CHAPTER 3
RESULTS

3.1 Slot Width d/h

In order to evaluate the integrals of equations 2.6 and 2.7, for arbitrary θ, a computer code was required. Firstly, the integral was re-written so that each expression of the integrand was expressed in terms of its modulus (r₁) and its argument (θ₁), i.e. it was re-written as,

\[ f(s) = r_1(s)e^{iθ_1(s)} \cdot r_2(s)e^{iθ_2(s)} \cdot r_3(s)e^{iθ_3(s)} \]  

Next a path of integration was required. One could not simply integrate from -1 to T along the x axis, since the integrand has a non-simple pole at x=0. To avoid this pole, the path of integration was changed to a box.

![Diagram showing path of integration](image)

Figure 3.1. Path of integration.
Since,

\[ \int_{\gamma_B} f = P.V. \int_{-1}^{1} f \]

(by an application of Cauchy's theorem) where the integrand is analytic inside the path of integration, \( \gamma_B \).

Along each section of the box, the path of integration was parametrized and the integral was written in the form:

\[ \int_{a}^{b} f(\gamma(t)) \cdot \gamma'(t) dt, \quad t \in (a, b) \]

Care was taken to break the intervals of integration up so that the proper branches of the arguments would be respected in each section of the complex plane. Once the integrand's arguments were properly determined and the parametrization determined, the integral was then re-written as a sum of integrals along the box, giving:

\[ \text{Im}z(\Delta) = \sum_{i=1}^{3} \int_{\gamma_i(t)} R(t) \cdot \sin(\Theta(t)) \, dt \]  \hspace{1cm} 3.2

\[ \text{Re}z(\Delta) = \sum_{i=1}^{3} \int_{\gamma_i(t)} R(t) \cdot \cos(\Theta(t)) \, dt \]  \hspace{1cm} 3.3
where
\[ R(t) = \text{modulus of integrand} \]
\[ R(t) = r_1(t) \cdot r_2(t) \cdot r_3(t) \]
and
\[ \Theta(t) = \text{argument of integrand} \]
\[ \Theta(t) = \theta_1(t) + \theta_2(t) + \theta_3(t) \]

If we consider a function \( \Lambda(\Delta) \) where
\[ \Lambda(\Delta) = \text{Im } z(\Delta) \]
then to find \( \Delta \) one must determine the root(s) of \( \Lambda \). To proceed the value of \( \Delta \) was incremented from zero upwards. For each increment, the value of \( \Lambda(\Delta) \) was evaluated for the corresponding path of integration. The incrementation was continued until a value of \( \Delta^* \) was found which gave a root for \( \Lambda \). This procedure was carried out on the Apollo computers using an integration subroutine called DQAG from the Slatec mathematical software library. DQAG is a Gauss-Konrad general purpose globally adaptive integration routine, complete with error control and integrand examiner (a procedure which examines the integrand to determine how many integration points should be chosen on a given interval). It was noted that as a function of \( \Delta \), \( \Lambda(\Delta) \) was monotonically increasing, so that \( \Delta^* \) was, in fact, the unique root of \( \Lambda \). In addition, \( \Delta \) was evaluated on a different path of integration with the exact values being reproduced. Once \( \Delta \) was established, then the value of \( d/h \) could be found by evaluation of Eq. 2.7. The results are shown in fig. 3.2.
3.2 Shape of Dividing Streamline

Once $\Delta$ is known, the shape of the dividing streamline could be found. To evaluate the integral in Eq. (2.8), the dependence on the parametrization variable $(t)$ was eliminated from the upper limit of integration. By a simple change of variables we obtain (see Appendix II):

$$x(t) = \frac{1}{\pi}(T(t) + 1)^2 \int_0^1 r \cdot \frac{[(T(t) + 1)r - (1 + \Delta)]^{\theta/\pi}}{[T(t) + 1 + r - 1]^{1+\theta/\pi}} \, dr$$  

with

$$T(t) = \frac{-te^{-at}}{\sin t}, te(0, \pi)$$
Again, expressing the functions in their real and imaginary parts allows the evaluation of \( z(t) = x(t) + iy(t) \) for \( t \in (0, \pi) \). Starting with \( t = 0 \) and incrementing in steps the integrals are evaluated until \( t = \pi \). The resulting values become the coordinates of the dividing streamline in the \( z \)-plane. As a check one should find that

\[
\pi(\pi) = \infty
\]

and

\[
y(\pi) = 1
\]

It was found that in fact \( x(3.1) \) was considerably larger than the other values and \( y(3.1) \) was around 0.99. For purposes of plotting the streamlines, the variables were rescaled so that the slot width is unity for all cases. The results are shown in figure 3.3.

![DIVIDING STREAMLINES](image)

**Figure 3.3. Plots of Dividing Streamlines.**
3.3 Mass Flow Out of Slot

To determine the mass flow out of the slot, one must rescale the variables, since in our analysis we scaled all variables such that the mass flow out of the slot was unity. By adjusting the slot width to be unity and keeping the value of $d/h$ fixed, we find that the mass flow out of the slot becomes $h/d$. Results are shown in fig. 3.4.

![Graph of Mass Flow From Slot](image-url)
3.4 Coefficient of Pressure

To find the coefficient of pressure across the slot surface, we can write Bernoulli's equation which holds everywhere, i.e.,

$$P(x) + 1/2\rho |u(x)|^2 = P_\infty + 1/2\rho U_\infty^2$$

Hence

$$C_P = \frac{P(x) - P_\infty}{1/2\rho U_\infty^2}$$

$$C_P = 1 - \frac{|u(x)|^2}{U_\infty^2}$$

and hence

$$C_P = 1 - |\zeta(T)|^2$$

$$C_P(x) = 1 - |(\frac{T}{T-\Delta})^{\theta/\pi}|^2$$  \(3.5\)

Next, one must determine the image of the slot surface in the T-plane. To do this, we assume that the slot surface in the T-plane can be parametrized as some curve $T(t) = \mu(t) + i\theta(t)$. We know for $t=0$, $T(t) = -1$ and since the dividing streamline is tangent to the slot surface at point (2) in the physical plane, the dividing streamline must be tangent to the slot surface in the T-plane. To check whether a point $T_0$ has its image on the slot surface, one must check if

$$x(T_0) = x + iy$$

has

$$y = 0, x \epsilon (0, d)$$
i.e. we must check

\[
x(t) = \frac{1}{\pi} (T(t) + 1)^2 \cdot \int_0^1 r \frac{(T(t) + 1)r - (1 + \Delta)^{\theta/\pi}}{[(T(t) + 1)r - 1]^{1+\theta/\pi}} \, dr
\]

has

\[
y(t) = 0, x(t) \in (0, d)
\]

\[
T(t) = \mu(t) + zh(t)
\]

Choosing \( \mu(t) = -1 + t \) we obtain

\[
x(t) = \frac{1}{\pi} (t + zh(t))^2 \cdot \int_0^1 r \frac{[(t + zh(t))r - (1 + \Delta)]^{\theta/\pi}}{[(t + zh(t))r - 1]^{1+\theta/\pi}} \, dr
\]

\[
= \frac{1}{\pi} (u + sv) \cdot \left[ \int_0^1 r_1 \cdot r_2 \cdot r_3 \cos \Theta \, dr + t \int_0^1 r_1 \cdot r_2 \cdot r_3 \sin \Theta \, dr \right]
\]

where

\[
r_1 = r
\]
\[
r_2 = ((t \cdot r - (1 + \Delta))^2 + r^2 h^2(t))^{\theta/2\pi}
\]
\[
r_3 = ((t \cdot r - 1)^2 + r^2 h^2(t))^{-1/2 - \theta/2\pi}
\]
\[
\Theta = \theta_1 + \theta_2 + \theta_3
\]
\[
\theta_1 = 0
\]
\[
\theta_2 = \frac{\theta}{\pi \cdot \tan^{-1} \frac{rh(t)}{r \cdot t - (1 + \Delta)}}
\]
\[
\theta_3 = (-1 - \theta/\pi) \cdot \tan^{-1} \frac{rh(t)}{r \cdot t - 1}
\]
\[
u = t^2 - h^2(t)
\]
\[
v = 2 \cdot t \cdot h(t)
\]
To proceed, \( t \) is incremented from 0 to \( \Delta + 1 \), \( h(t) \) is guessed until \( z(t) \) lies on the slot surface or specifically \( \text{Im} \, z(t) < 0.01 \), \( \text{Re} \, z(t) \in (0,d) \). So for each value of \( t \), a value of \( h(t) \) is found and hence the image of the slot surface is constructed in the \( T \)-plane. Once the points on the slot surface are found, then the values of

\[
C_P = 1 - |\zeta(T)|^2
\]

are easily calculated. Using the \( T \)-plane coordinates, \( T = a + ib \), for the points lying on the slot surface we have

\[
C_P = 1 - \left[ \frac{(a(a - \Delta) + b^2)^2 + b^2 \Delta^2}{(a - \Delta)^2 + b^2} \right]^{\theta/\pi}
\]

Results are shown in fig. 3.5.

![Graph of Coef of Pressure vs. Position for various \( \theta \)](image-url)
3.5 Velocity Across Slot

To find the velocity across the slot we note that

\[ u(x) + sv(x) = \overline{\zeta(T)} \]

hence

\[ u(x) + sv(x) = \left( \frac{T}{T - \Delta} \right)^{\theta/\pi} \]

for

\[ T = a + ib \]

\[ u + sv = \frac{[(a - \Delta) + b^2]^2 + b^2 \Delta^2]^{\theta/2\pi}}{[(a - \Delta)^2 + b^2]^{\theta/\pi}} \cdot e^{i(\theta/\pi)\Theta} \]

where

\[ \Theta = \tan^{-1} \frac{b\Delta}{a(a - \Delta) + b^2} \]

where the points \( T = a + ib \) are those found in the previous section. Inspection of the velocity fields show that these are not constant across the slot as shown in figures 3.6 to 3.9. The velocity vectors have been truncated at \( x/d = 0.98 \) to avoid the infinite velocities that occur at \( x/d = 1 \). Similar results are also observed in numerical simulations of similar (but not identical) slot geometries studied at U.B.C. [7],[19].
Velocity Field Across 20° Slot

Figure 3.6. Velocity Field Across Slot θ=20°

Velocity Field Across 40° Slot

Figure 3.7. Velocity Field Across Slot θ=40°
Velocity Field Across 90° Slot

Figure 3.8. Velocity Field Across Slot $\theta=90^\circ$

Velocity Field Across Orifice Slot

Figure 3.9. Velocity Field Across Slot $\theta=180^\circ$ (orifice)
3.6 Discharge Coefficient

The coefficient of discharge from a slot into a freestream can be expressed as (which may differ from other definitions):

\[ C_D = \frac{P_p - P_\infty}{\frac{1}{2}\rho V_s^2} \]  

where

- \( V_s = \) the average slot velocity defined as \( V_s = U_\infty h/d \)
- \( P_p = \) the plenum static pressure
- \( P_\infty = \) the freestream static pressure

Now since we have equal stagnation pressures

\[ P_p - P_\infty = P_{tp} - P_\infty \]

\[ = P_{t\infty} - P_\infty \]

\[ = \frac{1}{2}\rho U_\infty^2 \]

where

- \( P_{tp} = \) the plenum total pressure
- \( P_{t\infty} = \) the freestream total pressure.

This gives

\[ C_D = \frac{\frac{1}{2}\rho U_\infty^2}{\frac{1}{2}\rho V_s^2} = \frac{U_\infty^2}{V_s^2} \]
or

\[ \frac{U_{\infty}^2}{U_{\infty}^2 (h/d)^2} \]

hence

\[ C_D = \frac{d^2}{h^2} \]

This is plotted in fig 3.10. and may be compared with the values found experimentally as:

\[ C_D = A + B \cdot \frac{U_{\infty}^2}{V_s^2} \]

for similar slot geometries [19]. Presumably the presence of A and B are conditioned by viscous effects and by geometry.

![Figure 3.10. Discharge Coefficient For Various Slot Angles](image)