CHAPTER 2
PRESENT THEORY

2.1 DESCRIPTION AND JUSTIFICATION OF PRESENT MODEL

Since a 2-D model of an inviscid ideal fluid was to be considered, it seemed complex function methods were the obvious way to start. The goal was to examine the problem with a 90° slot and then eliminate this restriction to include arbitrary angles. For the case of equal stagnation pressures, it was obvious to start with Schwarz-Christoffel transformations to transform the slot flow to the upper half plane. The general background theory can be found in ref. [13],[14]. The required transformation between planes, for a 90° slot of finite depth (fig 2.1), is:

\[
\frac{dz}{dw} = K \sqrt{\frac{(w-a)(w-b)}{(w-c)(w-d)}}
\]

2.1

(where \(z\) is the physical plane and \(w\) is the upper half plane). The variables \(a,b,c,d\) represent the positions of the vertices in the upper half plane and \(K\) is a constant needed to properly orient the map. These must be chosen by trial and error in order to get the slot symmetric in the physical plane. This equation gives an elliptic integral which cannot be solved analytically. The slot was chosen to have a finite depth so that the velocity field could be prescribed at the bottom of the slot.
A computer code was written to solve and plot the transformation. However, the shape of the dividing streamline in the upperhalf plane did not seem analytically tractable and hence a computer code was needed to generate it. The problem was no longer an analytic model but a rather crude computational model.

To simplify matters, the slot was modified. It was suggested that the trailing edge of the slot was probably the most important aspect of the geometry so the leading edge was kept at a 90° corner and the trailing edge remained angled to form a cavity (see fig. 2.2).
Also, to include the tangential separation of the mainstream from the slot surface, it was hoped that by adding a source at the bottom of the cavity as well as a sink along the upstream wall, tangential separation could be modelled (ref [15]). However, it was not possible to find the location or the strength of the source that was needed. This becomes obvious when one remembers that the conformal map will preserve the angles between streamlines from plane to plane except at stagnation points. No matter where the source is, the dividing streamline intersects the upperhalf plane at 90°. Now, since the map is not conformal here (the injected flow has a stagnation point here), one does not find the dividing streamline leaving the slot at 90° but at 90° or π/2 radians multiplied by the local effect of the map (which turns out to be 3/2). So the dividing streamline separates at 135° regardless of where the sink lies. (see fig 2.3)

Figure 2.3 Dividing streamline.
One way, then, to achieve tangential separation was to have a small lip where the plane is turned through an angle of $180^\circ$ (see fig. 2.4). Hence, the idea was to incorporate a small lip in the cavity and use the computer to generate the shapes of the dividing streamlines. Again, the resulting transformations were difficult and the size of the lip was somewhat arbitrary; it was hard to determine the lip size so that it did not change the flow within the slot unrealistically. David Stropky [10] has used this idea on an infinitely deep slot but the lip size is still arbitrary. This route no longer seemed analytically feasible so the geometry and Schwarz-Christoffel methods were abandoned.

![Figure 2.4 Cavity with lip.](image)

A different technique, called the Helmholtz-Kirchoff method or hodograph method, was then considered. This method is better suited to the task since it does not concern itself with the shape of the solid boundaries but more with the velocity field. This allowed the model to have 2 pieces which were not connected. To achieve the goal of tangential separation, while not prescribing the velocity at the slot
surface but modelling the important aspects of the slot geometry and the inside of the slot as a plenum, the schematic of figure 2.5 was used.

![Diagram of slot geometry used in analysis.](image)

**Figure 2.5** Slot geometry used in analysis.

### 2.2 Helmholtz-Kirchoff Method

The Helmholtz-Kirchoff method is a conformal mapping procedure which maps a physical plane of interest into a simplified plane (usually the upperhalf plane) where the flow field can easily be analyzed [16],[17],[18]. To map to the upperhalf plane, the physical (z = x + iy) plane is examined in terms of its velocity field. Firstly, the physical plane is transferred to the hodograph plane having coordinates ζ = u - iv, where u and v represent the streamwise and normal velocity components. (This is simply the conjugate of the vector velocity.) Secondly, the physical plane is re-written in terms of potential coordinates or w = ϕ + iψ. For jet flows, this method is especially well suited since along the boundaries, in the physical plane, the flow is
inclined at a fixed angle and hence the argument of the complex velocity is constant. So the boundaries get mapped to straight lines in the hodograph plane. Also, lines of constant velocity get mapped to circular arcs in the hodograph plane. Similarly, streamlines get mapped to straight lines in the potential plane. The potential and hodograph plane are then mapped to the same upperhalf plane. The transformation between the upperhalf plane and the potential plane gives the shape of streamlines in the upperhalf pane. Then the upperhalf plane or T plane is mapped back to the physical plane since by definition:

\[
\frac{dw}{ds} = u - w = \zeta
\]

hence

\[
z = \int ds = \int \frac{dw}{\zeta}
\]

or

\[
z = \int \frac{1}{\zeta(T)} \frac{dw}{dT} dT
\]  \hspace{1cm} 2.2

So a mapping can be found from a complex physical plane to a simplified plane and hence the flow field can be analyzed by means of a simpler intermediate geometry.
2.3 Present Analysis

In the present analysis the physical plane (z-plane) is mapped to the upperhalf plane (T-plane) by means of the velocity potential (w-plane) and hodograph planes (ζ-plane). We consider the following physical plane where all velocities are scaled by the freestream velocity $U_\infty$ and distances by the asymptotic height of the jet downstream, $h$.

![Diagram of Physical or z-plane](image)

**Figure 2.6.** Physical or z-plane.

Hence at Position (1) velocity is 1
Position (3) velocity is 0
Position (2) velocity is less than 1
Position (4) velocity is arbitrarily large at an angle of $\theta$

Next, we examine the associated hodograph or ζ-plane
The points (1), (3), (4) are easily found in the hodograph plane by inspection but the dividing streamline is unknown in this plane. As well, we examine the flow field in terms of the potential or \( w \)-plane.

Here, as stated before, the streamlines become straight lines. One has the choice to arbitrarily decide the zero point for the velocity potential, \( \phi \), and this is chosen to be zero at point (2). As well, the zero for the stream function is chosen to be the streamline (1)-(2)-(1). The difference between
stream function values represents the mass flow between the stream functions. Hence, since the mass flow out of the slot is \( U_\infty h \) or 1 when scaled, the value of the stream function (3)-(4) must be -1.

Finally, we consider the upper half plane. Both the hodograph and potential planes are mapped onto this plane. The points (1), (2), (3), (4) can be arbitrarily chosen in the T-plane to simplify the mappings.

\[
T = \alpha + i \beta
\]

Figure 2.9. Upper half of T-plane.

To map the \( W \)-plane to the T-plane we use a Schwarz-Christoffel transformation of the polygon with vertices at points (2) and (3) having angles of \( 2\pi \) and 0, respectively.

The desired mapping is then

\[
\frac{dw}{dT} = K \frac{T + 1}{T}
\] 2.3
where (2) is chosen to be -1 in the T-plane and (3) is chosen to be 0 in the T-plane.

Further analysis of this mapping shows that the constant K has a value of $1/\pi$.

Next, to map the hodograph plane to the T-plane, we first open up the plane to the lower half $\Gamma$-plane by means of a power transformation.

![Diagram](a. Hodograph Plane) ![Diagram](b. Lower Half Plane)

Figure 2.10. Power transformation.

If we require

$$\Gamma = \zeta^a$$

then we require the line

$$se^{-i\theta}$$
to be mapped to $$te^{-i\pi}$$

we find

$$te^{-i\pi} = [se^{-i\theta}]^a$$
gives
\[ \alpha = \frac{\pi}{\theta} \]
hence
\[ \Gamma = \zeta^\frac{1}{2} \]

Next the lower half \( \Gamma \)-plane is mapped to the same upperhalf \( T \)-plane (as defined by the Schwarz-Christoffel mapping). This is achieved by means of a bilinear mapping with:

\[
\begin{align*}
\Gamma &= 0 & T &= 0 \\
\Gamma &= 1 & T &= \infty \\
\Gamma &= \infty & T &= \Delta
\end{align*}
\]
giving
\[ \Gamma = \frac{T}{T - \Delta} \]
and hence
\[ \zeta = \left( \frac{T}{T - \Delta} \right)^{\theta/\pi} \]

where \( \Delta \) is an unknown constant representing the position of the trailing edge of the slot, point (4), in the \( T \)-plane.

Finally, we map the upperhalf \( T \)-plane back to the physical plane remembering that for \( w = \phi + i\psi \) we have,
\[
\frac{dw}{dz} = \phi_e + \psi_e = u - iv = \zeta
\]

hence

\[
z = \int \frac{dw}{dT} \frac{1}{\zeta} dT
\]

becomes, using 2.3 and 2.4

\[
z = \frac{1}{\pi} \int_{T_0}^{T} \frac{s + 1}{s} \left(\frac{s - \Delta}{s}\right)^{\theta/\pi} ds + L
\]

If we let \(z=0\) be the leading edge of the slot, point (2), then \(z=0\) corresponds to \(T=-1\) so,

\[
z(T) = \frac{1}{\pi} \int_{-1}^{T} \frac{s + 1}{s} \left(\frac{s - \Delta}{s}\right)^{\theta/\pi} ds
\]

2.5

The figure 2.11 summaries the mapping sequence.

Figure 2.11. Mapping sequence.
2.4 Location of Delta

To find the value of the constant $\Delta$, we realize that the image of $\Delta$ in the physical plane is the downward corner, point (4), of the slot. Since the leading edge (point (2)) lies at 0, then for a level slot $z(\Delta) = d/h$ is a positive real number.

Hence we have the following two equations:

\[
\text{Im} z(\Delta) = \text{Im} \frac{1}{\pi} \int_{-1}^{\Delta} \frac{s + 1}{s} \left( \frac{s - \Delta}{s} \right)^{\theta/\pi} \, ds = 0 \quad 2.6
\]

\[
\text{Re} z(\Delta) = \text{Re} \frac{1}{\pi} \int_{-1}^{\Delta} \frac{s + 1}{s} \left( \frac{s - \Delta}{s} \right)^{\theta/\pi} \, ds = d/h \quad 2.7
\]

Equation 2.6 gives a constraint to find $\Delta$, once this is known, equation 2.7 will give the value of $d/h$.

2.5 Shape of Dividing Streamline

To find the shape of the dividing streamline we simply follow it through the different mappings from the $w$-plane to the $T$-plane and finally to the $Z$-plane. In the $w$-plane it is:

\[
\psi = 0, \phi \geq 0
\]
In the T-plane this becomes

\[ T(t) = \frac{-te^{-it}}{\sin t}, t \in (0, \pi) \]

The derivation of this result can be found in Appendix I.

Hence in the Z-plane it is

\[ z(T(t)) = z(t) = \frac{1}{\pi} \int_{-1}^{T(t)} \frac{T(t) + 1}{T(t)} \left( \frac{T(t) - \Delta}{T(t)} \right)^{\sigma/\pi} T(t) \, dt \]

2.8