability of investing now. Hence the accurate collection and analysis of end-of-season data on the state of the fish resource becomes important for investment planning as well as for the more traditional conservation purposes.

The implementation of optimal management programs clearly requires appropriate regulatory instruments. If left uncontrolled, or even partially regulated, experience indicates that fisheries frequently become substantially over-capitalized (in the sense that investment occurs above the optimal levels given by the $h(S)$ curve).

As described above, the escapement problem can often be reduced to determination of a single biomass target. Several mechanisms have been proposed to achieve such a target; allocated catch quotas, taxes on catch, and limited entry are the most frequently discussed. While the use of quotas and taxes
in a stochastic fishery has not yet been sufficiently researched from a theoretical standpoint, it can be hoped nevertheless that an appropriate combination of these policy instruments will be able to achieve optimal escapement targets.

Controlling the catching power of fishing fleets has been the subject of much debate in recent years. The need for such control is well-accepted, but the means to accomplish this are not clear. Recent work by Clark (1980) has set out a methodology for achieving the correct level of effort through taxes or quotas. Clark developed a model of entry to and exit from the fishery by individual fishermen. However, this model is based on the existence of a sufficiently large "background" capital stock from which vessels can be drawn when required. Regulatory methods or other means are needed to control the size of this "capital pool" itself.

Investment incentives, through tax measures, grants or "fisheries development loans", can be used to increase the capital stock. When the processing sector controls a considerable share of the harvesting capital, through renting or leasing of their vessels to individual fishermen, control of fleet investment behaviour can also be accomplished in part through direct regulations applied to the corporations involved. In cases where private interests have not met the social desire to develop a particular fish resource, due perhaps to differences between private and social costs (or discount rates), both investment incentives and direct government acquisition of fleet capital may be important.
The problem of fisheries development, particularly in developing countries, involves not only the acquisition of sufficient physical capital (which in itself may require foreign aid) but also the provision of human capital, through suitable training of the fishermen. This rather obvious point has significant implications for the irreversible investment problem; delays in bringing new investment on-line may be substantially increased by the need for lengthy training programs. At the very least, one would expect the "catchability" per unit of fishing effort to be rather low initially, but to increase over time as skills of the fishermen are improved. While our results suggest that this effect would lead, ceteris paribus, to a low initial capital stock, the problem faced by society is one of balancing low initial returns per unit effort with the need for "learning by doing" on the part of the fishermen.

If the fishery under consideration is faced with a capital stock above the optimal level given by the h(S) curve, our model prescribes a moratorium on investment until the fleet has depreciated sufficiently. Indeed, almost by definition, this is the only possible solution given nonmalleable capital; as pointed out by Clark, Clarke and Munro, there is no need for more drastic action. Scrapping excess capacity serves no purpose (with zero scrap value), and buy-back programs merely transfer ownership of the capital from private to government hands. In practice, however, buy-back programs may serve a useful purpose; catching power is permanently removed from the fishery, making
management easier (assuming no corresponding additions to fleet capacity occur!). Furthermore, the removed vessels can often be profitably utilized in other fisheries, particularly those of developing countries.

(d) **The role of uncertainty**

Turning now to effects of uncertainty in fisheries management, we consider separately the two types of uncertainty studied in this thesis, namely stochastic fluctuations and parameter uncertainty.

We have seen that even in a long run steady state fishery, fleet capacity in a stochastic environment should be expected to fluctuate over a fairly wide range. This range will be greater the slower-growing and the more variable is the resource stock. In particular, an optimal investment program should allow the capital stock to respond positively to unusually "good" years, either by permitting increased entry or by direct acquisition of extra capital. This is done in full knowledge that idle capacity will then be greater in the "bad" years. [The possibility that pressure from user groups may lead to the over-utilization of this new capacity is a real danger, but has not been included in the model discussed here.]

Given a deterministic investment model of a real-world fishery, one may wish to know the qualitative effect of randomness without undertaking a full stochastic analysis. Our results suggest the following guiding principle: if the ratio of unit capital cost to yearly operating costs seems fairly low,
and if the resource is reasonably fast-growing (as with prawn stocks), then investment is likely to be higher under uncertainty. This qualitative information may be useful in determining whether a fishery has indeed experienced over-investment, or whether apparent excess capacity is in fact optimal given the history of the fishery's development in the face of uncertain future stock sizes.

The irreversibility of investment increases the importance of inherent uncertainty in the fishery. This is particularly the case for fisheries with slow-growing resource stocks, where the occurrence of an unusually "bad" year may lead to capital lying idle for a substantial part of its economic life. However, in accordance with the work of other researchers, we have found that in many linear-cost risk neutral fisheries, optimal policies recognizing the stochastic nature of the fishery perform only slightly better than policies based on the corresponding deterministic model. In other words, in this case use of deterministic models is sufficient to produce policies with near-optimal performance (on average).

The implication of the latter result is not so much that the existence of uncertainty in the form of year-to-year fluctuations is irrelevant to management. Rather, it tells us that any investment strategy "near" the optimal will perform almost optimally. As pointed out by Ludwig (1980), the fractional loss incurred by using a non-optimal policy is given approximately by the square of the fractional deviation of the policy away from optimal. In other words, with linear costs and
risk neutrality, economic optimization is "forgiving"; other objectives (conservation, job creation, etc.) can be pursued with little loss in the fishery's economic value. [Of course, the underlying requirement is that the modified policy remain near the optimal strategy, with a deviation of roughly ±20 percent being reasonable.]

Lewis (1981) has shown that this "forgiving" nature need not apply when nonlinearities in costs or utility are included. Since our results show that, even with linear costs and risk neutrality, investment policies are strongly affected by uncertainty, the incorporation of additional nonlinearities may make the use of stochastic rather than deterministic policies particularly important to the fishery's performance. This will be the topic of future work.

While it appears reasonable to propose that stochastic models are unnecessary in studying linear-cost risk neutral fisheries, even in these cases such a simplification may lead to faulty results whenever an additional component is added to the problem, whether economic (mixed fleets, processing sector) or ecological (age-structured fish stocks, multi-species effects). It seems likely that the current pattern apparent in fisheries research, of first obtaining initial deterministic results analytically and then using numerical methods to expand these results to the stochastic case, in some sense represents optimal behaviour. As more and more realistic models are considered, analytic results become impossible even in the deterministic case; when numerical methods are called for, we have found that
obtaining full stochastic results can be as straightforward as finding the deterministic solution.

The three types of uncertainty important in fisheries management were outlined in Chapter III. While we have concentrated here on the problem of stochastic fluctuations in the resource stock, the fundamental uncertainties involved when model parameters are known only imprecisely, or even worse, when the correct structure of the fishery is unknown, are also of great interest. Parameter uncertainty may arise in the stock-recruitment function describing the behaviour of the fish stock or in the fishery's economic parameters (the fish price, the variable cost function, the future cost of capital, the discount rate, and so on). Of these possibilities, uncertainty in the population dynamics parameters is perhaps most serious, since these parameters enter directly into the dynamics of the fishery and are inherently uncertain (whereas the economic parameters will at least become known with time). It seems clear that such uncertainties must play an important role in determining optimal investment strategies; for example, initial errors in parameter estimates, combined with irreversible investment, could lead to long term over-capacity in a developing fishery, if simple-minded policies are used. [The potential importance of uncertainty in a parameter can be checked using the analysis of Chapters II and III. If the policy functions are fairly insensitive to variations in the parameter over its likely range, such uncertainty can be fairly safely ignored. However, one would not expect this to be the case in
general.]

Preliminary analysis of the investment problem under parameter uncertainty, while beyond the scope of this thesis, suggests a rich variety of possible behaviour and of possible investment strategies; this promises to be a fruitful area for further research.
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Appendix A

We wish to examine in more detail the long run equilibrium values for recruitment, escapement and capacity, denoted by $R, S, K$ respectively. For any given $K$, the equilibrium value of $S$ is the solution of:

\[ (A1) \quad S = \max \{ s(K), F(S)e^{-qTK} \}. \]

For fixed $S$, capacity in the delayed investment case is given by

\[ (A2) \quad K = \max \{ (1-\gamma)K, h(S) \} \quad \text{so that the equilibrium satisfies} \]

\[ \quad n+1 \quad n \]

\[ K = \max \{ (1-\gamma)K, h(S) \} = h(S). \]

The solution of (A1) and (A2) produces unique values $S$ and $K$, and hence $R = F(S)$. In the instantaneous investment case, (A2) becomes: $K = h(R) = h(F(S))$ and the solution follows similarly.

To relate these equilibria to the solutions of Modified Golden Rule equations, we shall consider the behaviour of the value function in the vicinity of the equilibrium $(R,K)$.

\[ V(R,K+dK) = pR(1-\exp{-qT(K+dK)}) -cT(K+dK) \]
\[ -\sigma(h(F[R \cdot \exp{-qT(K+dK)}])) - (1-\gamma)(K+dK) \delta \]
\[ +\sigma V(F[R \cdot \exp{-qT(K+dK)}], h(F[R \cdot \exp{-qT(K+dK)}])) \]
\[ = V(R,K) + [pqT \delta - cT + \sigma(1-\gamma) \delta] dK - \sigma qTSF'(S) V(R,K) dK \]

Hence:

\[ (A3) \quad V(R,K) = \frac{pqT(S - 1/(pq)[c-\sigma(1-\gamma) \delta/T])) - \sigma qTSF'(S) V(R,K)}{K} \]

Also:
\[ V(R+dR,K) = p(R+dR)(1-\exp[-qTK]) - cTK - \alpha \theta [h(S+\exp[-qTK]dR)] - (1-\gamma)K \]
\[ + \alpha V(S+\exp[-qTK]dR), h(S+\exp[-qTK]dR)) \]
\[ = V(R,K) + [p(1-\exp[-qTK]dR), \alpha \exp[-qTK]F'(S)V(R,K)]dR. \]

Solving for \( V(R,K) \) gives

\[ (A4) \quad V(R,K) = \frac{p[F(S)-S]}{(F(S)-\alpha SF'(S))} \]

Inserting (A4) in (A3), noting that \( V(R,K) = \theta \) and rearranging,

we see that \( S \) satisfies:

\[ (A5) \quad F'(S) \cdot [1-x_0/F(S)]/[1-x_0/S] = 1/\alpha \]

where \( x_0 = \{c + [1-\alpha + \alpha \gamma]\theta/T\} \).

Repeating the analysis for the delayed investment case, equation (A4) is unchanged while (A3) becomes:

\[ (A3') \quad V(R,K) = pqT(S - 1/pq[c-(1-\gamma)\theta/T]) - \alpha qTSF'(S)V(R,K) \]

and hence (A5) becomes:

\[ (A5') \quad F'(S) \cdot [1-x_{g*}/F(S)]/[1-x_{g*}/S] = 1/\alpha \]

where \( x_{g*} = 1/pq \{c + [(1-\alpha + \alpha \gamma)/\alpha]\theta/T\} = (1/pq)\{c + [1-\alpha + \alpha \gamma](\theta/\alpha T)\} \)

Therefore the optimal equilibrium escapement \( S \) satisfies a Modified Golden Rule equation where the unit variable cost has been appropriately changed to include the rental cost of capital. In particular, in \( x_{g*}, (1-\alpha)\theta \) and \( \alpha \gamma \theta \) are the yearly charges for interest and depreciation respectively.

Equation (A5) and (A5') are precisely the M.G.R. equations obtained by assuming that capital is perfectly malleable with unit cost \( \theta \) and \( \theta/\alpha \) respectively. This can be seen in the delayed investment case as follows (the instantaneous investment
case is similar):

\[
V(R_1, K_1) = \sum_{n=1}^{n-1} \sigma \left\{ (R - S) - \delta I \right\} + \sum_{n=1}^{n+1} \sigma \left\{ 0 \right\}.
\]

Now \( \sum_{n=1}^{n} \delta I = \sum_{n=1}^{n} \delta \left\{ K - (1-\gamma)K \right\} \)

\[
= \delta \left\{ \sum_{n=1}^{n} \frac{1}{\alpha} \right\} \left\{ K - (1-\gamma)K \right\}
\]

\[
= \delta \left\{ (1-\alpha+\alpha\gamma)/\alpha \right\} \sum_{n=1}^{n} K - \delta/\alpha K_1.
\]

Hence \( V(R_1, K_1) = \sum_{n=1}^{n} \sigma \left\{ (R - S) - \delta\left\{ (1-\alpha+\alpha\gamma)/\alpha \right\} K \right\} + \delta/\alpha K_1. \)

If capital is malleable, \( K \) is precisely equal to \( E \), the optimal instantaneous effort. Since \( E = (1/\alpha T) \log(R / S) \) and \( \pi(R, S) = p(R, S) - cTE \) we obtain:

\[
\sum_{n=1}^{n} \sigma \left\{ (R - S) - (1/\alpha T) \alpha \left\{ (1-\alpha+\alpha\gamma)/\alpha \right\} \log(R / S) \right\} + \delta/\alpha K_1.
\]

The maximization of (A6), subject to \( R = F(S) \), calls for a steady-state escapement given by (A5'), namely \( S = S \) (Clark, 1976a).
Table I

Parameter values used in the base case runs of the model for the prawn and whale fisheries. (c.d. = "catcher day")

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Prawn Fishery</th>
<th>Whale Fishery</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;Fish&quot; price (p)</td>
<td>0.9 A$/kg</td>
<td>7000 US$/BWU</td>
</tr>
<tr>
<td>Variable Cost (c)</td>
<td>1600 A$/week/vessel</td>
<td>5000 US$/c.d.</td>
</tr>
<tr>
<td>Capital cost (δ)</td>
<td>0.47 x 10^6 A$/vessel</td>
<td>10000 US$/(c.d./year)</td>
</tr>
<tr>
<td>Net Revenue in Alternative Fishery (ρ)</td>
<td>0 A$/week/vessel</td>
<td>0 US$/c.d.</td>
</tr>
<tr>
<td>Depreciation Rate (γ)</td>
<td>0.15</td>
<td>0.15</td>
</tr>
<tr>
<td>Discount Factor (α)</td>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>Catchability (q)</td>
<td>0.00179 /week/vessel</td>
<td>1.3 x 10^-4 /c.d.</td>
</tr>
<tr>
<td>Max. Season Length (T)</td>
<td>26.0 weeks</td>
<td>1.0 years</td>
</tr>
<tr>
<td>Nat. Mortality Rate (m)</td>
<td>0.05 /week</td>
<td>0.1 /year</td>
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<tr>
<td>Maximum Fecundity (a)</td>
<td>42.0</td>
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<tr>
<td>Maximum Recruitment (b)</td>
<td>7.0 x 10^6 kg</td>
<td>1.186 x 10^7 BWU</td>
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Table II

The optimal value function $V(R,K)$ for the base case prawn fishery. Recruitment is given in millions of kilograms, capacity in standardized vessels and "value" in millions of Australian dollars.

<table>
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<th>Capital</th>
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<td>17.9</td>
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</table>
Table III. Values of $g(\cdot,\cdot,\cdot)$

Fixing the value of $i$ ($i=1,\ldots,M$), let $x_i=x$, $x_{i+1}=x$ and

$\Delta=x_{i+1}-x_i$. Then the function $g(x)$ of III.D can be written as a

polynomial in $x$, $g(x)=\sum_{i}^{4} \sum_{\nu=1}^{4-\gamma} g(i,\nu,\gamma)x_{\nu}$

where the coefficients $g(i,\nu,\gamma)$ are given by:

<table>
<thead>
<tr>
<th>$\gamma=1$</th>
<th>$\gamma=2$</th>
<th>$\gamma=3$</th>
<th>$\gamma=4$</th>
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</thead>
<tbody>
<tr>
<td>$n=1$</td>
<td>$2/\Delta^3$</td>
<td>$3/\Delta^2-6x./\Delta^3$</td>
<td>$-6x./\Delta^2+6x^2./\Delta^3$</td>
</tr>
<tr>
<td>$n=2$</td>
<td>$1/\Delta^2$</td>
<td>$-3x./\Delta^2+1/\Delta$</td>
<td>$3x^2./\Delta^2-2x^2./\Delta$</td>
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<tr>
<td>$n=4$</td>
<td>$1/\Delta^2$</td>
<td>$-3x./\Delta^2-1/\Delta$</td>
<td>$3x^2./\Delta^2+2x./\Delta$</td>
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</table>
Table IV. Comparison of Value Functions for Optimal Deterministic and Stochastic Policies

The value function \( V(R,K) \) is evaluated in a stochastic \((\sigma=0.58)\) environment, using two sets of policy functions: (i) the optimal policies for the \( \sigma=0.58 \) stochastic environment, and (ii) the "optimal" policies if the fishery were deterministic. In the table, recruitment is given in millions of kilograms, capacity \( K \) is in standardized vessels, and value is measured in millions of Australian dollars.

### \( V(R,K) \) for the Optimal Stochastic Policy

<table>
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<tr>
<th>K/R</th>
<th>0.20</th>
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<th>3.00</th>
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<td>8.87</td>
<td>10.7</td>
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</table>

### \( V(R,K) \) for the Optimal Deterministic Policy

<table>
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