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SYMMETRIC QUASIDEFINITE MATRICES*

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Abstract. It is stated here that a symmetric matrix K is *quasidefinite* if it has the form

$$K = \begin{bmatrix} -E & A^T \\ A & F \end{bmatrix},$$

where E and F are symmetric positive definite matrices. Although such matrices are indefinite, it is shown that *any* symmetric permutation of a quasidefinite matrix yields a factorization LDL^T.

This result is applied to obtain a new approach for solving the symmetric indefinite systems arising in interior-point methods for linear and quadratic programming. These systems are typically solved either by reducing to a positive definite system or by performing a Bunch–Parlett factorization of the full indefinite system at every iteration. This is an intermediate approach based on reducing to a quasidefinite system. This approach entails less fill-in than further reducing to a positive definite system, but is based on a static ordering and is therefore more efficient than performing Bunch–Parlett factorizations of the original indefinite system.

Key words. matrix factorization, linear programming, interior-point methods

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1. Introduction.

We call a symmetric matrix K *quasidefinite* if it has the form

$$K = \begin{bmatrix} -E & A^T \\ A & F \end{bmatrix},$$

where $E \in \Re^{n\times n}$ and $F \in \Re^{m\times m}$ are positive definite matrices with $m,\ n \geq 0$. The fact that quasidefinite matrices are nonsingular is trivial. To see it, consider the following system of equations:

$$\begin{bmatrix} -E & A^T \\ A & F \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}. \tag{1.1}$$

The positive definiteness of E allows us to use the first set of equations to solve for x in terms of y:

$$x = -E^{-1}(b - A^T y). \tag{1.2}$$

Substituting for x into the second set of equations yields

$$y = (F + AE^{-1}A^T)^{-1}(c + AE^{-1}b). \tag{1.3}$$

Again, the positive definiteness of E and F assures that the matrix $S := F + AE^{-1}A^T$ in (1.3) is also positive definite (and therefore nonsingular). Hence, there exists a unique solution to (1.1) for any b and c, which implies that K is nonsingular.

The above argument suggests an algorithm for solving (1.1). Namely, first solve for y using (1.3) and then solve for x using (1.2). However, when K is large and sparse,

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computational efficiency critically depends on maintaining sparsity in the matrix to be inverted in (1.3). Unfortunately, forming S can entail considerable fill-in. For example, if A has a single dense column, this matrix is completely full. Of course, one could try to solve the system in the other order, first solving for x directly and then solving for y as a function of x, but this approach encounters analogous inefficiencies when A has dense rows. In fact, both methods perform poorly when A has dense columns and dense rows.

A preferable approach is to apply an ordering heuristic (such as minimum-degree or minimum-local-fill) that prevents fill-in during the factorization of the *entire* matrix K. The caveat, however, is that *general* indefinite matrices are not guaranteed to be factorizable. The quasidefinite matrix K is indefinite, and so it is not clear a priori that one can (symmetrically) rearrange its rows and columns and factor the system in the resulting order. The fundamental result in this paper is that any symmetric permutation of a quasidefinite matrix is *guaranteed* to be factorizable.

It should be emphasized that no claim is made that all possible factorizations are equally stable numerically. Indeed, it is simple to give examples where one factorization is much worse than another (see §2). However, our aim is to apply the results presented here to the efficient implementation of interior-point methods for linear and quadratic programming, and in such cases we argue that there is not much disparity in the quality of the possible factorizations (where quality is measured by the relative sizes of the elements of the factorization). In fact, in the end they are all bad and yet it is rather remarkable that it is not difficult to obtain results with precision approximately equal to the square root of machine precision.

In situations where the relative sizes of the elements of a matrix factorization vary widely, it is important to limit the mixing of addition and subtraction operations in the calculation of the factors. This observation implies that, in the interior-point context, one should pivot out all the elements in either the upper left block (or the lower right block) first. While it is true that such strategies (which were called *conservative strategies* in [24]) are the safest, it is often possible to allow a little mixing in cases where such an allowance has a significant impact on the efficiency of the algorithm. The main result of this paper is that mathematically this poses no problem (but on a finite-precision machine one must be cautious about the degree of mixing allowed).

We end this section with a simple but important result.

THEOREM 1.1. *The inverse of a quasidefinite matrix is quasidefinite.*

Proof. Simple calculations show that

$$\begin{bmatrix} -E & A^T \\ A & F \end{bmatrix}^{-1} = \begin{bmatrix} -\bar{E} & \bar{A}^T \\ \bar{A} & \bar{F} \end{bmatrix},$$

where

$$\bar{E} = E^{-1} + E^{-1}A^T((F + AE^{-1}A^T)^{-1}AE^{-1}, \tag{1.4}$$

$$\bar{A} = (F + AE^{-1}A^T)^{-1}AE^{-1},$$

and

$$\bar{F} = (F + AE^{-1}A^T)^{-1}.$$

Applying the Sherman–Morrison–Woodbury formula to the right-hand side in (1.4), we see that

$$\bar{E} = (E + A^TF^{-1}A)^{-1}$$

and it follows that the inverse is quasidefinite. □

In the next section, we state and prove our main result. In §3, we apply this result to the system of equations arising in interior-point methods for mathematical programming. Then in §4 we present a simple example that illustrates the type of numerical difficulties that can arise. Finally, in §5, we present some numerical results.

2. The main result. We say that the symmetric nonsingular matrix K is *factorizable* if there exists a diagonal matrix D and a unit lower triangular matrix L such that $K = LDL^T$. The resulting pair (L, D) is then called a *factorization* of K. Furthermore, we say that the symmetric matrix K is *strongly factorizable* if every symmetric permutation of K yields a factorizable matrix. Thus, when K is strongly factorizable, there exists a factorization $PKP^T = LDL^T$ for any permutation P. To illustrate we present the following two examples.

Example 1. The matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \tag{2.1}$$

is not factorizable. To see this, note that for a 2×2 system LDL^T is simply

$$LDL^T = \begin{bmatrix} d_{11} & l_{21}d_{11} \\ l_{21}d_{11} & l_{21}^2 d_{11} + d_{22} \end{bmatrix}.$$

The zero in the upper left corner of (2.1) requires that $d_{11} = 0$, which in turn implies that

$$LDL^T = \begin{bmatrix} 0 & 0 \\ 0 & d_{22} \end{bmatrix}$$

for all unit lower triangular matrices L. Hence, it is clear that no factorization of (2.1) exists.

Example 2. The matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

is factorizable but not strongly factorizable, since (2.1) is a symmetric permutation of this matrix.

When a factorization exists, it is unique. This is a well-known result. See, e.g., [8]. In the remainder of this section, we show that symmetric quasidefinite matrices form a class of strongly factorizable matrices.

THEOREM 2.1. *Symmetric quasidefinite matrices are strongly factorizable.*

Proof. Fix a permutation matrix P. It suffices to show that the leading principal submatrices of PKP^T are nonsingular. But these submatrices are of the form $\bar{P}K_S\bar{P}^T$, where K_S is a principal submatrix of K and $\bar{P}$ is a permutation matrix. A principal submatrix K_S is of the form

$$K_S = \begin{bmatrix} -E_S & A_S^T \\ A_S & F_S \end{bmatrix}, \tag{2.2}$$

where E_S and F_S are principal submatrices of E and F, respectively, and so are positive definite. Thus K_S is quasidefinite and hence nonsingular, as required. □

There exist symmetric quasidefinite matrices for which some permutations yield much better factorizations than others. For example, consider

$$\begin{bmatrix} -\epsilon & 1 \\ 1 & 1 \end{bmatrix}, \tag{2.3}$$

where $\epsilon > 0$ is a small number. This matrix is symmetric quasidefinite and hence is strongly factorizable, but the two possible factorizations (corresponding to the matrix itself and its symmetric permutation) have very different properties. Indeed, factoring the matrix as given yields

$$D = \begin{bmatrix} -\epsilon & 0 \\ 0 & 1+\frac{1}{\epsilon} \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ -\frac{1}{\epsilon} & 1 \end{bmatrix}, \tag{2.4}$$

whereas factoring the symmetrically permuted matrix gives

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -(1+\epsilon) \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}. \tag{2.5}$$

It is clear that (2.5) is much better than (2.4). To quantify this notion, we introduce an a priori measure of stability τ defined as

$$\tau = \frac{\| \, |L| \, |D| \, |L^T| \, \|}{\|K\|}, \tag{2.6}$$

where $\| \cdot \|$ denotes the L^∞ matrix norm and $| \cdot |$ denotes the matrix whose elements are the absolute values of the indicated matrix. If τ is close to one, the factorization is stable (see [10, p. 136]). Larger values indicate less stability. For the matrix in (2.3), $\tau = 1 + 1/\epsilon$ whereas for its symmetric permutation we get $\tau = (3+\epsilon)/2$, which is clearly much better.

Our primary interest in symmetric quasidefinite matrices arises in the context of interior-point methods for linear and quadratic programming. We show in §3 that matrices such as the one considered here do not arise in that context.

3. Application to interior-point methods.

Consider the following linear programming problem

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to } & Ax \geq b, \\ & x \geq 0, \end{aligned} \tag{3.1}$$

where A is an $m \times n$ matrix, c is an n-vector, and b is an m-vector. The dual of this problem is

$$\begin{aligned} \text{maximize} \quad & b^T y \\ \text{subject to } & A^T y \leq c, \\ & y \geq 0. \end{aligned}$$

Adding surplus variables w to the primal and slack variables z to the dual, we can rewrite the problems as follows

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to } & Ax - w = b, \\ & x, w \geq 0 \end{aligned} \tag{3.2}$$

and

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + z = c, \\ & y, z \geq 0. \end{array}$$

For problems presented in this form, the system of equations that must be solved at each iteration of the interior-point algorithm has the following form involving a quasidefinite matrix:

$$\begin{bmatrix} -X^{-1}Z & A^T \\ A & Y^{-1}W \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \sigma \\ \rho \end{bmatrix}, \tag{3.3}$$

where X, Y, W, and Z denote diagonal matrices with the components of x, y, w, and z on their diagonals, respectively (and, as an interior-point method, all the components of these vectors are strictly positive at every iteration).

For interior-point methods, the main computational burden lies in solving systems of the form in (3.3). Early implementations for linear programming did not operate on system (3.3) directly, but rather dealt with the symmetric positive definite system obtained from (3.3) by solving for Δx in terms of Δy using the first set of equations and then eliminating Δx from the second set:

$$\Delta x = -XZ^{-1}(\sigma - A^T \Delta y) \tag{3.4}$$

and

$$(AXZ^{-1}A^T + Y^{-1}W)\Delta y = (\rho + AXZ^{-1}\sigma). \tag{3.5}$$

The advantage of this approach is that the matrix $AXZ^{-1}A^T + Y^{-1}W$ is symmetric and positive definite, so that well-known and well-behaved methods such as Cholesky factorization (which was used in the implementations described in [4], [12]–[15], [17], [18], [21], [23]) or preconditioned conjugate gradient (used in the implementations described in [4], [11], [16]) can be used to solve systems involving this matrix. However, the disadvantage is that $AXZ^{-1}A^T$ can be quite dense compared to A if A has dense columns.

Recent papers have suggested that it might be better to solve indefinite systems such as (3.3) at every iteration. This suggestion was first put forth by researchers in Stanford's Systems Optimization Laboratory ([5], [19], [9]) and by Turner [20]. Subsequently, Fourer and Mehrotra [6] began experimenting with the indefinite system approach. All of these papers rely on doing a Bunch–Parlett ([3], [2]) factorization of the indefinite system.

Solving the indefinite system mitigates the fill-in caused when dense columns are present, but Bunch–Parlett factorizations tend to be more computationally burdensome. As such, solving the indefinite system offers an advantage when dense columns are present, but tends to be slower on most other problems.

We apply Theorem 2.1 to obtain a robust procedure that is not hampered by dense columns. Unfortunately, linear programs as formulated usually do not fit directly into the form given in (3.1). For example, some of the variables might not be constrained to be nonnegative (we call these *free variables*), and some of the constraints might be equality instead of inequality constraints. It turns out that the algorithm can be modified to handle free variables as follows. For each free variable x_j, simply set the corresponding dual slack variable z_j to zero everywhere it appears in the algorithm. Similarly, equality constraints are handled by setting w_i equal to zero for each equality constraint. This makes for a very simple algorithm, but regretably the matrix

$$K = \begin{bmatrix} -X^{-1}Z & A^T \\ A & Y^{-1}W \end{bmatrix} \tag{3.6}$$

is no longer quasidefinite as zeros have now appeared on the main diagonal. This problem is handled by introducing a two-tiered elimination scheme. In the first tier, we select pivot elements associated with some (or maybe even all) of the nonzero diagonal elements in (3.6). As we shall show, pivoting on these elements in any order is safe. Furthermore, the reduced system produced by symmetric Gaussian elimination is eventually guaranteed to be itself a quasidefinite system and so from that point on we can enter tier two and choose the remaining pivots in any order.

To make the above explanation more precise, we partition $X^{-1}Z$ and $Y^{-1}W$ into 2×2 blocks

$$X^{-1}Z = \begin{bmatrix} E_1 & \\ & E_2 \end{bmatrix},$$

$$Y^{-1}W = \begin{bmatrix} F_1 & \\ & F_2 \end{bmatrix},$$

putting all the zero elements (and perhaps some nonzeros) of $X^{-1}Z$ into E_2 and all the zero elements (and perhaps some nonzeros) of $Y^{-1}W$ into F_2. Then we partition system (3.3)

$$\begin{bmatrix} -E_1 & & A_{11}^T & A_{21}^T \\ & -E_2 & A_{12}^T & A_{22}^T \\ A_{11} & A_{12} & F_1 & \\ A_{21} & A_{22} & & F_2 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta y_1 \\ \Delta y_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \rho_1 \\ \rho_2 \end{bmatrix}. \tag{3.7}$$

Since E_1 and F_1 are positive definite, we move those blocks to the upper left-hand corner

$$\begin{bmatrix} -E_1 & A_{11}^T & & A_{21}^T \\ A_{11} & F_1 & A_{12} & \\ & A_{12}^T & -E_2 & A_{22}^T \\ A_{21} & & A_{22} & F_2 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta y_1 \\ \Delta x_2 \\ \Delta y_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \rho_1 \\ \sigma_2 \\ \rho_2 \end{bmatrix}. \tag{3.8}$$

Now, the upper left 2×2 block is quasidefinite and so can be used to solve (with pivots in any order) for Δx_1 and Δy_1:

$$\begin{bmatrix} \Delta x_1 \\ \Delta y_1 \end{bmatrix} = \begin{bmatrix} -E_1 & A_{11}^T \\ A_{11} & F_1 \end{bmatrix}^{-1} \left(\begin{bmatrix} \sigma_1 \\ \rho_1 \end{bmatrix} - \begin{bmatrix} & A_{21}^T \\ A_{12} & \end{bmatrix} \begin{bmatrix} \Delta x_2 \\ \Delta y_2 \end{bmatrix} \right).$$

Substituting this into the last equations in (3.8), we get the following system for Δx_2 and Δy_2:

$$\left(\begin{bmatrix} -E_2 & A_{22}^T \\ A_{22} & F_2 \end{bmatrix} - \begin{bmatrix} & A_{12}^T \\ A_{21} & \end{bmatrix} \begin{bmatrix} -E_1 & A_{11}^T \\ A_{11} & F_1 \end{bmatrix}^{-1} \begin{bmatrix} & A_{21}^T \\ A_{12} & \end{bmatrix} \right) \begin{bmatrix} \Delta x_2 \\ \Delta y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_2 \\ \rho_2 \end{bmatrix} - \begin{bmatrix} & A_{12}^T \\ A_{21} & \end{bmatrix} \begin{bmatrix} -E_1 & A_{11}^T \\ A_{11} & F_1 \end{bmatrix}^{-1} \begin{bmatrix} \sigma_1 \\ \rho_1 \end{bmatrix}. \tag{3.9}$$

In Theorem 1.1, we showed that the inverse of a quasidefinite matrix is quasidefinite. Hence, we introduce the following notation for the inverse appearing above:

$$\begin{bmatrix} -E_1 & A_{11}^T \\ A_{11} & F_1 \end{bmatrix}^{-1} = \begin{bmatrix} -\bar{E}_1 & \bar{A}_{11}^T \\ \bar{A}_{11} & \bar{F}_1 \end{bmatrix}.$$

Then the triple matrix product in (3.9) can be written as

$$\begin{bmatrix} & A_{12}^T \\ A_{21} & \end{bmatrix} \begin{bmatrix} -E_1 & A_{11}^T \\ A_{11} & F_1 \end{bmatrix}^{-1} \begin{bmatrix} & A_{21}^T \\ A_{12} & \end{bmatrix} = \begin{bmatrix} A_{12}^T \bar{F}_1 A_{12} & A_{12}^T \bar{A}_{11} A_{21}^T \\ A_{21} \bar{A}_{11}^T A_{12} & -A_{21} \bar{E}_1 A_{21}^T \end{bmatrix},$$

and we see that the system in (3.9) is quasidefinite if and only if

$$E_2 + A_{12}^T \bar{F}_1 A_{12} \quad \text{and} \quad F_2 + A_{21} \bar{E}_1 A_{21}^T$$

are positive definite. Clearly, the larger the dimension of E_1 and F_1, the greater the likelihood for this.

To summarize, our procedure is based on partitioning the rows and columns of A into two tiers. Elements belonging to the first tier are eliminated first (in any order) and then elements from the second tier are eliminated. As long as the tiers are chosen appropriately, this scheme is guaranteed to work. While it is certainly possible to go through once at the beginning and ensure that enough elements are put into tier one, experience has shown that simple, conservative heuristics work just as well (as long as they are conservative). For example, our code, which is called LOQO and is described in [22], [24], uses such a conservative approach. This code actually uses four tiers. The first tier corresponds to all variables except those that are free variables and those associated with dense columns. The second tier consists of all inequality constraints and the dense columns. The third tier then has the equality constraints and the fourth tier the free variables. Within tiers, elimination order is determined by one of the usual heuristics such as minimum-degree or minimum-local-fill. The heuristic for determining which columns to call dense works as follows. First, out of the n columns, look at the m sparsest. Multiply the density of the densest of these m columns by a number larger than one (10 is the default) and declare a column to be dense if and only if its density exceeds this threshold.

We now return to the question of numerical stability in the context of interior-point methods. It was proved in [1] that strict complementarity holds in the limit (at least in the case of continuous trajectories of the affine scaling algorithm but it seems to be true in general). In the present context, this means that the diagonal elements of $X^{-1}Z$ and $Y^{-1}W$ all tend to zero or infinity. In fact, numerical experience indicates that the rate at which the elements tend to zero or to infinity is the same from one element to the next. Hence, 2×2 matrices such as (2.3) do not arise. Instead, 2×2 matrices where both the diagonal elements go to zero, both go to infinity, or one goes to zero and the other goes to infinity are more relevant. For

$$\begin{bmatrix} -\epsilon & 1 \\ 1 & \epsilon \end{bmatrix},$$

we get $\tau = (1 + \epsilon + 2/\epsilon)/(1 + \epsilon) \approx 2/\epsilon$ and for its symmetric permutation τ is the same. On the other hand, for

$$\begin{bmatrix} -\epsilon & 1 \\ 1 & \frac{1}{\epsilon} \end{bmatrix},$$

we get $\tau = (1 + 3/\epsilon)/(1 + 1/\epsilon) \approx 3$ and for its symmetric permutation $\tau = 1$. The other cases are similar. What one observes is that the level of instability is essentially independent of the permutation.

However, the value of τ does not tell the whole story. In the next section, we consider a specific example that illustrates the situations that can arise.

4. An example. Consider the following linear programming problem:

$$\begin{gathered}\text{minimize } x_1 + x_2, \\[1em] x_1 + 2x_2 \geq 1, \\ 2x_1 + x_2 \geq 1, \\[1em] x_1 \geq 0, \quad x_2 \text{ free.}\end{gathered} \tag{4.1}$$

For this problem, the symmetric indefinite system that must be solved at each iteration involves a matrix whose lower triangular part has the following form:

$$\begin{bmatrix} -\epsilon_1 & & & \\ & 0 & & \\ 1 & 2 & \delta_1 & \\ 2 & 1 & & \delta_2 \end{bmatrix},$$

where ϵ_1, δ_1, and δ_2 are all positive and tending to zero (at roughly the same rate) as the iterations progress (the zero on the second diagonal position arises from the fact that x_2 is a free variable). The zero diagonal element could pose a problem and so any static ordering must anticipate this and defer this pivot till the end. Therefore, the lower triangular part of the matrix becomes

$$\begin{bmatrix} -\epsilon_1 & & & \\ 1 & \delta_1 & & \\ 2 & & \delta_2 & \\ & 2 & 1 & 0 \end{bmatrix}.$$

This matrix is factorizable since after the first pivot, the reduced matrix is symmetric quasidefinite. However, to see what happens, consider applying symmetric Gaussian elimination to this matrix. After eliminating the nonzeros under the first two columns, the lower triangle of the resulting 2×2 submatrix becomes

$$\begin{bmatrix} (\delta_2 + \frac{4}{\epsilon_1}) - \frac{4}{\epsilon_1^2(\delta_1 + \frac{1}{\epsilon_1})} & \\ 1 - \frac{4}{\epsilon_1(\delta_1 + \frac{1}{\epsilon_1})} & \frac{4}{(\delta_1 + \frac{1}{\epsilon_1})} \end{bmatrix}. \tag{4.2}$$

In the elimination process, the parenthesized expressions are evaluated before the other operations. Hence, in finite precision arithmetic, once each of the small parameters becomes smaller than the square root of the arithmetic's precision, these expressions simplify to

$$\delta_2 + \frac{4}{\epsilon_1} = \frac{4}{\epsilon_1} \quad \text{and} \quad \delta_1 + \frac{1}{\epsilon_1} = \frac{1}{\epsilon_1},$$

and so (4.2) becomes

$$\begin{bmatrix} 0 & \\ -3 & 4\epsilon_1 \end{bmatrix},$$

which clearly presents a problem for the next stage of the elimination. However, using exact arithmetic, (4.2) simplifies to

$$\begin{bmatrix} \delta_2 + \frac{4\delta_1}{1+\epsilon_1\delta_1} & \\ 1 - \frac{4}{1+\epsilon_1\delta_1} & \frac{4\epsilon_1}{1+\epsilon_1\delta_1} \end{bmatrix}. \tag{4.3}$$

From this exact expression it is clear that the given order and the order obtained by interchanging the last two pivots generate similar values for τ (since both diagonal entries are of the same magnitude). Hence, our estimate of instability τ, defined by (2.6), fails to differentiate between these two permutations. What has gone wrong is the common problem of mixing addition and subtraction of large numbers.

In addition, (4.3) suggests that whenever an exact zero appears on a diagonal it might be a good idea to set the value to either plus or minus the square root of the arithmetic's precision. In fact, this is what is done in our code (which is described in [24]) and we are able to solve (4.1) to full precision.

This example shows that instability can occur. However, when it does, one can still expect to get results as accurate as the square root of machine precision. This seems to hold true even for large problems. A partial explanation for this is the fact that interior-point methods have a certain autoscaling property (i.e., they attempt to follow the central trajectory), which helps to make the diagonal elements go to zero or to infinity all at the same rate.

5. Numerical experiments and conclusions. Using our code, we computed the ratio of the largest to the smallest of the absolute values of the diagonal elements of D on the last iteration of the algorithm. On the eighty or so test problems in the NETLIB suite [7] this ratio ranged between `1.0e+19` and infinity (infinity means that an exact zero appeared on the diagonal, which can happen when rank deficiency occurs due to primal degeneracy). These ratios are tabulated in Tables 1 and 2. Given such large values for this ratio, it is quite remarkable that the code was able to solve all but two problems (greenbeb and pilot87) to eight significant figures of accuracy (and pilot87 stopped just short with seven significant figures). These tests were performed on an IBM RS 6000, which implements the IEEE floating-point standard and therefore has 15 digits of precision (53 bits). It is also interesting to note that the two that ran into numerical trouble were not necessarily the ones with the largest ratios. It turns out that for many problems in this suite the matrix to which K in (3.6) is converging is actually a singular matrix (due to primal or dual degeneracy) and so numerical difficulties will exist regardless of the ordering.

We also computed the value of τ, defined by (2.6), for each of the test problems. For these computations, we scaled the matrix K by dividing each row and each column by the maximum between `1.0` and the square root of the corresponding diagonal element. Tables 3 and 4 show the value of τ on the last iteration. It turns out that in every case this was the largest value over all iterations of the algorithm. Again there seems to be no correlation between those problems that encountered numerical difficulties and those that had large τ values. This lack of correlation gives credence to the notion that numerical difficulty arises primarily from primal and dual degeneracy.

TABLE 1
Ratios of diagonals in factorization (problems 1-p).

Problem name	Ratio	Iterations	Primal infeas	Dual infeas	Significant figures
25fv47	5.8709e+37	29	4.61e−13	2.56e−13	8
80bau3b	Infinity	43	6.43e−10	1.28e−11	8
adlittle	2.2458e+21	14	8.87e−11	1.74e−16	8
afiro	4.1738e+24	13	6.00e−14	8.98e−14	8
agg	7.8694e+41	26	3.98e−12	3.62e−09	8
agg2	1.1431e+32	22	1.09e−14	4.99e−12	8
agg3	4.8834e+34	22	5.34e−15	5.24e−12	8
bandm	8.6392e+27	20	8.74e−11	5.95e−13	9
beaconfd	1.9200e+28	14	7.95e−11	6.14e−12	8
blend	5.8329e+24	14	1.90e−12	6.75e−11	9
bnl1	1.0244e+35	35	2.39e−12	2.23e−12	8
bnl2	3.4315e+54	40	1.62e−09	1.94e−11	8
boeing1	1.0681e+41	28	1.26e−08	2.09e−13	9
boeing2	5.1782e+38	28	1.47e−15	1.71e−10	8
bore3d	Infinity	17	1.72e−08	2.48e−16	8
brandy	3.3141e+31	22	7.33e−08	1.97e−11	9
capri	8.8349e+27	23	1.07e−12	4.52e−11	8
cycle	6.2422e+36	32	5.75e−09	3.27e−11	9
czprob	6.7245e+30	38	1.15e−11	1.08e−13	8
d2q06c	2.8169e+47	38	2.24e−10	1.50e−09	8
degen2	1.0067e+20	14	5.94e−10	3.75e−16	8
degen3	Infinity	17	9.54e−10	2.94e−12	8
e226	8.4768e+33	22	5.12e−13	9.02e−14	9
etamacro	2.0071e+31	30	3.19e−13	1.62e−14	8
fffff800	1.7007e+39	36	3.97e−13	2.53e−08	8
finnis	8.9977e+33	26	1.00e−13	4.91e−14	8
fit1d	1.6286e+22	21	4.81e−08	3.61e−15	9
fit1p	3.5353e+24	26	5.68e−10	5.52e−12	8
fit2d	5.1893e+19	24	1.50e−08	1.70e−16	8
fit2p	6.3007e+23	24	4.24e−11	1.86e−12	8
forplan	4.8253e+40	29	7.28e−14	1.33e−10	8
ganges	3.3747e+33	23	5.06e−12	3.88e−11	9
gfrdpnc	2.9347e+25	19	1.82e−10	3.61e−14	8
greenbea	8.1336e+44	50	2.30e−06	2.86e−12	8
greenbeb	2.8818e+28	30	1.80e−06	1.82e−10	3
grow15	2.3749e+33	23	2.35e−06	7.18e−14	10
grow22	3.1747e+33	27	7.99e−06	9.88e−15	10
grow7	6.0359e+31	20	2.62e−06	3.69e−13	10
israel	5.7337e+28	28	3.58e−16	4.36e−15	9
kb2	3.7487e+26	20	2.54e−06	5.84e−10	8
lotfi	1.1710e+44	25	1.83e−14	7.60e−12	8
maros	9.9261e+31	28	3.27e−10	9.37e−10	8
nesm	3.3790e+25	37	9.48e−13	2.61e−14	8
perold	1.7497e+36	39	1.79e−13	5.11e−11	9
pilot4	5.9526e+36	38	1.81e−11	1.51e−10	8
pilot87	1.7318e+50	45	3.45e−12	7.84e−10	7
pilotja	1.1399e+37	38	3.06e−12	5.56e−10	8
pilotnov	1.1636e+36	29	2.24e−11	6.77e−11	8
pilots	1.1568e+45	44	5.17e−12	1.07e−08	8
pilotwe	2.8908e+32	39	7.47e−12	2.37e−14	8

TABLE 2
Ratios of diagonals in factorization (problems r-z).

Problem name	Ratio	Iterations	Primal infeas	Dual infeas	Significant figures
recipe	1.6775e+31	13	3.21e−08	7.81e−11	9
sc105	6.9689e+26	14	1.89e−13	1.56e−12	8
sc205	8.9380e+30	17	2.38e−14	4.35e−12	9
sc50a	4.2398e+26	14	2.02e−14	9.54e−13	9
sc50b	2.2211e+26	13	1.20e−14	5.61e−13	9
scagr25	3.4668e+24	18	5.96e−13	8.69e−14	9
scagr7	7.6158e+21	15	2.55e−13	4.72e−12	8
scfxm1	2.0143e+33	26	6.42e−10	3.19e−10	8
scfxm2	4.1487e+35	28	2.05e−08	1.80e−11	9
scfxm3	1.2522e+36	28	2.29e−07	3.43e−11	8
scorpion	Infinity	16	1.66e−10	1.35e−15	8
scrs8	4.1479e+39	23	1.11e−10	1.90e−16	8
scsd1	3.9397e+20	15	1.04e−11	2.30e−16	9
scsd6	3.3169e+19	17	7.44e−11	3.50e−16	8
scsd8	9.2737e+19	17	2.43e−10	8.85e−16	8
sctap1	6.3855e+21	18	6.23e−11	3.55e−13	8
sctap2	Infinity	16	2.84e−10	1.43e−12	8
sctap3	9.0610e+18	16	1.76e−10	1.51e−12	8
seba	1.3798e+27	19	8.25e−09	4.10e−12	8
share1b	8.1766e+33	26	1.25e−10	2.12e−11	8
share2b	1.1939e+22	14	6.02e−09	1.19e−14	8
shell	5.0921e+27	25	5.99e−14	3.11e−12	8
ship04l	3.2699e+26	20	9.49e−11	1.22e−15	9
ship04s	6.1811e+25	19	1.76e−10	1.38e−15	8
ship08l	3.5321e+27	20	6.52e−11	2.02e−15	8
ship08s	4.0247e+27	20	4.07e−11	2.60e−15	9
ship12l	1.9722e+27	24	2.35e−10	4.88e−15	8
ship12s	4.9804e+27	22	1.72e−10	1.27e−14	8
sierra	1.2960e+34	21	1.06e−11	1.83e−12	8
stair	1.4413e+30	21	2.63e−11	1.78e−12	9
standata	4.0684e+30	23	1.34e−12	8.55e−14	9
standmps	1.0390e+34	32	3.12e−13	9.57e−13	8
stocfor1	7.6992e+24	17	2.13e−08	5.74e−11	8
stocfor2	1.0840e+28	31	2.83e−10	1.26e−10	8
tuff	1.3385e+36	25	2.09e−12	2.87e−13	8
vtpbase	9.4701e+28	28	2.17e−11	1.66e−06	9
wood1p	1.3333e+24	28	1.94e−06	3.45e−14	8
woodw	4.6615e+28	27	3.07e−09	4.97e−14	9

TABLE 3
A priori test of stability (problems 1-p).

Problem	$\|K\|$	$\|L\|$	$\|D\|$	last τ
25fv47	4.32e+02	1.08e+19	1.08e+19	2.51e+16
80bau3b	2.16e+02	3.98e+13	3.96e+13	7.28e+11
adlittle	1.03e+02	1.76e+09	3.03e+10	3.24e+08
afiro	3.36e+00	2.87e+12	2.04e+12	4.04e+12
agg	4.28e+02	3.62e+14	5.54e+16	4.68e+14
agg2	4.29e+02	1.20e+14	6.50e+13	2.14e+12
agg3	4.29e+02	2.08e+14	1.13e+14	3.68e+12
bandm	1.06e+03	1.38e+13	7.63e+13	9.54e+11
beaconfd	1.10e+03	2.72e+13	2.09e+15	5.00e+12
blend	1.05e+02	2.54e+12	2.42e+12	1.09e+11
bnl1	4.18e+02	1.45e+14	9.67e+13	9.55e+11
bnl2	5.07e+02	4.51e+26	4.51e+26	2.63e+24
boeing1	9.92e+02	5.93e+17	1.49e+14	1.03e+12
boeing2	1.03e+03	1.50e+14	1.29e+14	5.81e+11
bore3d	3.13e+03	2.98e+14	3.33e+14	2.51e+11
brandy	9.43e+02	9.69e+16	7.65e+15	3.85e+15
capri	5.61e+02	1.06e+14	1.05e+14	5.63e+11
cycle	3.37e+03	2.16e+18	2.03e+15	2.24e+15
czprob	1.48e+02	1.24e+13	1.20e+13	9.19e+10
d2q06c	7.32e+03	1.88e+19	1.48e+18	5.42e+15
degen2	3.51e+01	4.44e+08	1.29e+08	1.70e+08
degen3	9.37e+01	6.28e+09	3.33e+09	1.25e+09
e226	9.84e+02	1.74e+13	1.33e+13	1.71e+11
etamacro	3.08e+03	9.76e+12	1.91e+13	7.12e+10
fffff800	1.13e+05	8.10e+15	6.53e+15	8.91e+10
finnis	3.95e+01	1.34e+17	1.34e+17	3.39e+15
fit1d	3.25e+03	4.78e+10	7.31e+12	1.67e+10
fit1p	1.42e+05	1.50e+11	5.17e+11	3.24e+07
fit2d	3.22e+03	4.68e+08	8.24e+09	1.24e+08
fit2p	2.73e+05	7.94e+11	7.94e+11	2.90e+06
forplan	3.09e+03	1.83e+17	2.44e+20	2.35e+17
ganges	1.20e+01	6.84e+16	6.29e+16	2.02e+16
gfrdpnc	2.35e+03	7.44e+12	6.01e+12	1.12e+10
greenbea	1.00e+02	3.21e+22	3.61e+22	8.68e+20
greenbeb	1.41e+02	3.01e+14	2.39e+14	8.63e+12
grow15	5.21e+00	5.86e+16	4.01e+16	2.54e+16
grow22	5.21e+00	5.67e+16	5.51e+16	2.60e+16
grow7	5.21e+00	9.28e+15	6.36e+15	3.72e+15
israel	9.22e+03	3.11e+12	3.82e+12	1.86e+09
kb2	6.02e+02	1.48e+13	2.55e+13	1.55e+11
lotfi	4.83e+03	1.53e+22	1.53e+24	3.17e+20
maros	5.10e+04	1.58e+16	1.23e+16	4.56e+13
nesm	9.35e+01	1.21e+12	1.21e+12	5.46e+10
perold	7.85e+04	1.83e+17	1.91e+19	2.12e+15
pilot4	6.85e+04	1.02e+17	4.35e+17	6.18e+14
pilot87	1.10e+03	1.90e+16	7.55e+16	1.01e+14
pilotja	1.56e+06	1.36e+17	2.20e+18	4.55e+13
pilotnov	1.19e+07	2.06e+17	2.84e+18	1.64e+13
pilots	2.66e+02	9.69e+15	1.04e+16	9.73e+13
pilotwe	7.50e+03	2.27e+15	2.54e+17	8.56e+13

TABLE 4
A priori test of stability (problems r-z).

Problem	$\|K\|$	$\|L\|$	$\|D\|$	last τ
recipe	9.15e+02	5.90e+15	5.68e+15	3.15e+13
sc105	5.52e+00	3.66e+13	3.66e+13	3.83e+13
sc205	5.52e+00	1.88e+15	1.88e+15	1.88e+15
sc50a	5.50e+00	1.50e+13	1.29e+13	1.24e+13
sc50b	1.00e+01	6.51e+12	6.51e+12	3.24e+12
scagr25	1.70e+01	1.99e+12	1.90e+12	3.61e+11
scagr7	1.70e+01	1.43e+11	9.15e+10	5.38e+10
scfxm1	8.26e+02	1.86e+16	9.30e+17	2.34e+15
scfxm2	8.24e+02	9.11e+17	4.55e+19	1.15e+17
scfxm3	8.26e+02	1.57e+18	7.87e+19	1.98e+17
scorpion	6.85e+00	6.49e+07	6.34e+07	2.47e+07
scrs8	3.59e+02	8.15e+19	8.08e+19	4.80e+17
scsd1	4.32e+00	2.30e+09	2.30e+09	2.00e+09
scsd6	5.79e+00	2.43e+09	2.09e+09	1.43e+09
scsd8	4.31e+00	1.62e+10	1.04e+10	1.15e+10
sctap1	2.89e+02	7.26e+09	2.58e+09	1.37e+09
sctap2	4.65e+02	4.63e+09	1.46e+10	9.49e+08
sctap3	4.65e+02	5.09e+09	1.76e+10	1.04e+09
seba	1.34e+03	3.71e+13	3.71e+13	2.78e+10
share1b	2.05e+03	7.36e+16	1.12e+17	2.26e+14
share2b	7.89e+02	5.47e+11	3.43e+11	2.68e+11
shell	1.80e+01	6.40e+11	4.88e+11	1.55e+11
ship04l	7.10e+01	6.57e+09	3.25e+08	3.86e+08
ship04s	4.90e+01	8.03e+08	1.42e+07	6.86e+07
ship08l	7.10e+01	2.08e+08	1.53e+08	5.21e+06
ship08s	4.10e+01	7.76e+08	1.01e+08	5.76e+07
ship12l	5.70e+01	1.93e+09	5.94e+08	9.70e+07
ship12s	3.10e+01	3.18e+08	1.44e+08	2.70e+07
sierra	1.00e+05	4.82e+10	4.68e+10	1.43e+06
stair	3.40e+01	1.70e+15	1.70e+15	2.00e+14
standata	1.34e+02	6.19e+13	6.14e+13	5.43e+11
standmps	1.05e+03	8.54e+12	8.48e+12	9.36e+09
stocfor1	1.10e+03	2.22e+12	3.48e+12	1.05e+10
stocfor2	1.20e+03	8.15e+13	1.33e+14	5.46e+11
tuff	1.92e+03	5.09e+17	1.16e+17	1.31e+15
vtpbase	2.91e+02	2.41e+14	2.32e+14	1.16e+12
wood1p	1.91e+04	5.40e+12	6.54e+13	7.57e+11
woodw	6.24e+04	6.61e+12	3.16e+15	3.55e+11

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